FIXED POINT THEOREMS FOR GENERALIZED β-ψ-GERAGHTY CONTRACTION TYPE MAPS IN S-METRIC SPACE

K. ANTHONY SINGH AND M. R. SINGH

ABSTRACT. In this paper, we introduce the notion of generalized β-ψ-Geraghty contraction type maps and β-ψ-Geraghty contraction type maps in the context of S-metric space and establish some fixed point theorems for such maps. Our results (with some modifications) extend the fixed point results of Erdal Karapinar [19] to complete S-metric space. An example is also given to illustrate our result.

1. INTRODUCTION

The Banach contraction principle is one of the most important and fundamental results in fixed point theory. The study of fixed point problems is indeed a powerful tool in nonlinear analysis and the fixed point theory techniques have very useful applications in many disciplines such as Chemistry, Physics, Biology, Computer Science, Economics, Game Theory and many branches of Mathematics. Due to this, several authors have improved, generalized and extended this basic result of Banach by defining new contractive conditions and replacing the metric space by more general abstract spaces. Among such results was an interesting result by Geraghty [10] which generalized the Banach contraction principle in the setting of a complete metric space by considering an auxiliary function. Then Amini-Harandi and Emami characterized the result of Geraghty in the context of a partially ordered complete metric space. Caballero et al.[6] discussed the existence of a best proximity point of Geraghty contraction. Gordji et al. [11] defined the notion of ψ-Geraghty type contraction and obtained results extending the results of Amini-Harandi and Emami [2]. Recently Samet et al.[26] defined the notion of α-ψ-contractive mappings and obtained remarkable fixed point results. Inspired by this notion of α-ψ-contractive mappings, Karapinar and Samet [16] introduced the concept of generalized α-ψ-contractive mappings and obtained fixed point results for such mappings. Very recently, Cho et al.[8] defined the concept of α-Geraghty contraction type maps in the setting of a metric space starting from the definition of generalized α-Geraghty

contraction type maps and proved the existence and uniqueness of a fixed point of such maps in the context of a complete metric space. Further as generalizations of the type of maps defined by Cho et al.[8], Erdal Karapinar [19] introduced the concept of generalized α-ψ-Geraghty contraction type maps and α-ψ-Geraghty contraction type maps and proved fixed point results generalizing the results obtained by Cho et al. [8].

In this paper, motivated by the results of Erdal Karapinar [19], we define generalized β-ψ-Geraghty contraction type maps and β-ψ-Geraghty contraction type maps in the setting of S-metric space and obtain the existence and uniqueness of a fixed point of such mappings. Our results (with some modifications) extend the fixed point results of Erdal Karapinar [19] to complete S-metric space. We also give an example to illustrate our result.

2. PRELIMINARIES

In this section, we recall some basic definitions and related results on the topic in the literature.

Let $\mathfrak{F}$ be the family of all functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition

$$\lim_{n \to \infty} \beta(t_n) = 1 \text{ implies } \lim_{n \to \infty} t_n = 0.$$

By using such a map, Geraghty proved the following interesting result.

Theorem 2.1.[10] Let (X, d) be a complete metric space and let T be a mapping on X. Suppose there exists $\beta \in \mathfrak{F}$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

Then T has a unique fixed point $x^* \in X$ and $\{T^n x\}$ converges to x^* for each $x \in X$.

Definition 2.2.[26] Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be α-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Definition 2.3.[14] A map $T : X \to X$ is said to be triangular α-admissible if
(T1) T is α-admissible,
(T2) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$.

Lemma 2.4.[14] Let $T : X \to X$ be a triangular α-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.

Erdal Karapinar [19] defined the following class of auxiliary functions.

Let Ψ denote the class of functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:
(a) ψ is nondecreasing;
(b) ψ is subadditive, that is, $\psi(s + t) \leq \psi(s) + \psi(t)$;
(c) ψ is continuous;
(d) $\psi(t) = 0 \Leftrightarrow t = 0$.

Erdal Karapinar [19] also introduced the following contraction and proved the following interesting results i.e. Theorem 2.6., Theorem 2.8. and Theorem 2.9.

Definition 2.5.[19] Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called a generalized α-ψ-Geraghty contraction type map if there exists $\beta \in \mathfrak{F}$ such that for all $x, y \in X$,

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$ and $\psi \in \Psi$.

Theorem 2.6.[19] Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:

(1) T is a generalized α-ψ-Geraghty contraction type map,

(2) T is triangular α- admissible,

(3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,

(4) T is continuous.

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Definition 2.7.[19] Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. We say that the sequence $\{x_n\}$ is α-regular if the following condition is satisfied:

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Theorem 2.8.[19] Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:

(1) T is a generalized α-ψ-Geraghty contraction type map,

(2) T is triangular α- admissible,

(3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,

(4) $\{x_n\}$ is α-regular.

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

For the uniqueness of a fixed point of a generalized α-ψ- Geraghty contraction type map, Erdal Karapinar considered the following hypothesis.

(H1) For all $x,y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$. Here $Fix(T)$ denotes the set of fixed points of T.

Theorem 2.9.[19] Adding condition (H1) to the hypotheses of Theorem 2.6.(resp. Theorem 2.8.), we obtain that x^* is the unique fixed point of T.

3. Main Results

We now state and prove our main results.

And we first recall S-metric space and some of its properties.

Definition 3.1.[25] Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x,y,z,a \in X$,

(S1) $S(x,y,z) \geq 0$,

(S2) $S(x,y,z) = 0$ if and only if $x = y = z$,

(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

Example 3.2.[25] Let X be a nonempty set and d be an ordinary metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

Lemma 3.3.[25] In an S-metric space (X,S), we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.

Definition 3.4.[25] Let (X,S) be an S-metric space.

(1) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote this by $\lim_{n \to \infty} x_n = x$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $S(x_n, x_n, x_m)$ $\to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$.

(3) An S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Lemma 3.5.[25] Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 3.6.[25] Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X is convergent, then $\{x_n\}$ is a Cauchy sequence.

Lemma 3.7.[25] Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$, then $\lim_{n \to \infty} S(x_n, x_n, y_n)$ $= S(x, x, y)$.

Lemma 3.8.[24] Let $T : X \to Y$ be a map from an S-metric space X to an S-metric space Y. Then T is continuous at $x \in X$ if and only if $Tx_n \to Tx$ whenever $x_n \to x$.

Here we introduce the following new definitions.

Let Ω be the family of all functions $\theta : [0, \infty) \to [0, 1]$ which satisfy the following conditions

(1) $\theta(t) < 1$ for $t > 0$, and

(2) $\lim_{n \to \infty} \theta(t_n) = 1$ implies $\lim_{n \to \infty} t_n = 0$.

Remark 3.9. Here instead of the family $\mathfrak{F}$ we are introducing a more refined family Ω.

Definition 3.10. Let $A : X \to X$ be a map and $\beta : X \times X \times X \to \mathbb{R}$ be a function. Then A is said to be β-admissible if $\beta(x, x, y) \geq 1$ implies $\beta(Ax, Ax, Ay) \geq 1$.

Definition 3.11. A map $A : X \to X$ is said to be triangular β-admissible if

(A1) A is β-admissible,

(A2) $\beta(x, x, z) \geq 1$ and $\beta(z, z, y) \geq 1$ imply $\beta(x, x, y) \geq 1$.

Lemma 3.12. Let $A : X \to X$ be a triangular β-admissible map. Assume that there exists $x_1 \in X$ such that $\beta(x_1, x_1, Ax_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Ax_n$. Then we have $\beta(x_n, x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.

Proof. Since there exists $x_1 \in X$ such that $\beta(x_1, x_1, Ax_1) \geq 1$, we have from (A1) $\beta(x_2, x_2, x_3) = \beta(Ax_1, Ax_1, A^2 x_1) \geq 1$. By continuing this process, we get $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Let us suppose that $n, m \in \mathbb{N}$ with $n < m$. Since $\beta(x_n, x_n, x_{n+1}) \geq 1$ and $\beta(x_{n+1}, x_{n+1}, x_{n+2}) \geq 1$, from (A2), we get $\beta(x_n, x_n, x_{n+2}) \geq 1$. Again, since $\beta(x_n, x_n, x_{n+2}) \geq 1$ and $\beta(x_{n+2}, x_{n+2}, x_{n+3}) \geq 1$, we deduce that $\beta(x_n, x_n, x_{n+3}) \geq 1$. Continuing the process in this way, we get $\beta(x_n, x_n, x_m) \geq 1$.

Definition 3.13. Let (X, S) be an S-metric space and let $\beta : X \times X \times X \to \mathbb{R}$ be a function. Then a map $A : X \to X$ is called a generalized β-ψ- Geraghty contraction type map if there exists $\theta \in \Omega$ such that for all $x, y \in X$,

$$\beta(x, x, y)\psi(S(Ax, Ax, Ay)) \leq \theta(\psi(N(x, y)))\psi(N(x, y))$$

where

$$N(x, y) = \max\{S(x, x, y), S(x, x, Ax), S(y, y, Ay)\} \text{ and } \psi \in \Psi.$$

Theorem 3.14. Let (X, S) be a complete S-metric space, $\beta : X \times X \times X \to \mathbb{R}$ be a function and let $A : X \to X$ be a map. Suppose that the following conditions hold:

(i) A is a generalized β-ψ-Geraghty contraction type map,

(ii) A is triangular β-admissible,

(iii) there exists $x_1 \in X$ such that $\beta(x_1, x_1, Ax_1) \geq 1$,

(iv) A is continuous.

Then A has a fixed point $x^* \in X$ and $\{A^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\beta(x_1, x_1, Ax_1) \geq 1$. We construct a sequence of points $\{x_n\}$ in X such that $x_{n+1} = Ax_n$ for $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of A. Therefore we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By hypothesis, $\beta(x_1, x_1, x_2) \geq 1$ and the map A is triangular β- admissible. Therefore by Lemma 3.12., we have $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Then we have

$$\psi(S(x_{n+1}, x_{n+1}, x_{n+2})) = \psi(S(Ax_n, Ax_n, Ax_{n+1}))$$
$$\leq \beta(x_n, x_n, x_{n+1})\psi(S(Ax_n, Ax_n, Ax_{n+1}))$$
$$\leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1})) \qquad (1)$$

for all $n \in \mathbb{N}$.

Here we have

$$N(x_n, x_{n+1}) = \max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, Ax_n), S(x_{n+1}, x_{n+1}, Ax_{n+1})\}$$
$$= \max\{S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2})\}.$$

If $N(x_n, x_{n+1}) = S(x_{n+1}, x_{n+1}, x_{n+2})$, then from (1) and the definition of θ, we have $\psi(S(x_{n+1}, x_{n+1}, x_{n+2})) < \psi(S(x_{n+1}, x_{n+1}, x_{n+2}))$, which is a contradiction. Therefore we have

$$N(x_n, x_{n+1}) = S(x_n, x_n, x_{n+1}).$$

Thus, we have

$$\psi(S(x_{n+1}, x_{n+1}, x_{n+2})) \leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1}))$$
$$\leq \theta(\psi(S(x_n, x_n, x_{n+1})))\psi(S(x_n, x_n, x_{n+1}))$$
$$< \psi(S(x_n, x_n, x_{n+1}))$$

so that $S(x_{n+1}, x_{n+1}, x_{n+2}) < S(x_n, x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Thus the sequence $\{S(x_n, x_n, x_{n+1})\}$ is nonnegative and nonincreasing.

Now, we prove that $S(x_n, x_n, x_{n+1}) \to 0$ as $n \to \infty$.

It is clear that $\{S(x_n, x_n, x_{n+1})\}$ is a decreasing sequence which is bounded from below. Therefore, there exists $r \geq 0$ such that $\lim_{n \to \infty} S(x_n, x_n, x_{n+1}) = r$. We show that $r = 0$. And we suppose on the contrary that $r > 0$.

We have

$$\frac{\psi(S(x_{n+1}, x_{n+1}, x_{n+2}))}{\psi(S(x_n, x_n, x_{n+1}))} \leq \theta(\psi(S(x_n, x_n, x_{n+1}))) < 1.$$

Now by taking limit $n \to \infty$, we have

$$\lim_{n \to \infty} \theta(\psi(S(x_n, x_n, x_{n+1}))) = 1.$$

By the property of θ, we have

$$\lim_{n \to \infty} \psi(S(x_n, x_n, x_{n+1})) = 0 \Rightarrow \lim_{n \to \infty} S(x_n, x_n, x_{n+1}) = 0. \qquad (2)$$

Now we show that the sequence $\{x_n\}$ is a Cauchy sequence. Let us suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that, for all positive integers k, there exist $m_k > n_k > k$ with

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon. \qquad (3)$$

Let m_k be the smallest number satisfying the conditions above. Then we have

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \varepsilon. \tag{4}$$

By (3) and (4), we have

$$\varepsilon \leq S(x_{m_k}, x_{m_k}, x_{n_k}) \leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k})$$
$$< 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + \varepsilon$$

that is,

$$\varepsilon \leq S(x_{m_k}, x_{m_k}, x_{n_k}) < \varepsilon + 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) \text{ for all } k \in \mathbb{N}. \tag{5}$$

Then in view of (2) and (5), we have

$$\lim_{k \to \infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon. \tag{6}$$

Again, we have

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{n_k}, x_{n_k}, x_{m_k-1})$$
$$\leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + 2S(x_{n_k}, x_{n_k}, x_{n_k-1})$$
$$+ S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1})$$

and

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) \leq 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k})$$
$$+ 2S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + S(x_{m_k}, x_{m_k}, x_{n_k}).$$

Taking limit as $k \to \infty$ and using (2) and (6), we obtain

$$\lim_{k \to \infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \varepsilon. \tag{7}$$

By Lemma 3.12., we get $\beta(x_{n_k-1}, x_{n_k-1}, x_{m_k-1}) \geq 1$. Therefore, we have

$$\psi(S(x_{m_k}, x_{m_k}, x_{n_k})) = \psi(S(Ax_{m_k-1}, Ax_{m_k-1}, Ax_{n_k-1}))$$
$$\leq \beta(x_{n_k-1}, x_{n_k-1}, x_{m_k-1})\psi(S(Ax_{n_k-1}, Ax_{n_k-1}, Ax_{m_k-1}))$$
$$\leq \theta(\psi(N(x_{n_k-1}, x_{m_k-1})))\psi(N(x_{n_k-1}, x_{m_k-1})).$$

Here, we have

$$N(x_{n_k-1}, x_{m_k-1}) = \max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k-1}), S(x_{n_k-1}, x_{n_k-1}, Ax_{n_k-1}),$$
$$S(x_{m_k-1}, x_{m_k-1}, Ax_{m_k-1})\}$$
$$= \max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k-1}), S(x_{n_k-1}, x_{n_k-1}, x_{n_k}),$$
$$S(x_{m_k-1}, x_{m_k-1}, x_{m_k})\}.$$

And we see that

$$\lim_{k \to \infty} N(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

Now we have

$$\frac{\psi(S(x_{n_k}, x_{n_k}, x_{m_k}))}{\psi(N(x_{n_k-1}, x_{m_k-1}))} \leq \theta(\psi(N(x_{n_k-1}, x_{m_k-1}))) < 1.$$

By using (6) and taking limit as $k \to \infty$ in the above inequality, we obtain

$$\lim_{k \to \infty} \theta(\psi(N(x_{n_k-1}, x_{m_k-1}))) = 1.$$

So, $\lim_{k \to \infty} \psi(N(x_{n_k-1}, x_{m_k-1})) = 0 \Rightarrow \lim_{k \to \infty} N(x_{n_k-1}, x_{m_k-1}) = 0 = \varepsilon$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. As A is continuous, we have $Ax_n \to Ax^*$ i.e.

$\lim_{n\to\infty} x_{n+1} = Ax^*$ and so $x^* = Ax^*$. Hence, x^* is a fixed point of A.

In the following Theorem, we replace the continuity of A by a suitable condition.

Theorem 3.15. Let (X, S) be a complete S-metric space, $\beta : X \times X \times X \to \mathbb{R}$ be a function and let $A : X \to X$ be a map. Suppose that the following conditions hold:

(i) A is a generalized β-ψ-Geraghty contraction type map,

(ii) A is triangular β-admissible,

(iii) there exists $x_1 \in X$ such that $\beta(x_1, x_1, Ax_1) \geq 1$,

(iv) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\beta(x_{n_k}, x_{n_k}, x) \geq 1$ for all k.

Then A has a fixed point $x^* \in X$ and $\{A^n x_1\}$ converges to x^*.

Proof. The proof goes along similar lines of the proof of Theorem 3.14. We conclude that the sequence $\{x_n\}$ defined by $x_{n+1} = Ax_n$ for all $n \in \mathbb{N}$, converges to a point say $x^* \in X$. By the hypothesis (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\beta(x_{n_k}, x_{n_k}, x^*) \geq 1$ for all k.

Now for all k , we have

$$\psi(S(x_{n_k+1}, x_{n_k+1}, Ax^*)) = \psi(S(Ax_{n_k}, Ax_{n_k}, Ax^*))$$

$$\leq \beta(x_{n_k}, x_{n_k}, x^*)\psi(S(Ax_{n_k}, Ax_{n_k}, Ax^*))$$

$$\leq \theta(\psi(N(x_{n_k}, x^*)))\psi(N(x_{n_k}, x^*))$$

so that

$$\psi(S(x_{n_k+1}, x_{n_k+1}, Ax^*)) \leq \theta(\psi(N(x_{n_k}, x^*)))\psi(N(x_{n_k}, x^*)). \tag{8}$$

On the other hand, we have

$$N(x_{n_k}, x^*) = \max\{S(x_{n_k}, x_{n_k}, x^*), S(x_{n_k}, x_{n_k}, Ax_{n_k}), S(x^*, x^*, Ax^*)\}$$

$$= \max\{S(x_{n_k}, x_{n_k}, x^*), S(x_{n_k}, x_{n_k}, x_{n_k+1}), S(x^*, x^*, Ax^*)\}.$$

We suppose that $x^* \neq Ax^*$ so that $S(x^*, x^*, Ax^*) > 0$. Taking limit $k \to \infty$ in the above equality, we get

$$\lim_{k\to\infty} N(x_{n_k}, x^*) = S(x^*, x^*, Ax^*).$$

Now we have

$$\frac{\psi(S(x_{n_k+1}, x_{n_k+1}, Ax^*))}{\psi(N(x_{n_k}, x^*))} \leq \theta(\psi(N(x_{n_k}, x^*))) < 1.$$

And taking limit $k \to \infty$, we get $\lim_{k\to\infty} \theta(\psi(N(x_{n_k}, x^*))) = 1$.

So we have $\lim_{k\to\infty} \psi(N(x_{n_k}, x^*)) = 0$ which implies that $\lim_{k\to\infty} N(x_{n_k}, x^*) = 0$ i.e. $S(x^*, x^*, Ax^*) = 0$.

This is a contradiction. Therefore, we must have $x^* = Ax^*$.

For the uniqueness of a fixed point of a generalized β-ψ-Geraghty contraction type map, we consider the following hypothesis:

(G) For any two fixed points x and y of A, there exists $z \in X$ such that $\beta(x, x, z) \geq 1$, $\beta(y, y, z) \geq 1$ and $\beta(z, z, Az) \geq 1$.

Remark 3.16. Here we are using a condition stronger than the condition analogous to the condition (H1) of Erdal Karapinar because we observe that such a condition is not enough.

Theorem 3.17. Adding condition (G) to the hypotheses of Theorem 3.14. (or Theorem 3.15.), we obtain that x^* is the unique fixed point of A.

Proof. Due to Theorem 3.14.(or Theorem 3.15.), we obtain that $x^* \in X$ is a fixed point of A. Let $y^* \in X$ be another fixed point of A. Then by hypothesis (G), there exists $z \in X$ such that $\beta(x^*, x^*, z) \geq 1$, $\beta(y^*, y^*, z) \geq 1$ and $\beta(z, z, Az) \geq 1$. Since A is β-admissible, we get

$$\beta(x^*, x^*, A^n z) \geq 1 \text{ and } \beta(y^*, y^*, A^n z) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Then we have

$$\psi(S(x^*, x^*, A^{n+1} z)) \leq \beta(x^*, x^*, A^n z)\psi(S(Ax^*, Ax^*, AA^n z))$$
$$\leq \theta(\psi(N(x^*, A^n z)))\psi(N(x^*, A^n z)), \ \forall n \in \mathbb{N}.$$

Here we have

$$N(x^*, A^n z) = \max\{S(x^*, x^*, A^n z), S(x^*, x^*, Ax^*), S(A^n z, A^n z, AA^n z)\}$$
$$= \max\{S(x^*, x^*, A^n z), S(x^*, x^*, x^*), S(A^n z, A^n z, A^{n+1} z)\}$$
$$= \max\{S(x^*, x^*, A^n z), S(A^n z, A^n z, A^{n+1} z)\}$$

By Theorem 3.14.(or Theorem 3.15.), we deduce that the sequence $\{A^n z\}$ converges to a fixed point $z^* \in X$. Then taking limit $n \to \infty$ in the above equality, we get $\lim_{n\to\infty} N(x^*, A^n z) = S(x^*, x^*, z^*)$. And let us suppose that $z^* \neq x^*$. Then we have

$$\frac{\psi(S(x^*, x^*, A^{n+1} z))}{\psi(N(x^*, A^n z))} \leq \theta(\psi(N(x^*, A^n z))) < 1.$$

And taking limit $n \to \infty$, we get $\lim_{n\to\infty} \theta(\psi(N(x^*, A^n z))) = 1$. Therefore we have $\lim_{n\to\infty} \psi(N(x^*, A^n z)) = 0$. This implies $\lim_{n\to\infty} N(x^*, A^n z) = 0$, i.e. $S(x^*, x^*, z^*) = 0$ which is a contradiction. Therefore, we must have $z^* = x^*$. Similarly, we get $z^* = y^*$. Thus, we have $y^* = x^*$. Hence, x^* is the unique fixed point of A.

4. Consequences

Here we deduce some results as consequences of the results in the above section.

Definition 4.1. Let (X, S) be an S-metric space and let $\beta : X \times X \times X \to \mathbb{R}$ be a function. Then a map $A : X \to X$ is called a β-ψ-Geraghty contraction type map if there exists $\theta \in \Omega$ such that for all $x, y \in X$,

$$\beta(x, x, y)\psi(S(Ax, Ax, Ay)) \leq \theta(\psi(S(x, x, y)))\psi(S(x, x, y)),$$

where $\psi \in \Psi$.

Theorem 4.2. Let (X, S) be a complete S-metric space, $\beta : X \times X \times X \to \mathbb{R}$ be a function and let $A : X \to X$ be a map. Suppose that the following conditions hold:
(i) A is a β-ψ-Geraghty contraction type map,
(ii) A is triangular β-admissible,
(iii) there exists $x_1 \in X$ such that $\beta(x_1, x_1, Ax_1) \geq 1$,
(iv) A is continuous.
Then A has a fixed point $x^* \in X$ and $\{A^n x_1\}$ converges to x^*.

Theorem 4.3. Let (X, S) be a complete S-metric space, $\beta : X \times X \times X \to \mathbb{R}$ be a function and let $A : X \to X$ be a map. Suppose that the following conditions hold:
(i) A is a β-ψ-Geraghty contraction type map,
(ii) A is triangular β-admissible,
(iii) there exists $x_1 \in X$ such that $\beta(x_1, x_1, Ax_1) \geq 1$,

(iv) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\beta(x_{n_k}, x_{n_k}, x) \geq 1$ for all k.
Then A has a fixed point $x^* \in X$ and $\{A^n x_1\}$ converges to x^*.

For the uniqueness of a fixed point of a β-ψ-Geraghty contraction type map, we consider the following hypothesis (G1) which is weaker than hypothesis (G):
(G1) For any two fixed points x and y of A, there exists $z \in X$ such that $\beta(x, x, z) \geq 1$ and $\beta(y, y, z) \geq 1$.
Theorem 4.4. Adding condition (G1) to the hypotheses of Theorem 4.2.(or Theorem 4.3.), we obtain that x^* is the unique fixed point of A.
Proof. Due to Theorem 4.2.(or Theorem 4.3.), we obtain that $x^* \in X$ is a fixed point of A. Let $y^* \in X$ be another fixed point of A. Then by hypothesis (G1), there exists $z \in X$ such that $\beta(x^*, x^*, z) \geq 1$ and $\beta(y^*, y^*, z) \geq 1$.
Since A is β- admissible we get

$$\beta(x^*, x^*, A^n z) \geq 1 \text{ and } \beta(y^*, y^*, A^n z) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Then we have

$$\psi(S(x^*, x^*, A^{n+1}z)) \leq \beta(x^*, x^*, A^n z)\psi(S(Ax^*, Ax^*, AA^n z))$$
$$\leq \theta(\psi(S(x^*, x^*, A^n z)))\psi(S(x^*, x^*, A^n z))$$
$$< \psi(S(x^*, x^*, A^n z)) \text{ for all } n \in \mathbb{N}.$$

Thus the sequence $\{\psi(S(x^*, x^*, A^n z))\}$ is nonnegative and nonincreasing. Therefore there exists $r \geq 0$ such that $\lim_{n \to \infty} \psi(S(x^*, x^*, A^n z)) = r$. We show that $r = 0$. And we suppose on the contrary that $r > 0$.
We have

$$\frac{\psi(S(x^*, x^*, A^{n+1}z))}{\psi(S(x^*, x^*, A^n z))} \leq \theta(\psi(S(x^*, x^*, A^n z))) < 1.$$

Now by taking limit $n \to \infty$, we have

$$\lim_{n \to \infty} \theta(\psi(S(x^*, x^*, A^n z))) = 1.$$

By the property of θ, we have

$$\lim_{n \to \infty} \psi(S(x^*, x^*, A^n z)) = 0 \Rightarrow \lim_{n \to \infty} S(x^*, x^*, A^n z) = 0.$$

And this implies that $\lim_{n \to \infty} A^n z = x^*$.
Similarly, we have $\lim_{n \to \infty} A^n z = y^*$. Hence, we have $x^* = y^*$.

Here we give an example to illustrate Theorem 4.3.
Example 4.5. Let $X = [0, \infty)$ and let $S(x, y, z) = |x-z| + |y-z|$ for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. And let $\theta(t) = \frac{1}{1+2t}$ for all $t \geq 0$. Then $\theta \in \Omega$. Also let the function $\psi : [0, \infty) \to [0, \infty)$ be defined as $\psi(t) = \frac{t}{3}$. Then we have $\psi \in \Psi$.
Let a map $A : X \to X$ be defined by

$$Ax = \begin{cases} \frac{x}{4}, & (0 \leq x \leq 1); \\ 4x, & (x > 1). \end{cases}$$

And let a function $\beta : X \times X \times X \to \mathbb{R}$ be defined by

$$\beta(x, y, z) = \begin{cases} 1, & (0 \leq x, y, z \leq 1); \\ 0, & \text{otherwise}. \end{cases}$$

Condition (iii) of Theorem 4.3. is satisfied with $x_1 = 1$. And if $\{x_n\}$ be a sequence in X such that $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then we must have $x \in [0, 1]$. Therefore, by the definition of β, we must have $\beta(x_n, x_n, x) \geq 1$. Hence, condition (iv) of Theorem 4.3. is satisfied.

Let $x, y \in X$ such that $\beta(x, x, y) \geq 1$. Then we have $x, y \in [0, 1]$ and so $Ax \in [0, 1]$, $Ay \in [0, 1]$ and therefore $\beta(Ax, Ax, Ay) = 1$. Further if $\beta(x, x, z) \geq 1$ and $\beta(z, z, y) \geq 1$, then $x, y, z \in [0, 1]$. Therefore $\beta(x, x, y) \geq 1$. Hence A is triangular β-admissible and so condition (ii) of Theorem 4.3. is satisfied. We finally show that condition (i) of Theorem 4.3. is satisfied.

If $0 \leq x, y \leq 1$, then $\beta(x, x, y) = 1$ and we have

$$\theta(\psi(S(x, x, y)))\psi(S(x, x, y)) - \beta(x, x, y)\psi(S(Ax, Ax, Ay))$$

$$= \frac{\frac{1}{3}S(x, x, y)}{1 + \frac{2}{3}S(x, x, y)} - \frac{1}{3}S\left(\frac{x}{4}, \frac{x}{4}, \frac{y}{4}\right)$$

$$= \frac{\frac{2}{3}|x - y|}{1 + \frac{4}{3}|x - y|} - \frac{2}{3}\left|\frac{x}{4} - \frac{y}{4}\right|$$

$$= \frac{|x - y|(9 - 4|x - y|)}{6(3 + 4|x - y|)} \geq 0.$$

Therefore, we have

$$\beta(x, x, y)\psi(S(Ax, Ax, Ay)) \leq \theta(\psi(S(x, x, y)))\psi(S(x, x, y)) \text{ for } 0 \leq x, y \leq 1.$$

And if $0 \leq x \leq 1$, $y > 1$ or $0 \leq y \leq 1$, $x > 1$ or $x > 1$, $y > 1$, then $\beta(x, x, y) = 0$ and we have

$$\beta(x, x, y)\psi(S(Ax, Ax, Ay)) \leq \theta(\psi(S(x, x, y)))\psi(S(x, x, y)).$$

Thus all the conditions of Theorem 4.3. are satisfied and A has a unique fixed point $x^* = 0$. We also note that if $X = (0, 1]$, then X is not complete and A does not have a fixed point in X.

References

[1] Agarwal R.P., El-Gebeily M.A., ORegan D., Generalized contractions in partially ordered metric spaces, Appl. Anal., 87, 1-8 (2008).

[2] Amini-Harandi A., Emami H., A fixed point theorem for contraction type maps in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal., 72, 2238-2242 (2010).

[3] Aydi H., Karapinar E., Erhan I., Salimi P., Best proximity points of generalized almost ?-Geraghty contractive non-self-mappings, Fixed Point Theory Appl., Article ID 32 (2014).

[4] Banach S., Sur les operations dans les ensembles abstraits et leur applications aux equations integrals, Fundam. Math., 3, 133-181 (1922).

[5] Bilgili N., Karapinar E., Sadarangani K., A generalization for the best proximity point of Geraghty-contractions, J. Inequal. Appl., Article ID 286 (2013).

[6] Caballero J., Harjani J., Sadarangani K., A best proximity point theorem for Geraghty-contractions, Fixed Point Theory Appl., Article ID 231 (2012).

[7] Chaipunya P., Cho Y.E., Kumam P., Geraghty-type theorems in modular metric spaces with an application to partial differential equation, Adv. Differ. Equ., Article ID 83 (2012).

[8] Cho S.H., Bae J.S., Karapinar E., Fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl., Article ID 329 (2013).

[9] Cho S.H., Bae J.S., Common fixed point theorems for mappings satisfying property (E,A) on cone metric spaces, Math. Comput. Model, 53, 945-951 (2011).

[10] Geraghty M., On contractive mappings, Proc. Am. Math. Soc., 40, 604-608 (1973).

[11] Gordji M.E., Ramezami M., Cho Y.J., Pirbavafa S., A generalization of Geraghtys Theorem in partially ordered metric spaces and applications to ordinary differential equations, Fixed Point Theory Appl., Article ID 74 (2012).

[12] Hille E., Phillips R.S., Functional Analysis and Semi-groups, Amer. Math. Soc. Colloq. Publ.vol. 31. Am. Math. Soc., Providence (1957).

[13] Huang L.G., Zhang X., Cone metric space and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332(2), 1468-1476 (2007).

[14] Karapinar E., Kumam P., Salimi P., On α-ψ-Meir-Keeler contractive mappings, Fixed Point Theory Appl., Article ID 94 (2013).

[15] Karapinar E., Samet B., A note on α-Geraghty type contractions, Fixed Point Theory Appl., (2014) doi:10.1186/1687-1812-2014-26.

[16] Karapinar E., Samet B., Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal., Article ID 793486 (2012).

[17] Karapinar E., Edelstein type fixed point theorems, Fixed Point Theory Appl., Article ID 107 (2012).

[18] Karapinar E., On best proximity point of ψ-Geraghty contractions, Fixed Point Theory Appl., (2013) doi:10.1186/1687-1812-2013-200.

[19] Karapinar E., α-ψ-Geraghty contraction type mappings and some related fixed point results, Filomat, 28, 37-48 (2014).

[20] Khamsi M.A., Kreinovich V.Y., Fixed point theorems for dissipative mappings in complete probabilistic metric spaces, Math. Jpn., 44, 513-520 (1996).

[21] Mongkolkeha C., Cho Y.E., Kumam P., Best proximity points for Geraghtys proximal contraction mappings, Fixed Point Theory Appl., Article ID 180 (2013).

[22] Nihal Yilmaz Ozgur, Nihal Tas, Some fixed point theorems on S-metric spaces, Mat. Vesnik, 69(1), 39-52 (2017).

[23] Rhoades B.E., A comparison of various definitions of contractive mappings, Trans. Am. Math. Soc., 226, 257-290 (1977).

[24] S. Sedghi, N. V. Dung, Fixed point theorems on S-metric spaces, Mat. Vesnik, 66(1), 113-124 (2014).

[25] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, 64 (3), 258-266 (2012).

[26] Samet B., Vetro C., Vetro P., Fixed point theorems for α-ψ-contractive mappings, Nonlinear Anal., 75, 2154-2165 (2012).

K. Anthony Singh

Department of Mathematics, D.M. College of Science, Imphal-795001, India
    E-mail address: anthonykumam@manipuruniv.ac.in

M. R. Singh

Department of Mathematics, Manipur University, Canchipur, Imphal-795003, India
    E-mail address: mranjitmu@rediffmail.com