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A NOTE ON $*$-SEMIMULTIPLIERS IN PRIME RINGS WITH INVOLUTION

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Abstract. Let R be a $*$-ring and g be a surjective map of R. An additive mapping $F : R \to R$ is called a $*$-semimultiplier if (1) $F(xy) = F(x)g(y^*) = g(x^*)F(y)$ (2) $F(g(x)) = g(F(x))$ for all $x, y \in R$. In this paper, we introduce the notion of $*$-semimultiplier of a ring R, and investigate the commutativity of prime rings satisfying certain identities involving $*$-semimultiplier of R.

1. Introduction

Many considerable works have been done on left (right) multipliers in prime and semiprime rings during the last couple of decades([10-12]). An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + yd(x)$ holds for all $x, y \in R$. Following [5], an additive mapping $F : R \to R$ is called a *generalized derivation* on R if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for every $x, y \in R$. Obviously, a generalized derivation with $d = 0$ covers the concept of left multiplicars. Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R. The first result in this direction is due to E. C. Posner [9] who proved that if a ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. This result was subsequently, refined and extended by a number of authors. In [7], Bresar and Vuckman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation, generalized derivation. In this paper, we introduce the notion of a $*$-semimultiplier of R, and investigate the commutativity of prime $*$-rings satisfying certain identities involving $*$-semimultiplier of R.

2. Preliminaries

Throughout R will represent an associative ring with center $Z(R)$. For all $x, y \in R$, as a usual commutator, we shall write $[x, y] = xy - yx$, and $x \circ y = xy + yx$. Also,

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we make use of the following two basic identities without any specific mention:

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$

$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$$

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z.$$

Recall that R is *prime* if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $x \to x^*$ of R into itself is called an *involution* if the following conditions are satisfied (i) $(xy)^* = y^*x^*$ (ii) $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called an $*$*-ring* or *ring with involution.* Let R is a ring. An additive mapping $F : R \to R$ is called a *left multiplier* if $F(xy) = F(x)y$ holds for every $x, y \in R$. Similarly, an additive mapping $F : R \to R$ is called a *right multiplier* if $F(xy) = xF(y)$ holds for every $x, y \in R$. If F is both a left and a right multiplier of R, then it is called a *multiplier* of R.

Definition 2.1. ([8]) Let R be a ring. An additive mapping $F : R \to R$ is called a *semimultiplier* associated with a surjective function $g : R \to R$ if
(a) $F(xy) = F(x)g(y) = g(x)F(y)$,
(b) $F(g(x)) = g(F(x))$, for every $x, y \in R$.

3. ∗-semimultipliers in prime rings with involution

Definition 3.1. Let R be a $*$-ring. An additive mapping $F : R \to R$ is called a $*$*-semimultiplier* associated with a surjective function $g : R \to R$ if
(a) $F(xy) = F(x)g(y^*) = g(x^*)F(y)$,
(b) $F(g(x)) = g(F(x))$, for every $x, y \in R$.

Lemma 3.2. *Let R be a prime $*$-ring and let g be a surjective function. Suppose that F is a $*$-semimultiplier associated with g and $a \in R$. If $aF(x) = 0$ for every $x \in R$, then $a = 0$ or $F = 0$.*

Proof. By hypothesis, we have $aF(x) = 0$ for any $x \in R$. Replacing x by xr in the last relation, we get

$$ag(x^*)F(r) = 0, \ \forall\ x, r \in R. \tag{1}$$

Replacing x by x^* in (1), we have $ag(x)F(r) = 0$ for all $x, r \in R$. Since g is onto, we have $aRF(r) = \{0\}$ for all $r \in R$. Using the fact that R is prime, we have $a = 0$ or $F(r) = 0$ for all $r \in R$. That is, $a = 0$ or $F = 0$.

□

Theorem 3.3. *Let R be a semiprime $*$-ring and let g be an automorphism on R. Suppose that R admits a nonzero $*$-semimultiplier F associated with g. Then F maps from R to $Z(R)$.*

Proof. By hypothesis,

$$F(xy) = F(x)g(y^*) = 0, \ \forall\ x, y \in R. \tag{2}$$

Replacing y by yz with $z \in R$, in (2), we obtain

$$\begin{aligned} F(xyz) &= F(x(yz)) = F(x)(g(yz)^*) \\ &= F(x)g(z^*y^*) = F(x)g(z^*)g(y^*) \end{aligned} \tag{3}$$

Also, we have

$$F(xyz) = F((xy)z) = F(xy)g(z^*) \\ = F(x)g(y^*)g(z^*) \tag{4}$$

Comparing (3) with (4), we get

$$F(x)[g(z^*), g(y^*)] = 0, \forall\ x, y, z \in R. \tag{5}$$

Substituting z^* for z and y^* for y in (5), we obtain

$$F(x)[g(z), g(y)] = 0, \forall\ x, y, z \in R. \tag{6}$$

Taking $zF(x)$ in place of z in (6), we get

$$F(x)g(z)[g(F(x)), g(y)] + F(x)[g(z), g(y)]g(F(x)) = 0$$

for all $x, y, z \in R$. By using the relation (6), we get

$$F(x)g(z)[g(F(x)), g(y)] = 0, \forall\ x, y, z \in R. \tag{7}$$

Multiplying by $g(yF(x))$ on left side of (7), we have

$$0 = g(yF(x))F(x)g(z)[g(F(x)), g(y)] \\ = g(y)g(F(x))F(x)g(z)[g(F(x)), g(y)]. \tag{8}$$

Multiplying by $g(F(x)y)$ on left side of (7), we have

$$0 = g(F(x)y)F(x)g(z)[g(F(x)), g(y)] \\ = g(F(x))g(y)F(x)g(z)[g(F(x)), g(y)]. \tag{9}$$

Comparing (8) with (9), we obtain $[g(F(x), g(y)]F(x)g(z)[g(F(x), g(y)] = 0$ for all $x, y, z \in R$. That is, $[g(F(x), g(y)]R[g(F(x), g(y)] = \{0\}$ for all $x, y \in R$. Since R is semiprime, we have $[g(F(x), g(y)] = 0$ for all $x, y \in R$. Hence we get

$$0 = [g(F(x), g(y)] = g(F(x))g(y) - g(y)g(F(x)) \\ = g(F(x)y) - g(yF(x)) = g(F(x)y - yF(x)) \\ = g[F(x), y] \tag{10}$$

for all $x, y \in R$. Since g is an automorphism of R, we get $[F(x), y] = 0$ for all $x, y \in R$. Hence F is a mapping from R into $Z(R)$.

□

Theorem 3.4. *Let R be a prime $*$-ring and let g is an automorphism on R. Suppose that R admits a nonzero $*$-semimultiplier F associated with g, then R is commutative.*

Proof. By hypothesis,

$$F(xy) = F(x)g(y^*) = 0,\ \forall\ x, y \in R. \tag{11}$$

Replacing y by yz with $z \in R$, in (11), we obtain

$$F(xyz) = F(x(yz)) = F(x)(g(yz)^*) \\ = F(x)g(z^*y^*) = F(x)g(z^*)g(y^*) \tag{12}$$

Also, we have

$$F(xyz) = F((xy)z) = F(xy)g(z^*) \\ = F(x)g(y^*)g(z^*) \tag{13}$$

Comparing (12) with (13), we get

$$F(x)[g(z^*), g(y^*)] = 0, \forall\ x, y, z \in R. \tag{14}$$

Substituting z^* for z and y^* for y in (14), we obtain

$$F(x)[g(z), g(y)] = 0, \forall\ x, y, z \in R. \tag{15}$$

Substituting $g^{-1}(z)$ for z and $g^{-1}(y)$ for y in this relation, we get

$$F(x)[z, y] = 0, \forall\ x, y, z \in R. \tag{16}$$

Replacing z by zr in the last equation, we have $F(x)z[r, y] = 0$, which implies that $F(y)R[z, y] = \{0\}$ for every $x, y, z \in R$. Since R is prime, we have $F(y) = 0$ or $[r, z] = 0$ for every $r, y, z \in R$. Since $F \neq 0$, we have $[r, y] = 0$ for every $x, z \in R$, which implies that R is commutative.

□

Theorem 3.5. *Let R be a prime $*$-ring and $a \in R$ and let g be an automorphism on R. If R admits a $*$-semimultiplier F of R and $[F(x), a] = 0$, then $F(x) = 0$ or $a \in Z(R)$.*

Proof. By hypothesis, we have

$$[F(xy), a] = 0,\ \forall\ x, y \in R, \tag{17}$$

which implies that $[F(x)g(y^*), a] = 0$ for all $x, y \in R$. That is,

$$F(x)[g(y^*), a] = 0,\ \forall\ x, y \in R. \tag{18}$$

Substituting y^* for y in this relation, we have $F(x)[g(y), a] = 0$ for all $y \in R$. Substituting $g^{-1}(y)$ for y in this relation, we have $F(x)[y, a] = 0$ for all $y \in R$. Again, taking yx in stead of y in the last relation, we obtain

$$F(x)y[x, a] = 0,\ \forall\ x, y \in R. \tag{19}$$

This implies that $F(x)R[x, a] = \{0\}$ for all $x \in R$. Since R is prime, we have $F(x) = 0$ or $a \in Z(R)$.

□

Definition 3.6. Let R be a $*$-ring. An additive mapping $F : R \to R$ is called a *reverse $*$-semimultiplier* associated with a surjective function $g : R \to R$ if

(a) $F(xy) = F(y)g(x^*) = g(y^*)F(x)$,

(b) $F(g(x)) = g(F(x))$, for every $x, y \in R$.

Theorem 3.7. *Let R be a prime $*$-ring and let g is an automorphism on R. Suppose that R admits a reverse $*$-semimultiplier F associated with g. If $F([x, y]) = 0$ for all $x, y \in R$, then $F(x) = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$F([x, y]) = 0,\ \forall\ x, y \in R. \tag{20}$$

Replacing x by xz in (20), we have

$$\begin{aligned} 0 &= F([xz, y]) = F(x[z, y] + [x, y]z) \\ &= F([z, y])g(x^*) + F(z)g([x, y]^*) \\ &= F(z)g([x, y]^*) \end{aligned} \tag{21}$$

for all $x, y, z \in R$. Substituting $g^{-1}([x, y]^*)$ for $[x, y]$ in (21), we have $F(z)[x, y] = 0$ for all $x, y, z \in R$. Also, replacing y by yr in this relation, we have $F(z)y[x, r] = 0$

for all $r, x, y \in R$. This implies that $F(z)R[x, r] = \{0\}$ for all $r, x \in R$. Since R is prime, we have $F(z) = 0$ or $[x, r] = 0$ for all $r, x \in R$. Let $K = \{z \in R | F(z) = 0\}$ and $L = \{x \in R | [x, r] = 0, \forall\, r \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $F = 0$. If $L = R$, then R is commutative.

□

Theorem 3.8. *Let R be a prime $*$-ring and let g is an automorphism on R. Suppose that R admits a reverse $*$-semimultiplier F associated with g. If $F(x \circ y) = 0$ for all $x, y \in R$, then $F(x) = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$F(x \circ y) = 0,\ \forall\ x, y \in R. \tag{22}$$

Replacing x by xy in (22), we have

$$\begin{aligned} 0 &= F(xy \circ y) = F((x \circ y)y) \\ &= F(y)g((x \circ y)^*) \end{aligned} \tag{23}$$

for all $x, y \in R$. Substituting $(x \circ y)^*$ for $(x \circ y)$ in (23), we have $F(y)g(x \circ y) = 0$ for all $x, y, z \in R$. Also, replacing $x \circ y$ by $g^{-1}(x \circ y)$ in this relation, we have $F(y)(x \circ y) = 0$ for all $x, y \in R$. Replacing x by yx in the last equation, we have $F(y)y(x \circ y) = 0$, which implies that $F(y)R(x \circ y) = \{0\}$ for every $x, y \in R$. Since R is prime, we have $x \circ y = 0$ or $F(y) = 0$ for all $x, y \in R$. Let $K = \{y \in R | F(y) = 0\}$ and $L = \{y \in R | x \circ y = 0\}$ for all $x, y \in R$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In first case, $F = 0$. In second case, If $R = L$, we have $x \circ y = 0$ for all $x, y \in R$. Replacing x by xz in the last relation and using the fact that $yx = -xy$, we obtain $x[z, y] = 0$ for all $x, y, z \in R$. That is, $R[z, y] = \{0\}$. This implies that $[z, y]R[z, y] = \{0\}$ for all $y, z \in R$. Since R is prime, we have $[z, y] = 0$ for all $y, z \in R$, which means that R is commutative.

□

Theorem 3.9. *Let R be a prime $*$-ring and let g is an automorphism on R. Suppose that R admits a reverse $*$-semimultiplier F associated with g. If $[F(x), y] = 0$ for all $x, y \in R$, then $F(x) = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$[F(x), y] = 0,\ \forall\ x, y \in R. \tag{24}$$

Replacing x by xz in (24), we have

$$\begin{aligned} 0 &= [F(xz), y] = [F(z)g(x^*), y] \\ &= F(z)[g(x^*), y] + [F(z), y]g(x^*) \\ &= F(z)[g(x^*), y] \end{aligned} \tag{25}$$

for all $x, y, z \in R$. Substituting x^* for x in (25), we have $F(z)[g(x), y] = 0$ for all $x, y, z \in R$. Since g is onto, we have $F(z)[x, y] = 0$ for all $x, y, z \in R$. Also, replacing y by yr in this relation, we have $F(z)y[x, r] = 0$ for all $r, x, y, z \in R$. This implies that $F(z)R[x, r] = \{0\}$ for all $r, x, z \in R$. Since R is prime, we have $F(z) = 0$ or $[x, r] = 0$ for all $r, x, z \in R$. Let $K = \{z \in R | F(z) = 0\}$ and $L = \{x \in R | [x, r] = 0, \forall\ r \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but

$(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $F = 0$. If $L = R$, then R is commutative. □

Theorem 3.10. *Let R be a prime $*$-ring and let g is an automorphism on R. Suppose that R admits a reverse $*$-semimultiplier F associated with g. If $F(x) \circ y = 0$ for all $x, y \in R$, then $F(x) = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$F(x) \circ y = 0, \ \forall \ x, y \in R. \tag{26}$$

Replacing x by xz in (26), we have

$$\begin{aligned} 0 &= F(xz) \circ y = F(z)g(x^*) \circ y \\ &= (F(z) \circ y)g(x^*) + F(z)[g(x^*), y] \\ &= F(z)[g(x^*), y] \end{aligned} \tag{27}$$

for all $x, y, z \in R$. Substituting x^* for x in (27), we have $F(z)[g(x), y] = 0$ for all $x, y, z \in R$. Since g is onto, we have $F(z)[x, y] = 0$ for all $x.y, z \in R$. Also, replacing y by yr in this relation, we have $F(z)y[x, r] = 0$ for all $r, x, y, z \in R$. This implies that $F(z)R[x, r] = \{0\}$ for all $r, x, z \in R$. Since R is prime, we have $F(z) = 0$ or $[x, r] = 0$ for all $r, x, z \in R$. Let $K = \{z \in R | F(z) = 0\}$ and $L = \{x \in R | [x, r] = 0, \forall \ r \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $F = 0$. If $L = R$, then R is commutative. □

Theorem 3.11. *Let R be a prime $*$-ring and let g is an automorphism on R. Suppose that R admits a reverse $*$-semimultiplier $F \neq 0$ associated with g. If $[F(x), F(y)] = 0$ for all $x, y \in R$, then R is commutative.*

Proof. By hypothesis, we have

$$[F(x), F(y)] = 0, \ \forall \ x, y \in R. \tag{28}$$

Replacing x by xz in (28), we have

$$\begin{aligned} 0 &= [F(xz), F(y)] = [F(z)g(x^*), F(y)] \\ &= F(z)[g(x^*), F(y)] + [F(z), F(y)]g(x^*) \\ &= F(z)[g(x^*), F(y)] \end{aligned} \tag{29}$$

for all $x, y, z \in R$. Substituting x^* for x in (29), we have $F(z)[g(x), F(y)] = 0$ for all $x, y, z \in R$. Since g is onto, we have $F(z)[x, F(y)] = 0$ for all $x, y, z \in R$. Also, replacing x by xr in this relation, we have $F(z)x[r, F(y)] + F(z)[x, F(y)]r = F(z)x[r, F(y)] = 0$ for all $r, x, y, z \in R$. This implies that $F(z)R[r, F(y)] = \{0\}$ for all $r, y, z \in R$. Since R is prime, we have $F(z) = 0$ or $[r, F(y)] = 0$ for all $r, y, z \in R$. Since $F \neq 0$, we have $[r, F(y)] = 0$ for all $r, y \in R$. By the same methods as we used in the last part proof of Theorem 3.9, we get the required result. □

Theorem 3.12. *Let R be a prime $*$-ring and let g is an automorphism on R. Suppose that R admits a reverse $*$-semimultiplier $F \neq 0$ associated with g. If $F(x) \circ F(y) = 0$ for all $x, y \in R$, then R is commutative.*

Proof. By hypothesis, we have

$$F(x) \circ F(y) = 0, \ \forall \ x, y \in R. \tag{30}$$

Replacing x by xz in (30), we have

$$\begin{aligned} 0 &= F(xz) \circ F(y) = F(z)g(x^*) \circ F(y) \\ &= (F(z) \circ F(y))x^* + F(z)[g(x^*), F(y)] \\ &= F(z)[g(x^*), F(y)] \end{aligned} \tag{31}$$

for all $x, y, z \in R$. Substituting x^* for x in (31), we have $F(z)[g(x), F(y)] = 0$ for all $x, y, z \in R$. Since g is onto, we obtain $F(z)[x, F(y)] = 0$ for all $x, y, z \in R$. By the same methods as we used in the last part proof of Theorem 3.11, we get the required result.

□

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