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ON f-STATISTICAL CONVERGENCE IN RANDOM 2-NORMED SPACES

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Abstract. The idea of f-statistical convergence was introduced in Aizpuru et al. [2] and since then several generalizations and applications of this concept have been investigated by various authors. Recently Gürdal and Özgür [12] and Borgohain [4] studied f-statistical convergence in probabilistic normed space, and the generalized statistical convergence via moduli in normed space, respectively. In this paper we propose to study f-statistical convergence in random 2-normed space which seems to be a quite new and interesting idea.

1. Introduction

The probabilistic metric space was studied by Menger [20], which is an interesting and important generalization of the notion of a metric space. The theory of probabilistic normed (or metric) spaces was initiated and developed in [3, 25, 26, 27, 28] and, it was further extended to random/probabilistic 2-normed space by Golet [13] using the concept of 2-norm which is defined by Gähler [14, 15] and Gürdal and Pehlivan [10] studied statistical convergence in 2-normed spaces. Also, statistical convergence in 2-Banach spaces was studied by Gürdal and Pehlivan in [11]. Quite recently in [23, 24], generalized statistical convergence was studied for sequence spaces in probabilistic normed space by Savaş and Gürdal.

The concept of the statistical convergence of a sequence of real $S = \{s_n\}$ was first introduced by Fast [7] (see also [30]) as follows: let A be a subset of $\mathbb{N}$. Then the asymptotic density of A denoted by $\delta(A) := \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in A\}|$, where the vertical bars denote the cardinality of the enclosed set. A sequence $S = \{s_n\}_{k\in\mathbb{N}}$ is said to convergence statistically to s and we write $\lim_{n\to\infty} s_n = s$ $(stat)$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |s_k - s| \ge \varepsilon\}| = 0.$$

Properties of statistically convergent sequences were studied in [5, 8, 9, 18]. In [18], Kolk begins to study the applications of statistical convergence to Banach spaces. In [5] there are important results that relate the statistical convergence to classical properties of Banach spaces.

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We recall that $f:\mathbb{R}^{+}\rightarrow\mathbb{R}^{+}$ is called modulus function, or simply modulus, if it is satisfies:
(1) $f(s)=0$ if and only if $s=0$.
(2) $f\left(s+p\right)\leq f\left(s\right)+f\left(p\right)$ for every $s,p\in\mathbb{R}^{+}$
(3) f is increasing.
(4) f is continuous from the right at 0.
From these properties it is clear that a modulus function must be continuous on $\mathbb{R}^{+}$. Examples of modulus functions are $f\left(s\right)=\dfrac{s}{1+s}$ and $f\left(s\right)=s^{p}$ $(0<p\leq 1)$.

The notion of a modulus function was introduced by Nakano [22], Maddox [19] have introduced and discussed some properties of sequence space defined by using modulus function.

In this note we intend to unify these two approaches and define and study f-statistical convergence in random 2-normed spaces which is quite a new and interesting idea to work with.

2. Definitions and notations

First we recall some of the basic concepts, which will be used in this paper. All the concepts listed below are studied by Aizpuru et al. [2].

Let f be an unbounded modulus function. The f-density of a set $A\subseteq\mathbb{N}$ is defined by

$$\delta_{f}\left(A\right)=\lim_{n}\frac{f\left(\left|A\left(n\right)\right|\right)}{f\left(n\right)}$$

in case this limit exists.

Let X be a normed space and let $\left(s_{n}\right)_{n}$ be a sequence in X. We will say that the f-statistical limit of $\left(s_{n}\right)_{n}$ is $s\in X$, and write $f\text{-}st\lim s_{n}=s$, if for each $\varepsilon>0$ we have $\delta_{f}\left(\left\{i\in\mathbb{N}:\left\|s_{i}-s\right\|>\varepsilon\right\}\right)=0$.

Note that if $A\subseteq\mathbb{N}$ is finite we have that there exist $n_{0},p\in\mathbb{N}$ such that $\left|A\left(n\right)\right|=p$ if $n\geq n_{0}$ and it will be $\delta_{f}\left(A\right)=0$ for each unbounded f. Therefore, if $\lim s_{n}=s$ and f is an unbounded modulus function then $f\text{-}st\lim s_{n}=s$.

It is straightforward to see that $f\text{-}st\lim\left(s_{n}+p_{n}\right)=f\text{-}st\lim s_{n}+f\text{-}st\lim p_{n}$ and $\alpha f\text{-}st\lim s_{n}=f\text{-}st\lim\alpha s_{n}$, whenever $\alpha\in\mathbb{K}$ and the limits on the right sides exist. Also, it is easy to prove that for $X=\mathbb{K}$ we have $f\text{-}st\lim s_{i}p_{i}=f\text{-}st\lim s_{i}f\text{-}st\lim p_{i}$.

Although it is quite clear that $\delta(A)=1-\delta\left(\mathbb{N}\backslash A\right)$ whenever one of the sides exist, the situation is a bit different for unbounded moduli. First, assume $A\subseteq\mathbb{N}$ and $\delta_{f}\left(A\right)=0$. For every $n\in\mathbb{N}$

$$f\left(n\right)\leq f\left(\left|A\left(n\right)\right|\right)+f\left(\left|\left(\mathbb{N}\backslash A\right)\left(n\right)\right|\right)$$

and so

$$1\leq\frac{f\left(\left|A\left(n\right)\right|\right)}{f\left(n\right)}+\frac{f\left(\left|\left(\mathbb{N}\backslash A\right)\left(n\right)\right|\right)}{f\left(n\right)}\leq\frac{f\left(\left|A\left(n\right)\right|\right)}{f\left(n\right)}+1.$$

By taking limits we deduce that $\delta_{f}\left(\mathbb{N}\backslash A\right)=1$. On the other hand, the naturally expected reciprocal is false:

Example 1. *Let $f\left(x\right)=\log\left(x+1\right)$. If $E=\left\{n^{2}:n\in\mathbb{N}\right\}$ and $O=\mathbb{N}\backslash A$ then we have $\delta_{f}\left(E\right)=\delta_{f}\left(O\right)=1$. Moreover, if $S=\left\{n^{2}:n\in\mathbb{N}\right\}$ then $\delta_{f}\left(S\right)=\frac{1}{2}$, $\delta_{f}\left(\mathbb{N}\backslash S\right)=1$ and so $f\text{-}st\lim\chi_{(n)}$ does not even exist, whereas $st\lim\chi_{(n)}=0$.*

Let us note that for any unbounded modulus f and any $A\subseteq\mathbb{N}$ we have that $\delta_{f}\left(A\right)=0$ implies $\delta\left(A\right)=0$. Indeed, if $\delta_{f}\left(A\right)=0$ then for every $p\in\mathbb{N}$ there exists

$n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $f(|A(n)|) \leq \frac{1}{p} f(n) \leq \frac{1}{p} p f\left(\frac{1}{p} n\right) = f\left(\frac{1}{p} n\right)$, which implies $|A(n)| \leq \frac{1}{p} n$ and so $\delta(A) = 0$.

The concept of 2-normed spaces was introduced and studied by Siegfried Gähler, a German Mathematician who worked at German Academy of Science, Berlin, in a series of paper in German language published in Mathematische Nachrichten, see for example references [6, 14, 16]. This notion which is nothing but a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George, White Jr. of USA in 1969 entitled 2-Banach spaces [31]. In the same year Gähler published another paper on this theme in the same journal [16]. A.H. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with S. Gähler et al. [17] of 1975 also provide valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by Siddiqi [29]. A 2-normed space is a pair $(X, \|\cdot,\cdot\|)$, where X is a linear space of a dimension greater than one and $\|\cdot,\cdot\|$ is a real valued mapping on $X \times X$ such that the following conditions be satisfied:
(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent
(ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$,
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, whenever $x, y \in X$ and $\alpha \in \mathbb{R}$,
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

Observe that in any 2-normed space $(X, \|\cdot,\cdot\|)$ we have $\|x, y\| \geq 0$ and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Also, if x, y and z are linearly dependent, then $\|x, y + z\| = \|x, y\| + \|x, z\|$ or $\|x, y - z\| = \|x, y\| + \|x, z\|$. Given a 2-normed space $(X, \|\cdot,\cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence (x_n) in X is said to be convergent to x in X if $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for every $y \in X$.

Now we recall some of the basic concepts related to PN spaces, and we refer to [25, 26] for more details.

Definition 1. *Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $S = [0, 1]$ the closed unit interval. A mapping $f : \mathbb{R} \to S$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.*

We denote the set of all distribution functions by D^+ such that $f(0) = 0$. If $a \in \mathbb{R}_+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1, & \text{if } t > a, \\ 0, & \text{if } t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 2. *A triangular norm (t-norm) is a continuous mapping $* : S \times S \to S$ such that $(S, *)$ is an abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$ for all $a, b, c, d \in S$. A triangle function τ is a binary operation on D^+ which is commutive, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.*

Recently, Golet [5] defined the random 2-normed space as follows.

Definition 3. *Let X be a linear space of dimension greater than one, τ is a triangle function, and $F : X \times X \to D^+$. Then F is called a probabilistic 2-norm and (X, F, τ) a probabilistic 2-normed space if the following conditions are satisfied:*

(i) $F(x, y; t) = H_0(t)$ if x and y are linearly dependent, where $F(x, y; t)$ denotes the value of $F(x, y)$ at $t \in \mathbb{R}$,

(ii) $F(x, y; t) \neq H_0(t)$ if x and y are linearly independent,

(iii) $F(x, y; t) = F(y, x; t)$ for all $x, y \in X$,

(iv) $F(\alpha x, y; t) = F(x, y; \frac{t}{|\alpha|})$ for every $t > 0$, $\alpha \neq 0$ and $x, y \in X$,

(v) $F(x + y, z; t) \geq \tau(F(x, z; t), F(y, z; t))$ whenever $x, y, z \in X$, and $t > 0$.

If (v) is replaced by

(vi) $F(x+y, z; t_1+t_2) \geq F(x, z; t_1)*F(y, z; t_2)$ for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_+$; then $(X, F, *)$ is called a random 2-normed (also called fuzzy 2-normed) space (for short, FTN space).

As a standard example, we can give the following:

Example 2. *Let $(X, \|.,.\|)$ be a 2-normed space, and let $a * b = ab$ for all $a, b \in S$. For all $x, y \in X$ and every $t > 0$, consider*

$$F(x, y; t) = \frac{t}{t + \|x, y\|}.$$

Then observe that $(X, F, *)$ is a fuzzy 2-normed space.

We also recall that the concept of convergence and Cauchy sequence in a fuzzy 2-normed space is studied in [21].

Definition 4. *Let $(X, F, *)$ be a FTN space. Then, a sequence $x = \{x_k\}$ is said to be convergent to $L \in X$ with respect to the fuzzy norm F if, for every $\varepsilon > 0$ and $\eta \in (0, 1)$, there exists $k_0 \in \mathbb{N}$ such that $F_{x_k - L, z}(\varepsilon) > 1 - \eta$ whenever $k \geq k_0$ and nonzero $z \in X$. It is denoted by $F\text{-}\lim x = L$ or $x_k \to_F L$ as $k \to \infty$.*

Definition 5. *Let $(X, F, *)$ be a FTN space. Then, a sequence $x = \{x_k\}$ is said to be statistically convergent to $L \in X$ with respect to the fuzzy norm F if, for every $\varepsilon > 0$, $\eta \in (0, 1)$ and nonzero $z \in X$*

$$\delta\left(\{k \in \mathbb{N} : F_{x_k - L, z}(\varepsilon) \leq 1 - \eta\}\right) = 0$$

or equivalently

$$\delta\left(\{k \in \mathbb{N} : F_{x_k - L, z}(\varepsilon) > \eta\}\right) = 1.$$

It is denoted by st(FTN)-$\lim x = L$ or L is called the st(FTN)-limit of x.

Definition 6. *Let $(X, F, *)$ be a FTN space. Then, a sequence $x = \{x_k\}$ is called a statistically Cauchy sequence with respect to the fuzzy norm F if, for every $\varepsilon > 0$, $\eta \in (0, 1)$ and nonzero $z \in X$, there exists a number $k_0 \in \mathbb{N}$ such that*

$$\delta\left(\{k \in \mathbb{N} : F_{x_k - x_m, z}(\varepsilon) \leq 1 - \eta\}\right) = 0$$

for all $k, m \geq k_0$.

3. Main results

In this section we study the density on moduli with respect to the fuzzy norm F in the FTN-space and prove some important results. The results are analogues to those given by Aizpuru et al. [1, 2], Gürdal and Özgür [12] and Borgohain [4].

Following the line of Borgohain [4] we now introduce the following definition using modulus functions.

Definition 7. *Let $(X, F, *)$ be a FTN space. Then, a sequence $x = \{x_k\}$ is said to be f_{FTN}-statistically convergent to $L \in X$ with respect to the fuzzy norm F if, for every $\varepsilon > 0$, $\delta > 0$, $\eta \in (0, 1)$ and nonzero $z \in X$*

$$\lim_k \frac{f\left(\left|\left\{k \leq n : F_{x_k - L, z}(\varepsilon) \leq 1 - \eta\right\}\right|\right)}{f(k)} = 0.$$

We define it as $f_{\mathrm{FTN}}\text{-st-}\lim x = L$.

Corollary 1. *Let $(X, F, *)$ be a FTN space. For any unbounded modulus f, if $F\text{-}\lim x = L$, then $f_{FTN}\text{-}st\text{-}\lim x = L$. But the converse need not be true in general.*

Proof. Let $F\text{-}\lim x = L$. Then for every $\varepsilon > 0$ and $\eta \in (0, 1)$, there exists $k_0 \in \mathbb{N}$ such that

$$F_{x_k - L, z}(\varepsilon) > 1 - \eta$$

whenever $k \geq k_0$ and nonzero $z \in X$. Construct

$$A(\varepsilon) := \left\{k \leq n : F_{x_k - L, z}(\varepsilon) \leq 1 - \eta\right\},$$

which is a finite set of $\mathbb{N}$. Then we have that there exists $k_0, p \in \mathbb{N}$ such that $|A(\varepsilon)| = p$, if $k \geq k_0$, which will show that

$$\lim_k \frac{f(|A(\varepsilon)|)}{f(k)} = 0.$$

Hence $f_{\mathrm{FTN}}\text{-st-}\lim x = L$. □

The following example shows that the converse need not be true.

Example 3. *Let $X = \mathbb{R}^2$, with the 2-norm $\|x, z\| = \|x_1 z_2 - x_2 z_1\|$, $x = (x_1, x_2)$, $z = (z_1, z_2)$, and $a * b = ab$ for all $a, b \in S$. Let $F(x, z; t) = \frac{t}{t + \|x, z\|}$ for every $x, z \in X$, $z_2 \neq 0$, and every $\varepsilon > 0$. Now define a sequence,*

$$x_k := \begin{cases} (k, 0), & \text{if } k = n^2,\ k \leq n \\ (0, 0), & \text{otherwise} \end{cases}$$

and write

$$K_n(\eta, \varepsilon) := \left\{k \leq n : F_{x_k - L, z}(\varepsilon) \leq 1 - \eta\right\},\ 0 < \eta < 1;\ L = (0, 0).$$

We see that

$$F_{x_k - L, z}(\varepsilon) := \begin{cases} \frac{\varepsilon}{\varepsilon + k z_2}, & \text{if } k = n^2,\ k \leq n \\ 1, & \text{otherwise} \end{cases}.$$

Therefore $x = (x_k)$ is f_{FTN}-statistical convergent, i.e. $\lim_k \frac{f(|K_n(\eta, \varepsilon)|)}{f(k)} = 0$, but not convergent $(X, F, *)$.

The proofs of the following Theorems are easy and thus omitted.

Theorem 2. *Let $(X, F, *)$ be a FTN space. If a sequence $x = (x_k)$ is f_{FTN}-st-convergent, then the f_{FTN}-st-limit is unique.*

Corollary 3. *Let $(X, F, *)$ be a FTN space. For f and g two unbounded moduli, if $f_{FTN}\text{-}st\text{-}\lim x = L_1$ and $f_{FTN}\text{-}st\text{-}\lim x = L_2$ then $L_1 = L_2$.*

Theorem 4. *Let $(X, F, *)$ be a FTN space. Let $f_{FTN}\text{-}st\text{-}\lim x = L_1$ and $f_{FTN}\text{-}st\text{-}\lim y = L_2$. Then*

(i) $f_{FTN}\text{-}st\text{-}\lim (x + y) = L_1 + L_2$,

(ii) $f_{FTN}\text{-}st\text{-}\lim (\alpha x) = \alpha L_1$, *for any* $\alpha > 0$.

We now introduce our main theorem.

Theorem 5. *Let $(X, F, *)$ be a FTN space and f an unbounded modulus. Then f_{FTN}-st-$\lim x = L$ if and only if there exists a set $K = \{k_n : k_1 < k_2 < k_3 < ...\}$ with $\delta_f(K) = 1$ such that f_{FTN}-$\lim x_{k_n} = L$.*

Proof. Suppose that f_{FTN}-st-$\lim x = L$. Then for any $\varepsilon > 0$, $r \in \mathbb{N}$ and nonzero z in X, we have

$$K(r,\varepsilon) := \left\{n \in \mathbb{N} : F_{x_{k_n}-L,z}(\varepsilon) \geq 1 - \frac{1}{r}\right\},$$

and

$$M(r,\varepsilon) := \left\{n \in \mathbb{N} : F_{x_{k_n}-L,z}(\varepsilon) < \frac{1}{r}\right\}.$$

Then $\lim_n \frac{f(|K(r,\varepsilon)|)}{f(n)} = 0$,

$$M(1,\varepsilon) \supset M(2,\varepsilon) \supset ... \supset M(i,\varepsilon) \supset M(i+1,\varepsilon) \supset ..., \tag{1}$$

and

$$\lim_n \frac{f(|M(r,\varepsilon)|)}{f(n)} = 1,\ r \in \mathbb{N}. \tag{2}$$

Now we have to show that for $n \in M(r,\varepsilon)$, $\{x_{k_n}\}$ is f_{FTN}-$\lim x = L$. On contrary suppose that $\{x_{k_n}\}$ is not f_{FTN}-$\lim x = L$. Therefore there is $\eta > 0$ such that $F_{x_{k_n}-L,z}(\varepsilon) \geq \eta$ for infinitely many terms. Let $M(\eta,\varepsilon) := \{n \in \mathbb{N} : F_{x_{k_n}-L,z}(\varepsilon) < \eta\}$ and $\eta > \frac{1}{r}$, $r \in \mathbb{N}$. Then

$$\lim_n \frac{f(|M(\eta,\varepsilon)|)}{f(n)} = 0$$

and by (1), $M(r,\varepsilon) \subset M(\eta,\varepsilon)$. Thus $\lim_n \frac{f(|M(r,\varepsilon)|)}{f(n)} = 0$, which contradicts (2) and we get that $\{x_{k_n}\}$ is f_{FTN}-$\lim x = L$. Conversely, suppose that there exists a set $K = \{k_n : k_1 < k_2 < k_3 < ...\}$ with $\delta_f(K) = 1$ such that f_{FTN}-$\lim x_{k_n} = L$. Then there is a positive integer N such that $n > N$,

$$F_{x_n-L,z}(\varepsilon) > 1 - \eta.$$

Put $K(\eta,\varepsilon) := \{n \in \mathbb{N} : F_{x_n-L,z}(\varepsilon) \leq 1 - \eta\}$ and $K' = \{k_{N+1}, k_{N+2}, ...\}$. Then $\delta_f(K') = 1$ and $K(\eta,\varepsilon) \subseteq \mathbb{N} - K'$ which implies that $\delta_f(K(\eta,\varepsilon)) = 0$. Hence f_{FTN}-st-$\lim x = L$, as desired. □

Definition 8. *Let $(X, F, *)$ be a FTN space. Then, a sequence $x = \{x_k\}$ is said to be f_{FTN}-statistically Cauchy with respect to the fuzzy norm F if, for every $\varepsilon > 0$, $\delta > 0$, $\eta \in (0,1)$ and nonzero $z \in X$*

$$\lim_k \frac{f(|\{k \in \mathbb{N} : F_{x_k-x_N,z}(\varepsilon) \leq 1 - \eta\}|)}{f(k)} = 0.$$

We define it as f_{FTN}-st-Cauchy.

Theorem 6. *Let $(X, F, *)$ be a FTN space, f an unbounded modulus. Then f_{FTN}-statistically convergent if and only if it is f_{FTN}-statistically Cauchy sequence.*

Proof. Suppose that f_{FTN}-st-$\lim x = L$. hoose $r > 0$ such that $(1-r) * (1-r) > 1 - \eta$. Then, for all $\varepsilon > 0$ and nonzero z in X, we get $\lim_k \frac{f(|S(r,\varepsilon)|)}{f(k)} = 0$,

where

$$S(r,\varepsilon) = \left\{k \in \mathbb{N} : F_{x_k-L,z}\left(\frac{\varepsilon}{2}\right) \leq 1 - r\right\}.$$

This implies that $\lim_k \frac{f\left(\left|S^C(r,\varepsilon)\right|\right)}{f(k)} = 1$,

where

$$S^C(r,\varepsilon) = \left\{k \in \mathbb{N} : F_{x_k - L, z}\left(\frac{\varepsilon}{2}\right) > 1 - r\right\}.$$

Let $N \in S^C(r,\varepsilon)$. Then $F_{x_N - L, z}\left(\frac{\varepsilon}{2}\right) > 1 - r$. Now, let

$$B(\eta,\varepsilon) = \left\{k \in \mathbb{N} : F_{x_k - x_N, z}(\varepsilon) \leq 1 - \eta\right\}.$$

We need to show that $B(\eta,\varepsilon) \subset S(r,\varepsilon)$. Let $k \in B(\eta,\varepsilon)$. Then $F_{x_k - x_N, z}(\varepsilon) \leq 1-\eta$ and hence $F_{x_k - L, z}\left(\frac{\varepsilon}{2}\right) \leq 1 - r$, i.e. $k \in S(r,\varepsilon)$. Otherwise, if $F_{x_k - L, z}\left(\frac{\varepsilon}{2}\right) > 1 - r$ then

$$\begin{aligned} 1 - \eta \geq F_{x_k - x_N, z}(\varepsilon) &\geq F_{x_k - L, z}\left(\frac{\varepsilon}{2}\right) * F_{x_N - L, z}\left(\frac{\varepsilon}{2}\right) \\ &> (1 - r) * (1 - r) > 1 - \eta, \end{aligned}$$

which is not possible. Thus $B(\eta,\varepsilon) \subset S(r,\varepsilon)$, which implies that $x = \{x_k\}$ is f_{FTN}-st-convergent.

Suppose that $x = \{x_k\}$ is f_{FTN}-st-Cauchy but not f_{FTN}-st-convergent. Then there exists $N \in \mathbb{N}$ such that $\lim_k \frac{f(|B(\eta,\varepsilon)|)}{f(k)} = 0$ where

$$B(\eta,\varepsilon) = \left\{k \in \mathbb{N} : F_{x_k - x_N, z}(\varepsilon) \leq 1 - \eta\right\}.$$

From acceptance,

$$M(\eta,\varepsilon) = \left\{k \in \mathbb{N} : F_{x_k - L, z}\left(\frac{\varepsilon}{2}\right) > 1 - \eta\right\},$$

i.e. $\lim_k \frac{f\left(\left|M^C(\eta,\varepsilon)\right|\right)}{f(k)} = 1$. Since

$$F_{x_k - x_N, z}(\varepsilon) \geq 2F_{x_k - L, z}\left(\frac{\varepsilon}{2}\right) > 1 - \eta,$$

if $F_{x_k - L, z}\left(\frac{\varepsilon}{2}\right) > \frac{1-\eta}{2}$. Therefore $\lim_k \frac{f\left(\left|B^C(\eta,\varepsilon)\right|\right)}{f(k)} = 0$, i.e. $\lim_k \frac{f(|B(\eta,\varepsilon)|)}{f(k)} = 1$, which leads to a contradiction, since $x = \{x_k\}$ was f_{FTN}-statistically Cauchy sequence. Thus $x = \{x_k\}$ must be f_{FTN}-statistically convergent, as desired. The theorem is proved. □

Corollary 7. *Let $(X, F, *)$ be a FTN space, f an unbounded modulus. Then if $x = \{x_k\}$ is f_{FTN}-statistically Cauchy sequence then it has a Cauchy subsequence with respect to the fuzzy norm F.*

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