*Electronic Journal of Mathematical Analysis and Applications*
Vol. 8(1) Jan. 2020, pp. 199-208.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

---

# BEST APPROXIMATION OF A FUNCTION BY PRODUCT OPERATOR

H. K. NIGAM

Abstract. In this paper, we, obtain the best approximation of a function in generalized Zygmund class $Z_r^{(\lambda)}, r \geq 1$ [22], using $C^1 N_{p,q}$ operator of Fourier series. The result obtained in our first theorem generalizes the result of Lal [15]. Thus, the result of Lal [15] becomes a particular case of our theorem. Some important corollaries are also deduced from our main theorems.

## 1. Introduction

The studies of error estimation of a function $g$ in different Lipschitz classes by a trigonometric polynomial using single summability means have been done by the researchers [1], [3], [9], [14], [20]-[25] etc. in past few decades. The studies of error estimation of a function $g$ in different Lipschitz classes by the product means have been done by the researchers [10]-[12] [16], [18] etc. in recent past. Dikshit [5], for the first time, studied $|C^1 N_p|$ means of Fourier series. Dikshit [6, 7] also investigated $(F_1)$-effectiveness of $C^1 N_p$ method and its necessary condition is obtained by Kumar and Prasad [13]. Recently, Lal [15] has obtained the error estimates of a function in generalized Lipschitz class using $C^1 N_p$ means of Fourier series. The review of the above mentioned research works clearly suggests that the study of error estimates of a function $g$ in generalized Zygmund class $Z_r^{(\lambda)},\ r \geq 1$ using $C^1 N_{p,q}$ product means has not been done so far. Therefore, in this paper, we establish two theorems in order to obtain the best error estimates of a function $g$ in generalized Zygmund class $Z_r^{(\lambda)},\ r \geq 1$ using $C^1 N_{p,q}$ means of Fourier series. The result obtained in Theorem 1 generalizes the result of Lal [15]. Thus, the result of Lal [15] becomes a particular case of this theorem.

Let $g$ be a $2\pi$-periodic function and Lebesgue integrable on $[-\pi, \pi]$. The Fourier series of $g$ at a point $l$ is defined by

$$g(l) = \frac{a_0}{2} + \sum_{d=0}^{\infty} (a_d \cos dl + b_d \sin dl) \tag{1}$$

---

2010 *Mathematics Subject Classification.* 42A10, 41A10.

*Key words and phrases.* Degree of approximation, Generalized Zygmund class of function, $C^1$ means, $N_p$ means, $C^1 N_p$ product means, Fourier series.

Submitted Nov. 3, 2018.

with $d^{th}$ partial sums $s_d(l)$.
By following Hardy ([8], p.96), the $C^1$ transform is defined as the $d^{th}$ partial sum of $C^1$ means, which is given by

$$\begin{aligned} M_d = & \frac{s_0 + s_1 + s_2 + ..........s_d}{d+1} \\ = & \frac{1}{d+1}\sum_{k=0}^{d} s_k \to s \ as \ d \to \infty. \end{aligned} \tag{2}$$

then the Fourier series (1) is summable to $s$ by $C^1$ method.
By following Borwein [2] , let $\{p_d\}$ and $\{q_d\}$ be the sequence of constants, real or complex, such that

$$\begin{aligned} P_d = & \quad p_0 + p_1 + p_2 + ..........p_d = \sum_{\nu=0}^{d} p_\nu \to s \ as \ d \to \infty \\ Q_d = & \quad q_0 + q_1 + q_2 + ..........q_d = \sum_{\nu=0}^{d} p_\nu \to s \ as \ d \to \infty \\ R_d = & \quad p_0 q_d + p_1 q_{d-1} + p_2 q_{d-2} + ..........p_d q_0 = \sum_{\nu=0}^{d} p_\nu q_{d-\nu} \to s \ as \ d \to \infty. \end{aligned} \tag{3}$$

Given two sequences $\{p_d\}$ and $\{q_d\}$, convolution $(p * q)$ is defined as

$$R_d = (p * q)_d = \sum_{k=0}^{d} p_{d-k} q_k.$$

We write

$$M_d^{p,q} = \frac{1}{R_d}\sum_{k=0}^{d} p_{d-k} q_k s_k. \tag{4}$$

If $R_d \neq 0 \forall d$, then generalized Nörlund $(N, p, q)$ transform of the sequence $\{s_d\}$ is the sequence $\{M_d^{p,q}\}$. If $\{M_d^{p,q}\} \to s$ as $d \to \infty$, then the Fourier series (1) is summable to $s$ by $(N, p, q)$ method.
The product of $C^1$ means with $N_{p,q}$ means defines $C^1 N_{p,q}$ means and is given by

$$M_d^{C^1 N_{p,q}} = \frac{1}{d+1}\sum_{\nu=0}^{d} \frac{1}{R_\nu}\sum_{k=0}^{\nu} p_{\nu-k} q_k s_k. \tag{5}$$

If $M_d^{C^1 N_{p,q}} \to s$ as $d \to \infty$ then the Fourier series (1) is summable to $s$ by $C^1 N_{p,q}$ method.
Since $C^1$ and $N_{p,q}$ are regular methods so the regularity of $C^1$ and $N_{p,q}$ methods implies regularity of $C^1 N_{p,q}$ method.
**Remark 1**: $C^1 N_{p,q}$ means reduce to $C^1 N_p$ means if $q_d = 1 \forall d$.
The space of all functions ($2\pi$-periodic and integrable) be

$$L^r[0, 2\pi] = \left\{ g : [0, 2\pi] \to R; \int_0^{2\pi} | \ g(x) \ |^r dx < \infty \right\}, r \geq 1.$$

We define $\|\cdot\|$ by

$$\|g\|_r = \begin{cases} \left\{\frac{1}{2\pi}\int_0^{2\pi} \mid g(x) \mid^r dx\right\}^{\frac{1}{r}}, \ 1 \leq r < \infty \\ ess \sup\limits_{0<x<2\pi} |g(x)|, r = \infty. \end{cases}$$

As defined in Zygmund [27], $\lambda_1 : [0, 2\pi] \to R$ be an arbitrary function with $\lambda_1(l) > 0$ for $0 < l \leq 2\pi$ and $\lim\limits_{l\to 0+} \lambda_1(l) = \lambda_1(0) = 0$.

We also define

$$Z_r^{(\lambda_1)} = \left\{ g \in L^r[0, 2\pi] : r \geq 1, \sup_{l\neq 0} \frac{\|g(\cdot + l) + g(\cdot - l) - 2g(\cdot)\|_r}{\lambda_1(l)} < \infty \right\}$$

and

$$\|g\|_r^{(\lambda_1)} = \|g\|_r + \sup_{l\neq 0} \frac{\|g(\cdot + l) + g(\cdot - l) - 2g(\cdot)\|_r}{\lambda_1(l)}, r \geq 1.$$

Hence, the space $Z_r^{(\lambda_1)}$ is a Banach space under the norm $\|\|_r^{(\lambda_1)}$.
The completeness of the space $Z_r^{(\lambda_1)}$can be understood by considering the completeness of $L^r, r \geq 1$.
Now, We define

$$\|g\|_r^{(\lambda_2)} = \|g\|_r + \sup_{l\neq 0} \frac{\|g(\cdot + l) + g(\cdot - l) - 2g(\cdot)\|_r}{\lambda_2(l)}, r \geq 1.$$

**Remark 2**: $\lambda_1(l)$ and $\lambda_2(l)$ denote moduli of continuity of order two ([27]).
If $\frac{\lambda_1(l)}{\lambda_2(l)}$ be positive and non-decreasing, then

$$\|g\|_r^{(\lambda_2)} \leq \max\left(1, \frac{\lambda_1(2\pi)}{\lambda_2(2\pi)}\right) \|g\|_r^{(\lambda_1)} < \infty.$$

We observe that

$$Z_r^{(\lambda_1)} \subset Z_r^{(\lambda_2)} \subset L^r, r \geq 1.$$

**Remark 3**:
(i) If we take $r \to \infty$ in $Z_r^{(\lambda_1)}$ then $Z_r^{(\lambda_1)}$ reduces to $Z^{(\lambda_1)}$.
(ii) If we take $\lambda_1(l) = l^\alpha$ in $Z^{(\lambda_1)}$ then $Z^{(\lambda_1)}$ reduces to $Z_\alpha$.
(iii) If we take $\lambda_1(l) = l^\alpha$ in $Z_r^{(\lambda_1)}$ then $Z_r^{(\lambda_1)}$ reduces to $Z_{\alpha,r}$.
(iv) If we take $r \to \infty$ in $Z_{\alpha,r}$ then $Z_{\alpha,r}$ reduces to $Z_\alpha$.
(v) Let $0 \leq \delta_2 < \delta_1 < 1$, if $\lambda_1(l) = l^{\delta_1}$ and $\lambda_2(l) = l^{\delta_2}$ then $\frac{\lambda_1(l)}{\lambda_2(l)}$ is increasing, while $\frac{\lambda_1(l)}{l\lambda_2(l)}$ is decreasing.
The error estimation of function $g$ is given by

$$E_r(g) = \min \|g - l_d\|_r,$$

where $l_d$ is a trigonometric polynomial of degree $d$, [27].
We use the following notations:

$$\alpha_{(x)}(l) = g(x + l) + g(x - l) - 2g(x)$$

$$D_d(l) = \frac{1}{2\pi(d + 1)} \sum_{\nu=0}^{d} \frac{1}{R_\nu} \sum_{k=0}^{\nu} p_{\nu-k} q_k \frac{\sin(\nu - k + \frac{1}{2})l}{\sin \frac{l}{2}}$$

$$\tau \left(\text{Integral part of } \frac{1}{l}\right) = \left[\frac{1}{l}\right], R_\tau = R(1/l)$$

## 2. Main Theorems

**Theorem 1** Error estimation of the function g ($2\pi$-periodic) in generalized Zygmund class $Z_r^{(\lambda_1)}, r \geq 1$, by $C^1N_{p,q}$ means of Fourier series is given by

$$\inf_{M_d^{C^1N_{p,q}}} \|M_d^{C^1N_{p,q}}(g,\cdot) - g(\cdot)\|_r^{(\lambda_2)} = O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l^2\lambda_2(l)}\left\{\frac{1}{d+1} + l\right\} dl\right],$$

where $\lambda_1(l)$ and $\lambda_2(l)$ are as defined in remark 2 and $\frac{\lambda_1(l)}{\lambda_2(l)}$ is positive, non-decreasing.

**Theorem 2** Error estimation of the function g ($2\pi$-periodic) in generalized Zygmund class $Z_r^{(\lambda_1)}, r \geq 1$, by $C^1N_{p,q}$ means of Fourier series is given by

$$\inf_{M_d^{C^1N_{p,q}}} \|M_d^{C^1N_{p,q}}(g,\cdot) - g(\cdot)\|_r^{(\lambda_2)} = O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\{\log(d+1) + d(d+1)\}\right],$$

where $\frac{\lambda_1(l)}{l\lambda_2(l)}$ is non-decreasing in addition to the condition of Theorem 1.

## 3. Lemmas

**Lemma 1** [17]: Let $g \in Z_r^{(\lambda_1)}$, then for $0 < l \leq \pi$.

If $\lambda_1(l)$ and $\lambda_2(l)$ are as defined in remark 2, then
$\| \alpha_{(\cdot+z)}(l) + \alpha_{(\cdot-z)}(l) - 2\alpha_{(\cdot)}(l) \|_r = O\left(\lambda_2(|z|)\frac{\lambda_1(l)}{\lambda_2(l)}\right)$.

**Lemma 2** For $l \in \left(0, \frac{1}{d+1}\right)$, $|D_d(l)| = O(d+1)$

**Proof.** For $l \in \left(0, \frac{1}{d+1}\right)$, $\sin dl \leq dl, \sin(l/2) \geq l/\pi$ ([27]).

$$\begin{aligned}
|D_d(l)| =& \frac{1}{2\pi(d+1)}\left|\sum_{\nu=0}^{d}\frac{1}{R_\nu}\sum_{k=0}^{\nu} p_{\nu-k}q_k \frac{\sin\left(\nu-k+\frac{1}{2}\right)l}{\sin(l/2)}\right| \\
\leq& \frac{1}{2(d+1)}\sum_{\nu=0}^{d}\frac{1}{R_\nu}\sum_{k=0}^{\nu} p_{\nu-k}q_k \frac{\left(\nu-k+\frac{1}{2}\right)l}{l} \\
=& \frac{1}{4(d+1)}\sum_{\nu=0}^{d}\frac{1}{R_\nu}\sum_{k=0}^{\nu} p_{\nu-k}q_k(2\nu-2k+1) \\
\leq& \frac{1}{4(d+1)}\sum_{\nu=0}^{d}\frac{2\nu+1}{R_\nu}\sum_{k=0}^{\nu} p_{\nu-k}q_k \\
=& \frac{1}{4(d+1)}\sum_{\nu=0}^{d}(2\nu+1) \\
=& O(d+1).
\end{aligned}$$

**Lemma 3** For $l \in \left[\frac{1}{d+1}, \pi\right]$, $|D_d(l)| = O\left(\frac{\tau^2}{d+1}\right) + O\left(\frac{\tau R_\tau}{d+1} \sum_{\nu=\tau}^{d} \frac{1}{R_\nu}\right)$.

**Proof.** For $l \in \left[\frac{1}{d+1}, \pi\right]$, $\sin(l/2) \geq l/\pi$ ([27]).

$$\begin{aligned}|D_d(l)| =& \frac{1}{2\pi(d+1)} \left| \sum_{\nu=0}^{d} \frac{1}{R_\nu} \sum_{k=0}^{\nu} p_{\nu-k} q_k \frac{\sin\left(\nu - k + \frac{1}{2}\right) l}{\sin(l/2)} \right| \\ \leq& \frac{1}{2l(d+1)} \left| Im \sum_{\nu=0}^{d} \frac{1}{R_\nu} \sum_{k=0}^{\nu} p_{\nu-k} q_k e^{i\left(\nu-k+\frac{1}{2}\right)} \right|\end{aligned}$$

Using Abel's lemma,

$$\begin{aligned}\leq& \frac{1}{2l(d+1)} \left[ \left| \sum_{\nu=0}^{\tau-1} \frac{1}{R_\nu} \sum_{k=0}^{\nu} p_{\nu-k} q_k \mid e^{i(\nu-k)l} \mid \right| + \sum_{\nu=\tau}^{d} \frac{1}{R_\nu} \max_{0 \leq m \leq \nu} \left| \sum_{k=0}^{m} p_{\nu-k} q_k e^{i(\nu-k)l} \right| \right] \\ \leq& \frac{1}{2l(d+1)} \left[ \tau + R_\tau \sum_{\nu=\tau}^{d} \frac{1}{R_\nu} \right] \\ =& O\left(\frac{\tau^2}{d+1}\right) + O\left(\frac{\tau R_\tau}{d+1} \sum_{\nu=\tau}^{d} \frac{1}{R_\nu}\right).\end{aligned}$$

## 4. Proof of the Main Theorems

4.1. **Proof of Theorem 1.** Following [26], the integral representation of $s_d(g;x)$ is given by

$$s_d(g;x) - g(x) = \frac{1}{2\pi} \int_0^\pi \phi_x(l) \frac{\sin(d+\frac{1}{2})l}{\sin\frac{l}{2}} dl$$

Now denoting $C^1 N_{p,q}$ transform of $s_d(g;x)$ by $M^{C^1 N_{p,q}}$, we get

$$\begin{aligned}M_d^{C^1 N_{p,q}}(x) - g(x) &= \int_0^\pi \frac{\alpha_{(x)}(l)}{2\pi(d+1)} \left\{ \sum_{\nu=0}^{d} \frac{1}{R_\nu} \sum_{k=0}^{\nu} p_{\nu-k} q_k \frac{\sin\left(\nu - k + \frac{1}{2}\right) l}{\sin(l/2)} \right\} dl \\ &= \int_0^\pi \alpha_{(x)}(l) D_d(l) = \rho_d(l) (\text{say}). \qquad (6)\end{aligned}$$

Now,

$$\rho_d(x+z) + \rho_d(x-z) - 2\rho_d(x) = \int_0^\pi \left\{ \alpha_{(x+z)}(l) - \alpha_{(x-z)}(l) - 2\alpha_{(x)}(l) \right\} D_d(l) dl.$$

Using generalized Minkowski inequality [4], we can write

$$\begin{aligned}&\| \rho_d(\cdot + z) + \rho_d(\cdot - z) - 2\rho_d(\cdot) \|_r \\ &\leq \int_0^{\frac{1}{d+1}} \| \alpha_{(\cdot+z)}(l) - \alpha_{(\cdot-z)}(l) - 2\alpha_{(\cdot)}(l) \|_r \mid D_d(l) \mid dl \\ &+ \int_{\frac{1}{d+1}}^{\pi} \| \alpha_{(\cdot+z)}(l) - \alpha_{(\cdot-z)}(l) - 2\alpha_{(\cdot)}(l) \|_r \mid D_d(l) \mid dl \\ &= I_1 + I_2. \qquad (7)\end{aligned}$$

Now, using Lemmas 1 and 2,

$$\begin{aligned} I_1 =& O\left[\int_0^{\frac{1}{d+1}} \lambda_2(|z|)\frac{\lambda_1(l)}{\lambda_2(l)}(d+1)dl\right] \\ =& O\left[(d+1)\lambda_2(|z|)\int_0^{\frac{1}{d+1}} \frac{\lambda_1(l)}{\lambda_2(l)}dl\right] \\ =& O\left[(d+1)\lambda_2(|z|)\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\int_0^{\frac{1}{d+1}} dl\right] \\ =& O\left[\lambda_2(|z|)\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right]. \end{aligned} \tag{8}$$

Now, using Lemmas 1 and 3,

$$\begin{aligned} I_2 =& O\left[\int_{\frac{1}{d+1}}^{\pi} \lambda_2(|z|)\frac{\lambda_1(l)}{\lambda_2(l)}\left\{\left(\frac{\tau^2}{d+1}\right)+\left(\frac{\tau R_\tau}{d+1}\right)\sum_{\nu=\tau}^{d}\frac{1}{R_\nu}\right\}dl\right] \\ =& O\left[\frac{\lambda_2(|z|)}{(d+1)}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)}\left\{\tau^2+\tau R_\tau\sum_{\nu=\tau}^{d}\frac{1}{R_\nu}\right\}dl\right]. \end{aligned} \tag{9}$$

Combining (7), (8) and (9), we have

$$\begin{aligned} &\| \rho_d(\cdot+z)+\rho_d(\cdot-z)-2\rho_d(\cdot) \|_r \\ &= O\left[\lambda_2(|z|)\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right] + O\left[\frac{\lambda_2(|z|)}{(d+1)}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)}\left\{\tau^2+\tau R_\tau\sum_{\nu=\tau}^{d}\frac{1}{R_\nu}\right\}dl\right]. \\ &\sup_{z\neq 0}\frac{\| \rho_d(\cdot+z)+\rho_d(\cdot-z)-2\rho_d(\cdot) \|_r}{\lambda_2(|z|)} \\ &= O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right] + O\left[\frac{1}{(d+1)}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)}\left\{\tau^2+\tau R_\tau\sum_{\nu=\tau}^{d}\frac{1}{R_\nu}\right\}dl\right] \\ &= O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right] + O\left[\frac{1}{(d+1)}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)}\left\{\frac{1}{l^2}+\frac{1}{l}R_\tau\frac{(d+1)}{R_\tau}\right\}dl\right] \\ &= O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)}\left\{\frac{1}{l^2(d+1)}+\frac{1}{l}\right\}dl\right]. \end{aligned} \tag{10}$$

Again using Lemmas 2 and 3,

$$\begin{aligned}\| \rho_d(\cdot) \|_r \leq& \left[\int_0^{\frac{1}{d+1}} + \int_{\frac{1}{d+1}}^{\pi}\right] \| \alpha_{(\cdot)}(l) \|_r | D_d(l) | \, dl \\ =& O\left[(d+1)\int_0^{\frac{1}{d+1}} \lambda_1(l)dl\right] + O\left[\frac{1}{(d+1)}\int_{\frac{1}{d+1}}^{\pi}\left\{\tau^2 + \tau R_\tau \sum_{\nu=\tau}^{d}\frac{1}{R_\nu}\right\}\lambda_1(l)dl\right] \\ =& O\left[\lambda_1\left(\frac{1}{d+1}\right)\right] + O\left[\frac{1}{(d+1)}\int_{\frac{1}{d+1}}^{\pi}\left\{\frac{1}{l^2} + \frac{1}{l}R_\tau\frac{(d+1)}{R_\tau}\right\}\lambda_1(l)dl\right] \\ =& O\left[\lambda_1\left(\frac{1}{d+1}\right)\right] + O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l^2(d+1)}dl + \int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l}dl\right]. \qquad (11)\end{aligned}$$

Now, we have

$$\| \rho_d(\cdot) \|_r^{(\lambda_2)} = \| \rho_d(\cdot) \|_r + \sup_{z\neq 0}\frac{\| \rho_d(\cdot + z) + \rho_d(\cdot - z) - 2\rho_d(\cdot) \|_r}{\lambda_2(z)}.$$

From (10) and (11), we get

$$\begin{aligned}\| \rho_d(\cdot) \|_r^{(\lambda_2)} =& O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right] + O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{\lambda_2(l)}\left\{\frac{1}{l^2(d+1)} + \frac{1}{l}\right\}dl\right] \\ &+ O\left[\lambda_1\left(\frac{1}{d+1}\right)\right] + O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l^2(d+1)}dl + \int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l}dl\right].\end{aligned}$$

In view of monotonicity of $\lambda_2(l)$, we have
$\lambda_1(l) = \frac{\lambda_1(l)}{\lambda_2(l)}\lambda_2(l) \leq \lambda_2(\pi)\frac{\lambda_1(l)}{\lambda_2(l)} = O\left(\frac{\lambda_1(l)}{\lambda_2(l)}\right)$ for $0 < l \leq \pi$. Hence

$$\begin{aligned}\| \rho_d(\cdot) \|_r^{(\lambda_2)} =& O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right] + O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{\lambda_2(l)}\left\{\frac{1}{l^2(d+1)} + \frac{1}{l}\right\}dl\right] \\ =& O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right] + O\left[\frac{1}{d+1}\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l^2\lambda_2(l)}dl\right] + O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l\lambda_2(l)}dl\right]. \qquad (12)\end{aligned}$$

Since $\lambda_1$ and $\lambda_2$ are as defined in remark 2 and $\frac{\lambda_1(l)}{\lambda_2(l)}$ is positive, non-decreasing, therefore,

$$\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{\lambda_2(l)}\left\{\frac{1}{l^2(d+1)}\right\}dl \geq \frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\int_{\frac{1}{d+1}}^{\pi}\left\{\frac{1}{l^2(d+1)}\right\}dl \geq \frac{\lambda_1\left(\frac{1}{d+1}\right)}{2\lambda_2\left(\frac{1}{d+1}\right)}.$$

Then

$$\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)} = O\left[\frac{1}{d+1}\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l^2\lambda_2(l)}dl\right]. \qquad (13)$$

From (12) and (13), we get

$$\| \rho_d(\cdot) \|_r^{(\lambda_2)} = O\left[\frac{1}{d+1}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l^2\lambda_2(l)}dl\right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)}dl\right]$$

$$E_d(g) = O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l^2\lambda_2(l)}\left(\frac{1}{d+1}+l\right)dl\right]$$

This completes the proof of Theorem 1.

4.2. **Proof of Theorem 2.** Following the proof of Theorem 1,

$$E_d(g) = O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l^2\lambda_2(l)}\left(\frac{1}{d+1}+l\right)dl\right]$$

Since $\frac{\lambda_1(l)}{l\lambda_2(l)}$ is positive, non-decreasing, therefore by second mean value theorem of integral calculus,

$$E_d(g) = \left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\int_{\frac{1}{d+1}}^{\pi} \frac{1}{l}dl + \frac{(d+1)\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\int_{\frac{1}{d+1}}^{\pi} 1\ dl\right]$$

$$= O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\{\log(d+1)+(d+1)\}\right]$$

This completes the proof of Theorem 2.

## 5. Corollaries

**Corollary 1** Error estimates of function g ($2\pi$-periodic) in the class $Z_{\alpha,r}, r \geq 1$, using $C^1N_{p,q}$ means of Fourier Series is given by

$$\inf_{M_d^{C^1N_{p,q}}} \|M_d^{C^1N_{p,q}}(g,\cdot)-g(\cdot)\|_r^{(\lambda_2)} = \begin{cases} O\left\{(d+1)^{\delta_1-\delta_2}\right\}, 0 \leq \delta_2 < \delta_1 < 1 \\ O\left\{(d+1)^{-1}\log(d+1)+1\right\}, \delta_2 = 0, \delta_1 = 1 \end{cases}$$

**Proof.** Putting $\lambda_1(l) = l^{\delta_1}$ and $\lambda_2(l) = l^{\delta_2}$ in Theorems 1 and 2, the result follows.

**Corollary 2** If $q_d = 1$ for all d in Theorem 1, then error estimates of function g ($2\pi$-periodic) in the generalized Zygmund class $Z_r^{(\lambda_2)}, r \geq 1$, using $C^1N_p$ means of Fourier Series is given by

$$\inf_{M_d^{C^1N_p}} \|M_d^{C^1N_p}(g,\cdot) - g(\cdot)\|_r^{(\lambda_2)} = O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l^2\lambda_2(l)}\left(\frac{1}{d+1}+l\right)dl\right],$$

where $\lambda_1(l)$ and $\lambda_2(l)$ are as defined in remark 2 and $\frac{\lambda_1(l)}{\lambda_2(l)}$ is positive, non-decreasing.

**Corollary 3** If $q_d = 1$ for all d in Theorems 1 and 2, then error estimates of function g ($2\pi$-periodic)in the class $Z_{\alpha,r}, r \geq 1$, using $C^1N_p$ means of Fourier Series is given by

$$\inf_{M_d^{C^1N_p}} \|M_d^{C^1N_p}(g,\cdot)-g(\cdot)\|_r^{(\lambda_2)} = \begin{cases} O\left\{(d+1)^{\delta_1-\delta_2}\right\}, 0 \leq \delta_2 < \delta_1 < 1 \\ O\left\{(d+1)^{-1}\left(\log(d+1)+1\right)\right\}, \delta_2 = 0, \delta_1 = 1 \end{cases}$$

**Proof.** Putting $\lambda_1(l) = l^{\delta_1}$ and $\lambda_2(l) = l^{\delta_2}$ in Theorems 1 and 2, the result follows.

## 6. Particular Case

1. If we take $\lambda_1(l) = l^{\delta_1}$ and $\lambda_2(l) = l^{\delta_2}, r \to \infty$ and $\delta_2 = 0$ in Theorem 1 and also as per remark ([23], p. 6870), Theorem 1 of Lal [15] becomes a particular case of our Theorem 1.

**Acknowledgment**
The author expresses his gratitude towards his mother for her blessings. The author also expresses his gratitude towards his father in heaven, whose soul is always guiding and encouraging him. This research work is supported by Council of Scientific and Industrial Research, Government of India under the project 25/(0225)/13/EMR-II.

## References

[1] G. Alexits, Convergence problems of orthogonal series, Pergamon Press, London, 1961.
[2] D. Borwein, On Product of sequence, Journal of the London Mathematical Society, 33, 352-357, 1958.
[3] P. Chandra, Trigonometric approximation of functions in $L_p$ - norm, J. Math. Anal. Appl., 275(1), 13-26, 2002.
[4] C. K. Chui, An introduction to wavelets (Wavelet analysis and its application), 1, Academic Press, USA, 1992.
[5] H. P. Dikshit, Absolute $(C, 1)(N, p_n)$ summability of a Fourier series and its conjugate series, Pacific Journal of Mathematics, 26(2), 245-256, 1968.
[6] H. P. Dikshit, Summability of Fourier series by triangular matrix transformation, Pacific J. Math. 30, 399-410, 1969.
[7] H. P. Dikshit, Total and absolute total effective product method, Ann. Math. Pura. Appl. 94, 11-31, 1972.
[8] G. H. Hardy, Divergent series, first edition, Oxford University Press, 1949.
[9] H. H. Khan, On degree of approximation of functions belonging to the class Lip $(\alpha, p)$, Indian J. Pure Appl. Math., 5(2), 132-136, 1974.
[10] H. K. Nigam., Degree of approximation of a function belonging to weighted $W(L_r, \xi(t))$ class by (C, 1)(E, q) means, Tamkang J. of Mathematics, 42(1), 31-37, 2011.
[11] H. K. Nigam, Degree of approximation of functions belonging to $Lipa$ class and $W(L_r, \xi(t))$ class by product summability method, Surveys in Math. and its Appl., 5, 113-122, 2010.
[12] H. K. Nigam and K. Sharma, Degree of approximation of a class of a function by $(C, 1)(E, q)$ means of Fourier series, IAENG International Journal of Applied Mathematics, 41(2), 128-132, 2011.
[13] A. Kumar and G. V. Prasad, A necessary condition for F1-effectiveness of $(C, 1)(N, p_n)$ method, J. Indian Math. Soc. (N.S.), 54, 139-146, 1989.
[14] L. Leindler, Trigonometric approximation in $L_p$- norm, J. Math. Anal. Appl., 302(1), 129-136, 2005.
[15] S. Lal, Approximation of functions belonging to the generalized Lipschitz class by $C_1 N_p$ summability method of Fourier series, Applied Mathematics and Computation, 209, 346-350, 2009.
[16] S. Lal, On degree of approximation of functions belonging to the Weighted $W(L_p, \xi(t)))$ class by (C, 1)(E, 1) means, Tamkang Journal of Mathematics, 30, 47-52, 1999.
[17] S. Lal and A. Mishra, The method of summation $(E, 1)(N, p_n)$ and trigonometric approximation of function in generalized Hölder metric, J. Indian Math. Soc., 80(1-2), 87-98, 2013.
[18] S. Lal and J. K. Kushwaha, Degree of approximation of Lipschitz function by product summability method, International Mathematical Forum, 43(4), 2101-2107, 2009.
[19] L. McFadden, Absolute Nörlund summability, Duke Math. J., 9, 168-207, 1942.
[20] K. Qureshi, On the degree of approximation of a periodic function $f$ by almost Nörlund means, Tamkang J. Math. 12(1), 35-38, 1981.
[21] K. Qureshi, On the degree of approximation of a function belonging to the class Lip$\alpha$ , Indian J. pure Appl. Math., 13(8), 560-563, 1982.

[22] K. Qureshi and H. K. Neha, A class of functions and their degree of approximation, Ganita, 41(1), 37-42, 1990.
[23] B.E. Rhoades, K. Ozkoklu and I. Albayrak, On the degree of approximation of functions belonging to Lipschitz class by Hausdorff means of its Fourier series, Applied Mathematics and Computation, 217, 6868-6871, 2011.
[24] B. N. Sahney, and D. S. Goel, On the degree of continuous functions, Ranchi University Math. Jour., 4, 50-53, 1973.
[25] B. N. Sahney and V. Rao, Errors bounds in the approximation of functions, Bull. Australian Math. Soc. 6, 11-18, 1972.
[26] E. C. Titchmarsh, The Theory of functions, Oxford University Press, 402-403, 1939.
[27] A. Zygmund, Trigonometric series, Cambridge Univ. Press, Cambridge, 3rd rev. ed., 2002.

H. K. Nigam
Central University of South Bihar,, Gaya, Bihar, India
*E-mail address*: hknigam@cusb.ac.in