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GROWTH OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES ON THE BASIS OF CENTRAL INDEX

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Abstract. In the present paper we study the comparative growth properties of composite entire functions of several complex variables on the basis of central index.

1. Introduction, Definitions and Notations

We denote complex n-space by $\mathbb{C}^n$ and indicate its elements (points):

$$(z_1, z_2, ..., z_n), (|z_1|, |z_2|, ..., |z_n|), (r_1, r_2, ..., r_n), (k_1, k_2, ..., k_n)$$

by their corresponding symbols $z, |z|, r, k$ etc. Throughout $\Omega = \Omega_n$ stands for a nonempty open complete n-circular region in C^n(see §3.3 of [2]) with center at $(0, 0, ..., 0)$, the zero element of $\mathbb{C}^n$.

We write

$$|\Omega| = \{r : r = |z| \text{ for } z \in \Omega\}$$

and

$$\Omega^+ = \{r : r \in |\Omega|, \text{ no } r_j = 0,\ 1 \le j \le n\}$$

and regard these as subsets of the n-dimensional Euclidean space $\mathbb{R}^n$.

For any $r, s \in \mathbb{R}^n$, we say that

(i) $r \le s$ or $s \ge r$, if and only if $r_j \le s_j$ for $1 \le j \le n$,

(ii) $r < s$ or $s > r$, if and only if $r \le s$ but r is not equal to s
and

(iii) $r << s$ or $s >> r$, if and only if $r_j < s_j$ for $1 \le j \le n$.

A function $f(z)$, $z \in \mathbb{C}^n$ is said to be analytic at a point $\xi \in \mathbb{C}^n$ if it can be expanded in some neighborhood of ξ as an absolutely convergent power series. If we assume $\xi = (0, 0, ..., 0)$, then $f(z)$ has representation(see [4] and [6])

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$$f(z)=\sum_{k=(0,0,\ldots,0)}^{\infty} a_{k_1,k_2,\ldots,k_n} z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} = \sum_{|k|=0}^{\infty} a_k z^k,$$

where $k=(k_1,k_2,.....,k_n)$ belongs to $\mathcal{N}=\{k : k\in\mathbb{C}^n$, each k_j is rational integer$\}$ and $|k|=k_1+k_2+.....+k_n$.

For $r>(0,0,...,0)$, the maximum term $\mu(r)=\mu(r,f)$, the maximum modulus $M(r)=M(r,f)$ and the central index $\nu(r)=\nu(r,f)=(\nu_1(r,f),\nu_2(r,f),...,\nu_n(r,f))$ of entire function $f(z)$ are given by (see [4] and [5])

$$\mu(r)=\mu(r,f)=\max_{k\in\mathcal{N}}\{|a_k|\,r^k\}$$

$$M(r)=M(r,f)=\max_{|z|=r}|f(z)|$$

and

$$\nu_j(r)=\nu_j(r,f)=\left\{\begin{array}{c}\max\left[k_j : |a_k|\,r^k=\mu(r)\right], \text{ if } \mu(r)>0\\ 0, \text{ if } \mu(r)=0, \text{ for } 1\le j\le n.\end{array}\right\}$$

Also, the central index $\nu(r,f)$ for which maximum term is achieved

$$|\nu(r,f)|=\nu_1(r,f)+\nu_2(r,f)+...+\nu_n(r,f).$$

Definition 1 ([2], p.339) The order ρ_f and lower order λ_f of an entire function $f(z)=f(z_1,z_2,.....,z_n)$ are defined as follows

$$\rho_f=\limsup_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[2]} M(r_1,r_2,...,r_n,f)}{\log(r_1 r_2 ... r_n)}$$

and

$$\lambda_f=\liminf_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[2]} M(r_1,r_2,...,r_n,f)}{\log(r_1 r_2 ... r_n)}.$$

where

$$\log^{[k]} x=\log\left(\log^{[k-1]} x\right) \text{ for } k=1,2,3,... \text{ and } \log^{[0]} x=x.$$

Also one can define hyper order and hyper lower order of entire function of n-complex variables in the following way:

Definition 2 The hyper order $\overline{\rho}_f$ and the hyper lower order $\overline{\lambda}_f$ of an entire function f are defined as follows:

$$\overline{\rho}_f=\limsup_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[3]} M(r_1,r_2,...,r_n,f)}{\log(r_1 r_2 ... r_n)}$$

and

$$\overline{\lambda}_f=\liminf_{r_1,r_2,...,r_n\to\infty}\frac{\log^{[3]} M(r_1,r_2,...,r_n,f)}{\log(r_1 r_2 ... r_n)}.$$

In this paper we wish to establish the order (lower order) and hyper order (hyper lower order) of an entire function of several complex variables can also be defined in terms of central index. During the past few decades, many authors (see for e.g.[1] and [3]) investigated the growth of entire functions of a single complex variable on the basis of central index. Here our aim is to study the comparative growth properties of composite entire functions of several complex variables with respect to left (right) factor based on their central index.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1[4]: Let $p, r \in |\Omega|$ and let $\mu(p)$ and $\mu(r)$ be both positive. Then the line integral,

$$I = \int_{p}^{r} \sum_{j=1}^{n} \frac{\nu_j(x)}{x_j} dx_j$$

taken over any connected polygon in $|\Omega|$ with sides parallel to the axes and from p to r,

(i) exists,

(ii) is independent of the polygon and

(iii) is such that $\log \mu(r) = \log \mu(p) + I$.

Lemma 2[4]: Let $r \in |\Omega|$. Let $p \in |C^n|$ and be such that $p >> (1, 1, ..., 1)$, while $pr = (p_1 r_1, p_2 r_2, \ldots, p_n r_n) \in |\Omega|$.

Let

$$N_j = \max_{r \leq t \leq pr} \nu_j(t) \text{ for } 1 \leq j \leq n.$$

Then

$$(i)\ \mu(r) \leq M(r) \leq \mu(r) \prod_{j=1}^{n} \left[N_j + \frac{p_j}{p_j - 1} \right],$$

(ii) $\mu(r) = M(r)$, if and only if the series $\sum_{|k|=0}^{\infty} a_k r^k$ has at most one non vanishing term,

(iii) the last relation in (i) is an equality if and only if $\mu(r) = 0$.

Lemma 3 Let $f(z)$ be an entire function of n-complex variables with order ρ_f, then

$$\rho_f = \limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log |\nu(r_1, r_2, ..., r_n, f)|}{\log(r_1 r_2 ... r_n)}.$$

Proof. Set

$$f(z) = \sum_{k=(0,0,\ldots,0)}^{\infty} a_{k_1, k_2, \ldots, k_n} z_1^{k_1} z_2^{k_2} ... z_n^{k_n} = \sum_{|k|=0}^{\infty} a_k z^k.$$

By Lemma 1, we see the maximum term $\mu(r)$ of f satisfies

$$\log \mu(r) = \log \mu(p) + \int_{p}^{r} \sum_{j=1}^{n} \frac{\nu_j(x)}{x_j} dx_j \quad (1)$$

Since Krishna, J.G. ([4], Corollary 2.9) proved that $\nu_j(r)$ is increasing and right continuous in j-th variable for $1 \leq j \leq n$. Therefore, for any $p, r \in |\Omega|$ such that

$\mu(r) > 0$ and $p >> (1, 1, ..., 1)$, we get for $1 \leq j \leq n$,

$$\nu_j(r) \leq \frac{1}{\log p_j} \int_p^r \nu_j(r_1, ..., r_{j-1}, ..., r_n) \frac{dx_j}{x_j}. \tag{2}$$

From (1) and (2) we get

$$\log \mu(r) \geq \log \mu(p) + \sum_{j=1}^{n} \nu_j(r) \log p_j \tag{3}$$

By Lemma 2, we have

$$\mu(r, f) \leq M(r, f) \tag{4}$$

It follows from (3) and (4) that

$$\sum_{j=1}^{n} \nu_j(r) \log p_j \leq \log M(r, f) + C \tag{5}$$

where $C(> 0)$ is a suitable constant.

As $p >> (1, 1, ..., 1)$ i.e., $p = (p_1, p_2, ..., p_n) >> (1, 1, ..., 1)$, choosing $p_j = 2$ for $1 \leq j \leq n$, we get

$$\sum_{j=1}^{n} \nu_j(r) \log 2 \leq \log M(r, f) + C$$

$$\Rightarrow |\nu(r, f)| \log 2 \leq \log M(r, f) + C$$

By this and Definition 1, we have

$$\limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log |\nu(r_1, r_2, ..., r_n, f)|}{\log(r_1 r_2 ... r_n)} \leq \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log^{[2]} M(r_1, r_2, ..., r_n, f)}{\log(r_1 r_2 ... r_n)} = \rho_f \tag{6}$$

On the other hand, by choosing $p_j = 2$ for $1 \leq j \leq n$ i.e., $p = (2, 2, ..., 2)$ in (i) of Lemma 2, we have

$$M(r, f) \leq \mu(r, f) \prod_{j=1}^{n} [N_j + 2], \text{where } N_j = \max_{r \leq t \leq pr} \nu_j(t) \text{ for } 1 \leq j \leq n$$

$$\Rightarrow M(r, f) \leq \left|a_{\nu(r,f)}\right| r^{\nu(r,f)} \prod_{j=1}^{n} [N_j + 2] \tag{7}$$

Since $\{|a_k|\}$ is bounded, from (7) we get

$$\begin{aligned} \log M(r, f) &\leq \sum_{j=1}^{n} \nu_j(r) \log r_j + \sum_{j=1}^{n} \log N_j + C_1 \\ &\leq \sum_{j=1}^{n} |\nu(r, f)| \log r_j + \sum_{j=1}^{n} \log N_j + C_1 \\ &\leq |\nu(r, f)| \log(r_1 r_2 ... r_n) + \log(N_1 N_2 ... N_n) + C_1 \end{aligned}$$

$$\Rightarrow \log^{[2]} M(r, f) \leq \log |\nu(r, f)| + \log^{[2]}(r_1 r_2 ... r_n) + \log^{[2]}(N_1 N_2 ... N_n) + C_2$$

where $C_j(>0)(j=1,2)$ are suitable constants.

By this and Definition 1, we get

$$\rho_f = \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]} M(r_1,r_2,...,r_n,f)}{\log(r_1r_2...r_n)} \leq \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log|\nu(r_1,r_2,...,r_n,f)|}{\log(r_1r_2...r_n)} \tag{8}$$

By (6) and (8), Lemma 3 follows. □

Proceeding similarly as in Lemma 3, we can prove the following result:

Lemma 4 Let $f(z)$ be an entire function of n-complex variables with lower order λ_f, then

$$\lambda_f = \liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log|\nu(r_1,r_2,...,r_n,f)|}{\log(r_1r_2...r_n)}.$$

Lemma 5 Let $f(z)$ be an entire function of n-complex variables with order $\overline{\rho}_f$, then

$$\overline{\rho}_f = \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,f)|}{\log(r_1r_2...r_n)}.$$

Proof. Set

$$f(z) = \sum_{k=(0,0,....,0)}^{\infty} a_{k_1,k_2,.....,k_n} z_1^{k_1} z_2^{k_2}...z_n^{k_n} = \sum_{|k|=0}^{\infty} a_k z^k,$$

By Lemma 1, we see the maximum term $\mu(r)$ of f satisfies

$$\log\mu(r) = \log\mu(p) + \int_p^r \sum_{j=1}^n \frac{\nu_j(x)}{x_j} dx_j \tag{9}$$

Since Krishna, J.G. ([5], Corollary 2.9) proved that $\nu_j(r)$ is increasing and right continuous in j-th variable for $1 \leq j \leq n$. Therefore, for any $p, r \in |\Omega|$ such that $\mu(r) > 0$ and $p >> (1,1,...,1)$, we get for $1 \leq j \leq n$,

$$\nu_j(r) \leq \frac{1}{\log p_j} \int_p^r \nu_j(r_1,...,r_{j-1},...,r_n)\frac{dx_j}{x_j}. \tag{10}$$

From (9) and (10) we get

$$\log\mu(r) \geq \log\mu(p) + \sum_{j=1}^n \nu_j(r)\log p_j \tag{11}$$

By Lemma 2, we have

$$\mu(r,f) \leq M(r,f) \tag{12}$$

It follows from (11) and (12) that

$$\sum_{j=1}^n \nu_j(r)\log p_j \leq \log M(r,f) + C, \tag{13}$$

where $C(>0)$ is a suitable constant.

As $p >> (1,1,...,1)$ i.e., $p = (p_1,p_2,...,p_n) >> (1,1,...,1)$, choosing $p_j = 2$ for $1 \leq j \leq n$, we get

$$\sum_{j=1}^n \nu_j(r)\log 2 \leq \log M(r,f) + C$$

$$\Rightarrow |\nu(r,f)| \log 2 \leq \log M(r,f) + C$$

By this and Definition 2, we have

$$\limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]} |\nu(r_1,r_2,...,r_n,f)|}{\log(r_1r_2...r_n)} \leq \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[3]} M(r_1,r_2,...,r_n,f)}{\log(r_1r_2...r_n)} = \overline{\rho}_f. \tag{14}$$

On the other hand, by choosing $p_j = 2$ for $1 \leq j \leq n$ i.e., $p = (2,2,...,2)$ in (i) of Lemma 2, we have

$$M(r,f) \leq \mu(r,f) \prod_{j=1}^{n} [N_j + 2],$$

$$\text{where } N_j = \max_{r\leq t\leq pr} \nu_j(t) \text{ for } 1 \leq j \leq n$$

$$\Rightarrow M(r,f) \leq \left|a_{\nu(r,f)}\right| r^{\nu(r,f)} \prod_{j=1}^{n} [N_j + 2] \tag{15}$$

Since $\{|a_k|\}$ is bounded, from (15) we get

$$\begin{aligned} \log M(r,f) &\leq \sum_{j=1}^{n} \nu_j(r) \log r_j + \sum_{j=1}^{n} \log N_j + C_1 \\ &\leq \sum_{j=1}^{n} |\nu(r,f)| \log r_j + \sum_{j=1}^{n} \log N_j + C_1 \\ &\leq |\nu(r,f)| \log(r_1r_2...r_n) + \log(N_1N_2...N_n) + C_1 \end{aligned}$$

$$\Rightarrow \log^{[2]} M(r,f) \leq \log|\nu(r,f)| + \log^{[2]}(r_1r_2...r_n) + \log^{[2]}(N_1N_2...N_n) + C_2$$

where $C_j(>0)(j = 1,2)$ are suitable constants.

By this and Definition 2, we get

$$\overline{\rho}_f = \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[3]} M(r_1,r_2,...,r_n,f)}{\log(r_1r_2...r_n)} \leq \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]} |\nu(r_1,r_2,...,r_n,f)|}{\log(r_1r_2...r_n)} \tag{16}$$

By (14) and (16), Lemma 5 follows. □

Proceeding similarly as in Lemma 5, we can prove the following result:

Lemma 6 Let $f(z)$ be an entire function of n-complex variables with order $\overline{\lambda}_f$, then

$$\overline{\lambda}_f = \liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]} |\nu(r_1,r_2,...,r_n,f)|}{\log(r_1r_2...r_n)}.$$

3. Statement and Proof of main Theorems

In this section we present the main results of the paper.

Theorem 1 Let f and g be two entire functions of n-complex variables. Also let $0 < \lambda_{fog} \leq \rho_{fog} < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then

$$\frac{\lambda_{fog}}{\rho_g} \leq \liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log|\nu(r_1,r_2,...,r_n,fog)|}{\log|\nu(r_1,r_2,...,r_n,g)|} \leq \min\{\frac{\lambda_{fog}}{\lambda_g}, \frac{\rho_{fog}}{\rho_g}\}$$

$$\leq \max\{\frac{\lambda_{fog}}{\lambda_g},\frac{\rho_{fog}}{\rho_g}\} \leq \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log|\nu(r_1,r_2,...,r_n,fog)|}{\log|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\rho_{fog}}{\lambda_g}.$$

Proof. Using respectively Lemma 3 and Lemma 4 for the entire function g, we have for arbitrary positive ε and for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log|\nu(r_1,r_2,...,r_n,g)| \leq (\rho_g+\varepsilon)\log(r_1r_2...r_n) \tag{17}$$

$$\text{and } \log|\nu(r_1,r_2,...,r_n,g)| \geq (\lambda_g-\varepsilon)\log(r_1r_2...r_n). \tag{18}$$

Also, for a sequence of values of each of $r_1, r_2, ..., r_n$ tending to infinity

$$\log|\nu(r_1,r_2,...,r_n,g)| \leq (\lambda_g+\varepsilon)\log(r_1r_2...r_n) \tag{19}$$

$$\text{and } \log|\nu(r_1,r_2,...,r_n,g)| \geq (\rho_g-\varepsilon)\log(r_1r_2...r_n). \tag{20}$$

Using respectively Lemma 3 and Lemma 4 for the composite entire function fog, we have for arbitrary positive ε and for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log|\nu(r_1,r_2,...,r_n,fog)| \leq (\rho_{fog}+\varepsilon)\log(r_1r_2...r_n) \tag{21}$$

$$\text{and } \log|\nu(r_1,r_2,...,r_n,fog)| \geq (\lambda_{fog}-\varepsilon)\log(r_1r_2...r_n). \tag{22}$$

Again, for a sequence of values of each of $r_1, r_2..., r_n$ tending to infinity

$$\log|\nu(r_1,r_2,...,r_n,fog)| \leq (\lambda_{fog}+\varepsilon)\log(r_1r_2...r_n) \tag{23}$$

$$\text{and } \log|\nu(r_1,r_2,...,r_n,fog)| \geq (\rho_{fog}-\varepsilon)\log(r_1r_2...r_n). \tag{24}$$

Now from (17) and (22) it follows for all sufficiently large values of $r_1, r_2..., r_n$ that

$$\frac{\log|\nu(r_1,r_2,...,r_n,fog)|}{\log|\nu(r_1,r_2,...,r_n,g)|} \geq \frac{\lambda_{fog}-\varepsilon}{\rho_g+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log|\nu(r_1,r_2,...,r_n,fog)|}{\log|\nu(r_1,r_2,...,r_n,g)|} \geq \frac{\lambda_{fog}}{\rho_g}. \tag{25}$$

Again, combining (18) and (23) we get for a sequence of values of each of $r_1, r_2..., r_n$ tending to infinity

$$\frac{\log|\nu(r_1,r_2,...,r_n,fog)|}{\log|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\lambda_{fog}+\varepsilon}{\lambda_g-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log|\nu(r_1,r_2,...,r_n,fog)|}{\log|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\lambda_{fog}}{\lambda_g}. \tag{26}$$

Similarly, from (20) and (21) it follows for a sequence of values of each of $r_1, r_2..., r_n$ tending to infinity that

$$\frac{\log|\nu(r_1,r_2,...,r_n,fog)|}{\log|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\rho_{fog}+\varepsilon}{\rho_g-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log|\nu(r_1,r_2,...,r_n,fog)|}{\log|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\rho_{fog}}{\rho_g}. \tag{27}$$

Now combining (25), (26) and (27) we get that

$$\frac{\lambda_{fog}}{\rho_g} \leq \liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, g)|} \leq \min\{\frac{\lambda_{fog}}{\lambda_g}, \frac{\rho_{fog}}{\rho_g}\}. \tag{28}$$

Now, from (19) and (22) we obtain for a sequence of values of each of $r_1, r_2 ..., r_n$ tending to infinity that

$$\frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, g)|} \geq \frac{\lambda_{fog} - \varepsilon}{\lambda_g + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, g)|} \geq \frac{\lambda_{fog}}{\lambda_g}. \tag{29}$$

Again, from (18) and (21) it follows for all sufficiently large values of $r_1, r_2 ..., r_n$ that

$$\frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, g)|} \leq \frac{\rho_{fog} + \varepsilon}{\lambda_g - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, g)|} \leq \frac{\rho_{fog}}{\lambda_g}. \tag{30}$$

Similarly, combining (17) and (24) we get for a sequence of values of each of $r_1, r_2 ..., r_n$ tending to infinity that

$$\frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, g)|} \geq \frac{\rho_{fog} - \varepsilon}{\rho_g + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, g)|} \geq \frac{\rho_{fog}}{\rho_g}. \tag{31}$$

Therefore, combining (29), (30) and (31) we get that

$$\max\{\frac{\lambda_{fog}}{\lambda_g}, \frac{\rho_{fog}}{\rho_g}\} \leq \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, g)|} \leq \frac{\rho_{fog}}{\lambda_g}. \tag{32}$$

Thus the theorem follows from (28) and (32). □

Example 1 Considering $f = z,\ g = \exp z$ and $n = 1$ one can easily verify that the sign '$\leq$' in Theorem 1 cannot be replaced by '$<$' only.

Remark 1 If we take $0 < \lambda_f \leq \rho_f < \infty$ instead of $0 < \lambda_g \leq \rho_g < \infty$ and the other conditions remain the same then also Theorem 1 holds with g replaced by f in the denominator as we see in the next theorem.

Theorem 2 Let f and g be two entire functions of n-complex variables. Also let $0 < \lambda_{fog} \leq \rho_{fog} < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then

$$\frac{\lambda_{fog}}{\rho_f} \leq \liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, f)|} \leq \min\{\frac{\lambda_{fog}}{\lambda_f}, \frac{\rho_{fog}}{\rho_f}\}$$

$$\leq \max\{\frac{\lambda_{fog}}{\lambda_f}, \frac{\rho_{fog}}{\rho_f}\} \leq \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log|\nu(r_1, r_2, ..., r_n, fog)|}{\log|\nu(r_1, r_2, ..., r_n, f)|} \leq \frac{\rho_{fog}}{\lambda_f}.$$

Example 2 Considering $f = \exp z$, $g = z$ and $n = 1$ one can easily verify that the sign '$\leq$' in Theorem 2 cannot be replaced by '$<$' only.

Theorem 3 Let f and g be two entire functions of n-complex variables. Also let $0 < \overline{\lambda}_{fog} \leq \overline{\rho}_{fog} < \infty$ and $0 < \overline{\lambda}_g \leq \overline{\rho}_g < \infty$. Then

$$\frac{\overline{\lambda}_{fog}}{\overline{\rho}_g} \leq \liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \min\{\frac{\overline{\lambda}_{fog}}{\overline{\lambda}_g},\frac{\overline{\rho}_{fog}}{\overline{\rho}_g}\}$$

$$\leq \max\{\frac{\overline{\lambda}_{fog}}{\overline{\lambda}_g},\frac{\overline{\rho}_{fog}}{\overline{\rho}_g}\} \leq \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\overline{\rho}_{fog}}{\overline{\lambda}_g}.$$

Proof. Using respectively Lemma 5 and Lemma 6 for the entire function g, we have for arbitrary positive ε and for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log^{[2]}|\nu(r_1,r_2,...,r_n,g)| \leq (\overline{\rho}_g+\varepsilon)\log(r_1r_2...r_n) \tag{33}$$

$$\text{and } \log^{[2]}|\nu(r_1,r_2,...,r_n,g)| \geq (\overline{\lambda}_g-\varepsilon)\log(r_1r_2...r_n). \tag{34}$$

Also, for a sequence of values of each of $r_1, r_2, ..., r_n$ tending to infinity

$$\log^{[2]}|\nu(r_1,r_2,...,r_n,g)| \leq (\overline{\lambda}_g+\varepsilon)\log(r_1r_2...r_n) \tag{35}$$

$$\text{and } \log^{[2]}|\nu(r_1,r_2,...,r_n,g)| \geq (\overline{\rho}_g-\varepsilon)\log(r_1r_2...r_n). \tag{36}$$

Using respectively Lemma 5 and Lemma 6 for the composite entire function fog, we have for arbitrary positive ε and for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)| \leq (\overline{\rho}_{fog}+\varepsilon)\log(r_1r_2...r_n) \tag{37}$$

$$\text{and } \log^{[2]}|\nu(r_1,r_2,...,r_n,fog)| \geq (\overline{\lambda}_{fog}-\varepsilon)\log(r_1r_2...r_n). \tag{38}$$

Again, for a sequence of values of each of $r_1, r_2..., r_n$ tending to infinity

$$\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)| \leq (\overline{\lambda}_{fog}+\varepsilon)\log(r_1r_2...r_n) \tag{39}$$

$$\text{and } \log^{[2]}|\nu(r_1,r_2,...,r_n,fog)| \geq (\overline{\rho}_{fog}-\varepsilon)\log(r_1r_2...r_n). \tag{40}$$

Now, from (33) and (38) it follows for all sufficiently large values of $r_1, r_2..., r_n$ that

$$\frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \geq \frac{\overline{\lambda}_{fog}-\varepsilon}{\overline{\rho}_g+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \geq \frac{\overline{\lambda}_{fog}}{\overline{\rho}_g}. \tag{41}$$

Again, combining (34) and (39) we get for a sequence of values of each of $r_1, r_2..., r_n$ tending to infinity

$$\frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\overline{\lambda}_{fog}+\varepsilon}{\overline{\lambda}_g-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\overline{\lambda}_{fog}}{\overline{\lambda}_g}. \tag{42}$$

Similarly, from (36) and (37) it follows for a sequence of values of each of $r_1, r_2 ..., r_n$ tending to infinity that

$$\frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\overline{\rho}_{fog}+\varepsilon}{\overline{\rho}_g-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\overline{\rho}_{fog}}{\overline{\rho}_g}. \tag{43}$$

Now, combining (41), (42) and (43) we get that

$$\frac{\overline{\lambda}_{fog}}{\overline{\rho}_g} \leq \liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \min\{\frac{\overline{\lambda}_{fog}}{\overline{\lambda}_g}, \frac{\overline{\rho}_{fog}}{\overline{\rho}_g}\}. \tag{44}$$

Now, from (35) and (38) we obtain for a sequence of values of each of $r_1, r_2 ..., r_n$ tending to infinity that

$$\frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \geq \frac{\overline{\lambda}_{fog}-\varepsilon}{\overline{\lambda}_g+\varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \geq \frac{\overline{\lambda}_{fog}}{\overline{\lambda}_g}. \tag{45}$$

Again, from (34) and (37) it follows for all sufficiently large values of $r_1, r_2 ..., r_n$ that

$$\frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\overline{\rho}_{fog}+\varepsilon}{\overline{\lambda}_g-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\overline{\rho}_{fog}}{\overline{\lambda}_g}. \tag{46}$$

Similarly, combining (33) and (40) we get for a sequence of values of each of $r_1, r_2 ..., r_n$tending to infinity that

$$\frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \geq \frac{\overline{\rho}_{fog}-\varepsilon}{\overline{\rho}_g+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \geq \frac{\overline{\rho}_{fog}}{\overline{\rho}_g}. \tag{47}$$

Therefore, combining (45), (46) and (47) we get that

$$\max\{\frac{\overline{\lambda}_{fog}}{\overline{\lambda}_g}, \frac{\overline{\rho}_{fog}}{\overline{\rho}_g}\} \leq \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[2]}|\nu(r_1,r_2,...,r_n,fog)|}{\log^{[2]}|\nu(r_1,r_2,...,r_n,g)|} \leq \frac{\overline{\rho}_{fog}}{\overline{\lambda}_g}. \tag{48}$$

Thus the theorem follows from (44) and (48). □

Example 3 Considering $f = z$, $g = \exp(\exp z)$ and $n = 1$ one can easily verify that the sign ' $\leq$' in Theorem 3 cannot be replaced by ' $<$' only.

Remark 2 If we take $0 < \overline{\lambda}_f \leq \overline{\rho}_f < \infty$ instead of $0 < \overline{\lambda}_g \leq \overline{\rho}_g < \infty$and the other conditions remain the same then also Theorem 3 holds with g replaced by f in the denominator as we see in the next theorem.

Theorem 4 Let f and g be two entire functions of n-complex variables. Also let $0 < \overline{\lambda}_{fog} \leq \overline{\rho}_{fog} < \infty$ and $0 < \overline{\lambda}_f \leq \overline{\rho}_f < \infty$. Then

$$\frac{\overline{\lambda}_{fog}}{\overline{\rho}_f} \leq \liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[2]} |\nu(r_1, r_2, ..., r_n, fog)|}{\log^{[2]} |\nu(r_1, r_2, ..., r_n, f)|} \leq \min\{\frac{\overline{\lambda}_{fog}}{\overline{\lambda}_f}, \frac{\overline{\rho}_{fog}}{\overline{\rho}_f}\}$$

$$\leq \max\{\frac{\overline{\lambda}_{fog}}{\overline{\lambda}_f}, \frac{\overline{\rho}_{fog}}{\overline{\rho}_f}\} \leq \limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[2]} |\nu(r_1, r_2, ..., r_n, fog)|}{\log^{[2]} |\nu(r_1, r_2, ..., r_n, f)|} \leq \frac{\overline{\rho}_{fog}}{\overline{\lambda}_f}.$$

Example 4 Taking $f = \exp(\exp z)$, $g = z$ and $n = 1$ one can easily verify that the sign ' $\leq$' in Theorem 4 cannot be replaced by ' $<$' only.

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