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SOME GROWTH PROPERTIES OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES ON THE BASIS OF THEIR (p,q)-φ RELATIVE GOL'DBERG ORDER AND (p,q)-φ RELATIVE GOL'DBERG LOWER ORDER

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Abstract. In this paper our primary concern is to discuss some basic properties of entire functions of several complex variables based upon (p,q)-φ relative Gol'dberg order and (p,q)-φ relative Gol'dberg lower order, where p and q are any two positive integers and $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function.

1. Introduction

Usually, the complex and real n-spaces are denoted by the respective symbols $\mathbb{C}^n$ and R^n. In addition, let us assume that the points $(z_1, z_2, \cdots, z_n)$, $(m_1, m_2, \cdots, m_n)$ of $\mathbb{C}^n$ or I^n be represented by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers. Then the modulus of z, denoted by $|z|$, is defined as $|z| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then the expression $z_1^{m_1} \cdots z_n^{m_n}$ will be denoted by z^m where $\|m\| = m_1 + \cdots + m_n$.

Consider $D \subseteq \mathbb{C}^n$ to be an arbitrary bounded complex n-circular domain with center at the origin of coordinates. Then for any entire function $f(z)$ of n complex variables and $R > 0$, $M_{f,D}(R)$ may be defined as $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$ where a point $z \in D_R$ if and only if $\frac{z}{R} \in D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists such that $\lim_{R\to\infty} M_{f,D}^{-1}(R) = \infty$.

For $k \in \mathbb{N}$, we define $\exp^{[k]} R = \exp\left(\exp^{[k-1]} R\right)$ and $\log^{[k]} R = \log\left(\log^{[k-1]} R\right)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} R = R$, $\log^{[-1]} R = \exp R$, $\exp^{[0]} R = R$ and $\exp^{[-1]} R = \log R$. Further we assume that throughout the present paper p, q and m always denote positive integers. Also throughout

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the paper an entire function $f(z)$ of n-complex variables will stand for an entire function $f(z)$ for any bounded complete n-circular domain D with center at origin in $\mathbb{C}^n$. Taking this into account, we recall that Datta et al. [5] defined the concept of (p,q)-th Gol'dberg order and (p,q)-th Gol'dberg lower order of an entire function $f(z)$ of n-complex variables where $p \geq q$ in the following way:

$$\rho_D^{(p,q)}(f) = \limsup_{R\to+\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}$$

and

$$\lambda_D^{(p,q)}(f) = \liminf_{R\to+\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}.$$

For $p = 2$ and $q = 1$, the symbols $\rho_D^{(2,1)}(f)$ and $\lambda_D^{(2,1)}(f)$ are respectively denoted by $\rho_D(f)$ and $\lambda_D(f)$ which are actually classical growth indicators (see e.g. [8, 9]). However in the line of Gol'dberg (see e.g. [8, 9]), it may be easily established that $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ are independent of the choice of the domain D, and therefore one can write $\rho^{(p,q)}(f)$ and $\lambda^{(p,q)}(f)$ instead of $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$) respectively.

In [12], Shen et al. introduced the definition of (p,q)-φ order of an entire function. For details about (p,q)-φ order, one may see [12]. Consequently the definition of (p,q)-φ Gol'dberg order and (p,q)-φ Gol'dberg lower order of an entire function $f(z)$ of n-complex variables are established in [4] which are as follows:

Definition 1. [4] *Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Then the (p,q)-φ Gol'dberg order $\rho_D^{(p,q)}(f,\varphi)$ and (p,q)-φ Gol'dberg lower order $\lambda_D^{(p,q)}(f,\varphi)$ of an entire function $f(z)$ of n-complex variables are defined as*

$$\rho_D^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \varphi(R)}$$

and

$$\lambda_D^{(p,q)}(f,\varphi) = \liminf_{R\to+\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \varphi(R)}.$$

The above definition avoids the restriction $p \geq q$. However, an entire function $f(z)$ for which $\rho_D^{(p,q)}(f,\varphi)$ and $\lambda_D^{(p,q)}(f,\varphi)$ are the same is called a function of regular (p,q)-φ Gol'dberg growth. Otherwise, $f(z)$ is said to be irregular (p,q)-φ Gol'dberg growth. For any non-decreasing unbounded function $\varphi(R) : [0,+\infty) \to (0,+\infty)$, if it is assumed that $\lim_{R\to+\infty} \frac{\log^{[q]} \varphi(\alpha R)}{\log^{[q]} \varphi(R)} = 1$ for all $\alpha > 0$, then one can easily verify that $\rho_D^{(p,q)}(f,\varphi)$ and $\lambda_D^{(p,q)}(f,\varphi)$ are independent of the choice of the domain D, and therefore one can use the symbols $\rho^{(p,q)}(f,\varphi)$ and $\lambda^{(p,q)}(f,\varphi)$ instead of $\rho_D^{(p,q)}(f,\varphi)$ and $\lambda_D^{(p,q)}(f,\varphi)$) respectively.

Concerning this we just state the following definition:

Definition 2. *An entire function $f(z)$ of n-complex variables is said to have index-pair (p,q)-φ if $b < \rho^{(p,q)}(f,\varphi) < \infty$ and $\rho^{(p-1,q-1)}(f,\varphi)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if*

$0 < \rho^{(p,q)}(f,\varphi) < \infty$, *then*

$$\begin{cases} \rho^{(p-n,q)}(f,\varphi) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f,\varphi) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f,\varphi) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f,\varphi) < \infty$,

$$\begin{cases} \lambda^{(p-n,q)}(f,\varphi) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f,\varphi) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f,\varphi) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

If $\varphi(R) = R$ and $p \geq q$, then definition 1 coincides with the definition of (p,q)-th Gol'dberg order and (p,q)-th Gol'dberg lower order introduced by Datta et al. [5]. Consequently for $\varphi(R) = R$, Definition 2 reduces to the the definition of index-pair (p,q) of an entire function $f(z)$ of n-complex variables. For detail about index-pair (p,q) of an entire function $f(z)$ of n-complex variables, one may see [3].

However for any two entire functions $f(z)$ and $g(z)$ of n-complex variables, Mondal et al. [10] introduced the concept relative Gol'dberg order of $f(z)$ with respect to $g(z)$. In the case of relative Gol'dberg order, it therefore seems reasonable to define suitably the (p,q)-th relative Gol'dberg order. With this in view one can introduce the following definition in the light of index-pair.

Definition 3. [3] *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Then the (p,q)-th relative Gol'dberg order $\rho_{g,D}^{(p,q)}(f)$ and (p,q)-th relative Gol'dberg lower order $\lambda_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as*

$$\begin{matrix} \rho_{g,D}^{(p,q)}(f) \\ \lambda_{g,D}^{(p,q)}(f) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} R}.$$

Definition 3 avoids the restriction $p \geq q$ of Definition 1.3 of [1]. In view of Theorem 2.1 of [1] one can easily prove that $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$ are independent of the choice of the domain D, and therefore one can write $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ instead of $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$.

Further an entire function $f(z)$ of n-complex variables for which $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ are the same is called a function of regular relative (p,q) Gol'dberg growth with respect to an entire function $g(z)$ of n-complex variables. Otherwise, $f(z)$ is said to be irregular relative (p,q) Gol'dberg growth.with respect to $g(z)$.

Now in order to make some progress in the study of relative Gol'dberg order, in [4], the definition of (p,q)-φ relative Gol'dberg order and the (p,q)-φ relative Gol'dberg lower order in the light of index-pair are given which are as follows:

Definition 4. [4] *Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Also let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables. The (p,q)-φ relative Gol'dberg order and the (p,q)-φ relative Gol'dberg lower order of $f(z)$ with respect to $g(z)$ are defined as*

$$\rho_{g,D}^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \varphi(R)}$$

and

$$\lambda_{g,D}^{(p,q)}(f,\varphi) = \liminf_{R\to+\infty} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \varphi(R)}.$$

Further an entire function $f(z)$ of n-complex variables for which $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ are the same is called a function of regular (p,q)-φ relative Gol'dberg growth with respect to an entire function $g(z)$ of n-complex variables. Otherwise, $f(z)$ is said to be irregular (p,q)-φ relative Gol'dberg growth with respect to $g(z)$.

With time various authors {cf. [1, 2, 3, 5, 6, 7, 10, 11]} gradually enrich the study of growth properties of entire functions of several complex variables introducing different growth indicators such as Gol'dberg order, (p,q)-th Gol'dberg order, relative (p,q)-th Gol'dberg order etc. as tools. In this paper our primary concern is to discuss some basic properties of entire functions of several complex variables based upon (p,q)-φ relative Gol'dberg order and (p,q)-φ relative Gol'dberg lower order.

2. **Main Result**

In this section we present the main result of the paper. Further in order to establish our result, we assume that the nondecreasing unbounded function $\varphi(R) : [0,+\infty) \to (0,+\infty)$ always satisfies $\lim_{R\to+\infty} \frac{\log^{[q]}\varphi(\alpha R)}{\log^{[q]}\varphi(R)} = 1$ for all $\alpha > 0$. Since, Biswas et al. [4] have already shown that $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ are independent of the choice of the domain D when $\varphi(R) : [0,+\infty) \to (0,+\infty)$ is a nondecreasing unbounded function and satisfies $\lim_{R\to+\infty} \frac{\log^{[q]}\varphi(\alpha R)}{\log^{[q]}\varphi(R)} = 1$ for all $\alpha > 0$, so after this we shall always use the notations $\rho_{g}^{(p,q)}(f,\varphi)$ and $\lambda_{g}^{(p,q)}(f,\varphi)$ instead of $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ respectively.

Theorem 1. *Let us consider $f(z)$, $g(z)$ and $h(z)$ are any three entire functions of n-complex variables. Also let $0 < \lambda_h^{(m,q)}(f,\varphi) \le \rho_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \le \rho_h^{(m,p)}(g) < \infty$. Then*

$$\frac{\lambda_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)} \le \lambda_g^{(p,q)}(f,\varphi) \le \min\left\{\frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}\right\}$$
$$\le \max\left\{\frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}\right\} \le \rho_g^{(p,q)}(f,\varphi) \le \frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}.$$

Proof. From the definitions of $\rho_g^{(p,q)}(f,\varphi)$ and $\lambda_g^{(p,q)}(f,\varphi)$ it follows that

$$\log \rho_g^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty}\left(\log^{[p+1]} M_{g,D}(R) - \log^{[q+1]}\varphi(M_{f,D}(R))\right), \quad (1)$$

$$\log \lambda_g^{(p,q)}(f,\varphi) = \liminf_{R\to+\infty}\left(\log^{[p+1]} M_{g,D}(R) - \log^{[q+1]}\varphi(M_{f,D}(R))\right). \quad (2)$$

Now from the definitions of $\rho_h^{(m,q)}(f,\varphi)$ and $\lambda_h^{(m,q)}(f,\varphi)$, we obtain that

$$\log \rho_h^{(m,q)}(f,\varphi) = \limsup_{R\to+\infty}\left(\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi(M_{f,D}(R))\right), \quad (3)$$

$$\log \lambda_h^{(m,q)}(f,\varphi) = \liminf_{R\to+\infty}\left(\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi\left(M_{f,D}(R)\right)\right). \quad (4)$$

Similarly, from the definitions of $\rho_h^{(m,p)}(g)$ and $\lambda_h^{(m,p)}(g)$, we get that

$$\log \rho_h^{(m,p)}(g) = \limsup_{R\to+\infty}\left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right), \quad (5)$$

$$\log \lambda_h^{(m,p)}(g) = \liminf_{R\to+\infty}\left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right). \quad (6)$$

Therefore in view of (2), (4) and (5), it follows that

$$\log \lambda_g^{(p,q)}(f,\varphi) = \liminf_{R\to+\infty}\Big[\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi\left(M_{f,D}(R)\right) - \left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right)\Big]$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f,\varphi) \geq \Big[\liminf_{R\to+\infty}\left(\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi\left(M_{f,D}(R)\right)\right) - \limsup_{R\to+\infty}\left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right)\Big]$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f,\varphi) \geq \left(\log \lambda_h^{(m,q)}(f,\varphi) - \log \rho_h^{(m,p)}(g)\right). \quad (7)$$

In the similar way, from (1), (3) and (6), it follows that

$$\log \rho_g^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty}\Big[\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi\left(M_{f,D}(R)\right) - \left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right)\Big]$$

$$i.e.,\ \log \rho_g^{(p,q)}(f,\varphi) \leq \Big[\limsup_{R\to+\infty}\left(\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi\left(M_{f,D}(R)\right)\right) - \liminf_{R\to+\infty}\left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right)\Big]$$

$$i.e.,\ \log \rho_g^{(p,q)}(f,\varphi) \leq \left(\log \rho_h^{(m,q)}(f,\varphi) - \log \lambda_h^{(m,p)}(g)\right). \quad (8)$$

Again, in view of (2) we obtain that

$$\log \lambda_g^{(p,q)}(f,\varphi) = \liminf_{R\to+\infty}\Big[\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi\left(M_{f,D}(R)\right) - \left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right)\Big].$$

Assuming $A = \left(\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi\left(M_{f,D}(R)\right)\right)$ and $B = \left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right)$, we get from above that

$$\log \lambda_g^{(p,q)}(f,\varphi) \leq \min\left(\liminf_{R\to+\infty} A + \limsup_{R\to+\infty} -B, \limsup_{R\to+\infty} A + \liminf_{R\to+\infty} -B\right)$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f,\varphi) \leq \min\left(\liminf_{R\to+\infty} A - \liminf_{R\to+\infty} B, \limsup_{R\to+\infty} A - \limsup_{R\to+\infty} B\right).$$

Therefore in view of (3), (4), (5) and (6) it follows from above that

$$\log \lambda_g^{(p,q)}(f,\varphi) \leq \min\left\{\log \lambda_h^{(m,q)}(f,\varphi) - \log \lambda_h^{(m,p)}(g), \log \rho_h^{(m,q)}(f,\varphi) - \log \rho_h^{(m,p)}(g)\right\}. \quad (9)$$

Further from (1) we obtain that

$$\log \rho_g^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty}\Big[\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi(M_{f,D}(R)) - \left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right)\Big].$$

By taking $A = \left(\log^{[m+1]} M_{h,D}(R) - \log^{[q+1]}\varphi(M_{f,D}(R))\right)$ and $B = \left(\log^{[m+1]} M_{h,D}(R) - \log^{[p+1]} M_{g,D}(R)\right)$, it follows from above that

$$\log \rho_g^{(p,q)}(f,\varphi) \geq \max\left(\liminf_{R\to+\infty} A + \limsup_{R\to+\infty} -B, \limsup_{R\to+\infty} A + \liminf_{R\to+\infty} -B\right)$$

$$i.e.,\ \log \rho_g^{(p,q)}(f,\varphi) \geq \max\left(\liminf_{R\to+\infty} A - \liminf_{R\to+\infty} B, \limsup_{R\to+\infty} A - \limsup_{R\to+\infty} B\right).$$

Therefore in view of (3), (4), (5) and (6), we get from above that

$$\log \rho_g^{(p,q)}(f,\varphi) \geq \max\left\{\log \lambda_h^{(m,q)}(f,\varphi) - \log \lambda_h^{(m,p)}(g), \log \rho_h^{(m,q)}(f,\varphi) - \log \rho_h^{(m,p)}(g)\right\}. \quad (10)$$

Hence from (7), (8), (9) and (10), the conclusion of the theorem is established. □

In view of Theorem 1, one can easily verify the following corollaries:

Corollary 1. *Let us consider $f(z)$, $g(z)$ and $h(z)$ are any three entire functions of n-complex variables. Also let $0 < \lambda_h^{(m,q)}(f,\varphi) = \rho_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$. Then*

$$\lambda_g^{(p,q)}(f,\varphi) = \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)} \quad \text{and} \quad \rho_g^{(p,q)}(f,\varphi) = \frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}.$$

Corollary 2. *Let us consider $f(z)$, $g(z)$ and $h(z)$ are any three entire functions of n-complex variables. Also let $0 < \lambda_h^{(m,q)}(f,\varphi) \leq \rho_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g) < \infty$. Then*

$$\lambda_g^{(p,q)}(f,\varphi) = \frac{\lambda_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)} \quad \text{and} \quad \rho_g^{(p,q)}(f,\varphi) = \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}.$$

Corollary 3. *Let us consider $f(z)$, $g(z)$ and $h(z)$ are any three entire functions of n-complex variables. Also let $0 < \lambda_h^{(m,q)}(f,\varphi) = \rho_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g) < \infty$. Then*

$$\lambda_g^{(p,q)}(f,\varphi) = \rho_g^{(p,q)}(f,\varphi) = \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}.$$

Moreover when $\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)$, *then*

$$\lambda_g^{(p,q)}(f,\varphi) = \rho_g^{(p,q)}(f,\varphi) = 1 \ .$$

Corollary 4. *Let us consider* $f(z)$, $g(z)$ *and* $h(z)$ *are any three entire functions of* n*-complex variables. Also let* $0 < \lambda_h^{(m,q)}(f,\varphi) \le \rho_h^{(m,q)}(f,\varphi) < \infty$. *Then*

$$\begin{aligned} (i)\ \lambda_g^{(p,q)}(f,\varphi) &= \infty \text{ when } \rho_h^{(m,p)}(g) = 0\ ,\\ (ii)\ \rho_g^{(p,q)}(f,\varphi) &= \infty \text{ when } \lambda_h^{(m,p)}(g) = 0\ ,\\ (iii)\ \lambda_g^{(p,q)}(f,\varphi) &= 0 \text{ when } \rho_h^{(m,p)}(g) = \infty \end{aligned}$$

and

$$(iv)\ \rho_g^{(p,q)}(f,\varphi) = 0 \text{ when } \lambda_h^{(m,p)}(g) = \infty.$$

Corollary 5. *Let us consider* $f(z)$, $g(z)$ *and* $h(z)$ *are any three entire functions of* n*-complex variables. Also let* $0 < \lambda_h^{(m,p)}(g) \le \rho_h^{(m,p)}(g) < \infty$. *Then*

$$\begin{aligned} (i)\ \rho_g^{(p,q)}(f,\varphi) &= 0 \text{ when } \rho_h^{(m,q)}(f,\varphi) = 0\ ,\\ (ii)\ \lambda_g^{(p,q)}(f,\varphi) &= 0 \text{ when } \lambda_h^{(m,q)}(f,\varphi) = 0\ ,\\ (iii)\ \rho_g^{(p,q)}(f,\varphi) &= \infty \text{ when } \rho_h^{(m,q)}(f,\varphi) = \infty \end{aligned}$$

and

$$(iv)\ \lambda_g^{(p,q)}(f,\varphi) = \infty \text{ when } \lambda_h^{(m,q)}(f,\varphi) = \infty.$$

In the line of Theorem 1, the following remark may be proved and so we omit its proof.

Remark 1. *Let us consider* $f(z)$ *and* $g(z)$ *are any two entire functions of* n*-complex variables. Also let* $0 < \lambda^{(m,q)}(f,\varphi) \le \rho^{(m,q)}(f,\varphi) < \infty$ *and* $0 < \lambda^{(m,p)}(g) \le \rho^{(m,p)}(g) < \infty$. *Then*

$$\frac{\lambda^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)} \le \lambda_g^{(p,q)}(f,\varphi) \le \min\left\{\frac{\lambda^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}, \frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}\right\}$$
$$\le \max\left\{\frac{\lambda^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}, \frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}\right\} \le \rho_g^{(p,q)}(f,\varphi) \le \frac{\rho^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}.$$

Remark 2. *From the conclusion of Theorem 1, we may write that* $\rho_g^{(p,q)}(f,\varphi) = \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}$ *and* $\lambda_g^{(p,q)}(f,\varphi) = \frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}$ *when* $\lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g)$. *Similarly* $\rho_g^{(p,q)}(f,\varphi) = \frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}$ *and* $\lambda_g^{(p,q)}(f,\varphi) = \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}$ *when* $\lambda_h^{(m,q)}(f,\varphi) = \rho_h^{(m,q)}(f,\varphi)$.

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