Electronic Journal of Mathematical Analysis and Applications
Vol. 8(1) Jan. 2020, pp. 309-315
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

SPHERICAL FUNCTIONS OF TYPE δ ON NILPOTENT LIE GROUPS

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Abstract. Let N be a connected, simply connected two-step nilpotent Lie group and K a compact subgroup of $Aut(N)$, the group of automorphisms of N. Let δ denote a unitary equivalence class of irreducible unitary representations of K. In this paper, we give an explicit formula for spherical functions of type δ for the semidirect $K \propto N$.

1. Introduction

The notion of Gelfand pair introduced by I. M. Gelfand in 1950, permitted the generalization of Harmonic Analysis to the theory of commutative Banach algebras. In fact, the characters of the commutative Banach algebra $L^1(G\backslash\backslash K)$ are identified with some functions called spherical functions which work as exponential functions in general case. The notion of spherical function in this context has been sufficiently studied (see [2, 3, 5]). It has been extended to spherical functions of type δ, where δ is a unitary equivalence class of irreducible representations of K, by some authors such as R. Godement, W. Barker, G. Warner or A. Wawrzynczyk. For spherical functions of type δ, $L^1(G\backslash\backslash K)$ is replaced by the convolution algebra $I_{c,\delta}(G)$ of compactly supported, central and χ_δ-invariant functions with complex-valued on G. Note that in this situation, the last algebra is not necessarily commutative. The classical spherical functions are the spherical functions of type 1_K, the trivial and one dimensional representation of K. This extension of spherical function has been studied for n-dimensional sphere (see [10]), for semi-simple Lie groups (see [6, 7, 12, 9]) and in the more general case of reductive Lie groups (see [4]). The main goal of this paper is to construct the spherical functions of type δ for a semi-direct product of a connected nilpotent Lie group and a compact subgroup. In section 2, we give the notations and the definitions necessary for a better understanding of our paper. We recall also the decomposition of a two-step nilpotent Lie group obtained by C. Benson and al. in [2] thanks to Kirillov theory. Section 3 is devoted to the construction of spherical functions of type δ on $G = K \propto N$. First, we show that any spherical function of type δ is obtained from a unitary irreducible

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2010 *Mathematics Subject Classification.* 22E25, 22E27, 43A30, 43A90.
Key words and phrases. Spherical function of type δ, representations of groups , Nilpotent Lie group.
Submitted June 11, 2019. Revised Sep. 10, 2019.

representation of N. It generalizes the results of [11] where the authors used the unitary characters of N. Then, thanks to the decomposition mentioned above, we prove that the spherical functions of type δ for two-step nilpotent Lie groups are obtained from those of Heisenberg groups. Finally, since the representations of Heisenberg groups are well-known, we obtain an explicit formula for spherical functions of type δ on two-step nilpotent Lie groups.

2. Preliminaries

Let N be a connected, simply connected two-step nilpotent Lie group and K a compact subgroup of $Aut(N)$, the group of automorphisms of N. e will designate the identity element of N and e_K the identity element of K. We write the action by $k \in K$ on $x \in N$ as $k.x$. We designate by dn a Haar measure on N and dk the normalized Haar measure on K. We denote by η the Lie algebra of N and by $\widehat{N}$ the unitary dual of N. Let $l \in \eta^*$, where η^* is the algebraic dual of η, and let π_l be the unitary irreducible representation of N associated with l by the Kirillov method. The subgroup K acts on $\widehat{N}$ by the following way: $k.\pi(x) = \pi(k.x)$ for all $k \in K$, $x \in N$ and $\pi \in \widehat{N}$. Let K_l denote the stabilizer of π_l by this action. Let's put $z_l = Ker(l \mid z)$ where z is the center of η and $l \mid z$ is the restriction of l to z. Let's consider $<,>_l$ a K-invariant Euclidean structure on N. We denote by m the orthogonal complement of z in η and we set $a_l = \{X \in m : ad(X)m \in z_l\}$ where ad is the adjoint representation of η. We have $\eta = a_l \oplus b_l \oplus z_l^{'} \oplus z_l$ where $z_l^{'}$ and b_l are respectively the orthogonal supplementary of z_l in z and of a_l in m. For more details, the reader can refer to [1]. The authors of [1] prove that K_l preserves a_l, b_l, $z_l^{'}$ and z_l. We set $h_l = b_l \oplus z_l'$. Let A_l, H_l and Z_l be the connected Lie groups associated respectively with a_l, h_l and z_l. We have $N = A_l \times H_l \times Z_l$. H_l is either abelian or a Heisenberg group and A_l is an abelian group (see [1]). We put $G := K \propto N$, the semidirect product of K and N, with the following group law: for all (k, n), $(k', n') \in G$, $(k, n)(k', n'){:=}(kk', nk.n')$. Let δ denote a unitary equivalence class of irreducible unitary representations of K and let put $\chi_\delta{:=}\, d(\delta)\xi_\delta$, where $d(\delta)$ is the degree of δ and ξ_δ the character of δ. χ_δ is the normalized trace of δ. Let $I_c(G)$ denote the set of all K-central functions that is, the set of all continuous, complex-valued functions f on G and compactly supported such that:

$$f(k\tilde{k}k^{-1}, k.x) \quad = \quad f(\tilde{k}, x), \forall k, \tilde{k} \in K, \forall x \in N,$$

and $I_\delta(G)$ denotes the set of all continuous, complex- valued functions f on G and compactly supported such that: $\chi_\delta * f{:=}f * \chi_\delta{=}f$ where,

$$\begin{aligned} \chi_\delta * f(\tilde{k}, x) \quad &:= \quad \int_K \chi_\delta(k^{-1}) f(k\tilde{k}, k.x) dk \\ f * \chi_\delta(\tilde{k}, x) \quad &= \quad \int_K \chi_\delta(k^{-1}) f(k\tilde{k}, x) dk. \end{aligned}$$

Let us put $I_{c,\delta}(G) := I_c(G) \cap I_\delta(G)$. $I_{c,\delta}(G)$ is a subalgebra of $\mathcal{K}(G)$, where $\mathcal{K}(G)$ is the convolution algebra of all continuous, complex-valued functions on G and compactly supported. For all $f \in \mathcal{K}(G)$, we put

$$f_K(\tilde{k}, x) \quad = \quad \int_K f(k\tilde{k}k^{-1}, k.x) dk.$$

Then the map $f \mapsto \chi_\delta * f_K$ is a continuous projection of $\mathcal{K}(G)$ on $I_{c,\delta}(G)$. Let E denote a finite dimensional vector space. A spherical function ϕ on G of type δ is

a quasi-bounded continuous function on G with values in $End_{\mathbb{C}}(E)$ such that:
(i) $\phi(kxk^{-1}) = \phi(x) \forall k \in K$ and for all $x \in G$.
(ii) $\chi_\delta * \phi = \phi = \phi * \chi_\delta$
(iii) The map $U_\phi : f \mapsto \int_G f(x)\phi(x)dx$ is an irreducible representation of the algebra $I_{c,\delta}(G)$.
The dimension of E is called the height of ϕ (cf.[8]). Denote by $S^m_\delta(G)$ the space of all spherical functions of type δ and height m. In [8], the authors introduced the notion of spherical Fourier transform of type δ defined by,

$$\mathfrak{F}f(\phi) = \int_G f(x)\phi(x)dx, \forall \phi \in S^m_\delta(G), f \in I_{c,\delta}(G).$$

Denoting $\check{\delta}$ as the contragredient representation of δ and letting $\mu_{\check{\delta}}$ be an arbitrary element in the equivalence class of $\check{\delta}$, we put $F_{\check{\delta}} := Hom(E_{\check{\delta}}, E_{\check{\delta}})$ where $E_{\check{\delta}}$ is the space of the representation $\mu_{\check{\delta}}$. Then we denote by $\mathfrak{U}_{c,\delta}(G)$, the space of functions $\psi : G \mapsto F_{\check{\delta}}$, continuous and compactly supported such that $\psi(k_1(\tilde{k}, x)k_2) = \mu_{\check{\delta}}(k_1)\psi(\tilde{k}, x)\mu_{\check{\delta}}(k_2)$. In [12], it is shown that $I_{c,\delta}(G)$ is isomorphic to $\mathfrak{U}_{c,\delta}(G)$, thanks to the map $\psi^\delta : f \mapsto \psi^\delta_f$ from $I_{c,\delta}(G)$ to $\mathfrak{U}_{c,\delta}(G)$ where $\psi^\delta_f(\tilde{k}, x) := \int_K \mu_{\check{\delta}}(k^{-1})f(k.(\tilde{k}, x))dk$. We designate by $C_c(N, F_{\check{\delta}})$, the convolution algebra of continuous functions on N, compactly supported and with values in $F_{\check{\delta}}$. We set

$$\mathfrak{U}_{c,\delta}(N) := \{\psi \in C_c(N, F_{\check{\delta}}) : \psi(k.x) = \mu_{\check{\delta}}(k)\psi(x)\mu_{\check{\delta}}(k^{-1})\}.$$

We proved in [11] that $I_{c,\delta}(K \propto N)$ and $\mathfrak{U}_{c,\delta}(N)$ are isomorphic as convolution algebras. The isomorphism is given by $F^\delta : f \mapsto F^\delta_f$ from $I_{c,\delta}(K \propto N)$ to $\mathfrak{U}_{c,\delta}(N)$, where $F^\delta_f(x) := \int_K \mu_{\check{\delta}}(k^{-1})f(k, k.x)dk$. If δ is trivial and one dimensional, this turn to the well-known isomorphism between the algebra of K-invariant functions on N and the algebra of K-biinvariant functions on $K \propto N$.

3. Explicit Formula for Spherical Functions

In the following result, we prove the existence of an irreducible unitary representation π of N which gives the spherical function of type δ.

Theorem 3.1. Let ϕ_δ be a spherical function of type δ and let $\mu_{\check{\delta}}$ be an arbitrary element in the equivalence class of $\check{\delta}$. Then there exists a unitary irreducible representation π of N such that $\phi_\delta(\tilde{k}, x) = \int_K \pi(k^{-1}.x) \otimes \mu_\delta(k^{-1}\tilde{k}^{-1}k)dk$

Proof. Let ϕ_δ be a spherical function of type δ. We know by definition that the map $U_\phi : f \mapsto \int_G f(x)\phi(x)dx$ is an irreducible representation of the algebra $I_{c,\delta}(G)$ in a finite-dimensional vector space V. But since $I_{c,\delta}(G)$ is isomorphic to $\mathfrak{U}_{c,\delta}(N)$, we consider U_ϕ as irreducible representation of $\mathfrak{U}_{c,\delta}(N)$. We have $\mathfrak{U}_{c,\delta}(N) \subset C_c(N, F_{\check{\delta}})$. Hence, thanks to lemma 1.1.1.6 (p. 23 [12]), there exists a irreducible representation (L, E_L) of $C_c(N, F_{\check{\delta}})$ and a finite- dimensional closed invariant subspace M of E_L such that $(L \mid \mathfrak{U}_{c,\delta}(N), M)$ is equivalent to U_ϕ. We have $C_c(N, F_{\check{\delta}}) \simeq C_c(N) \otimes F_{\check{\delta}}$. Let $\widetilde{L}$ be a irreducible representation of $C_c(N)$. There exists $(\pi, \mathcal{H}_\pi) \in \widehat{N}$ such that $\widetilde{L}(f) = \int_N f(x)\pi(x)dx$, for $f \in C_c(N)$. Thus a representation of $C_c(N, F_{\check{\delta}})$ is written for $F \in C_c(N, F_{\check{\delta}})$ by $L(F) = \int_N \pi(x) \otimes F(x)dx$. It comes that for

$f \in I_{c,\delta}(G)$,

$$\begin{aligned}
U_\phi(F_f^\delta) =& U_\phi(\chi_\delta * f_K)\\
&= \int_G \int_K \chi_\delta * f_K(k, k.x)\pi(x) \otimes \mu_{\check{\delta}}(k^{-1})dkdx\\
&= \int_G \int_K \int_K \int_K \chi_\delta(\tilde{k}^{-1}) f(k_1\tilde{k}^{-1}kk_1^{-1}, k_1\tilde{k}^{-1}k.x)\pi(x) \otimes \mu_{\check{\delta}}(k^{-1})dk_1 d\tilde{k}dkdx\\
&= \int_G \int_K \int_K \int_K \chi_\delta(\tilde{k}^{-1}k^{-1}) f(k_1\tilde{k}k_1^{-1}, x)\pi(k_1^{-1}.x) \otimes \mu_{\check{\delta}}(k^{-1})dk_1 d\tilde{k}dkdx\\
&= \int_G \int_K \int_K \int_K \chi_\delta(k_1^{-1}\tilde{k}^{-1}k_1k^{-1}) f(\tilde{k}, x)\pi(k_1^{-1}.x) \otimes \mu_{\check{\delta}}(k^{-1})dk_1 d\tilde{k}dkdx\\
&= \int_G \int_K \int_K f(\tilde{k}, x)\pi(k_1^{-1}.x) \otimes \mu_\delta(k_1^{-1}\tilde{k}^{-1}k_1)d\tilde{k}dk_1dx\\
&= \int_G \int_K f(\tilde{k}, x)(\int_K \pi(k_1^{-1}.x) \otimes \mu_\delta(k_1^{-1}\tilde{k}^{-1}k_1)dk_1)d\tilde{k}dx\\
&= \int_G \int_K f(\tilde{k}, x)\phi_\delta(\tilde{k}, x)d\tilde{k}dx
\end{aligned}$$

where

$$\phi_\delta(\tilde{k}, x) = \int_K \pi(k^{-1}.x) \otimes \mu_\delta(k^{-1}\tilde{k}^{-1}k)dk$$

Corrolary 3.2. ϕ_δ is unitary and positive definite.

Proof. It is straightforward to show that ϕ_δ is unitary. We put on $\mathcal{H}_\phi \otimes E_\delta$ the following inner product: $< h \otimes \zeta, h' \otimes \zeta' >=< h, h' >< \zeta, \zeta' >$. For $n \in \mathbb{N}$, let's consider $(c_1, c_2, ..., c_n) \in \mathbb{C}^n$, $(x_1, x_2, ..., x_n) \in G^n$ and $(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) \in K^n$. We have

$$\begin{aligned}
&\sum_{1\le i,j\le n} c_i\bar{c}_j \langle \phi_\delta((\tilde{k}_j, x_j)^{-1}(\tilde{k}_i, x_i))u \otimes v, u \otimes v\rangle\\
&= \sum_{1\le i,j\le n} c_i\bar{c}_j \langle \int_K \pi(k^{-1}\tilde{k}_j^{-1}.(x_j^{-1}x_i))u \otimes \mu_\delta(k^{-1}\tilde{k}_i^{-1}\tilde{k}_jk)v, u \otimes v\rangle dk\\
&= \int_K \sum_{1\le i,j\le n} c_i\bar{c}_j \langle \pi(k^{-1}\tilde{k}_j^{-1}.x_i)u, \pi(k^{-1}\tilde{k}_j^{-1}.x_j)u\rangle \langle \mu_\delta(\tilde{k}_jk)v, \mu_\delta(\tilde{k}_ik)v\rangle dk\\
&= \int_K \sum_{1\le i,j\le n} c_i\bar{c}_j \langle \pi(k^{-1}.x_i)u, \pi(k^{-1}.x_j)u\rangle \langle \mu_\delta(k)v, \mu_\delta(\tilde{k}_i\tilde{k}_j^{-1}k)v\rangle dk\\
&= \int_K \sum_{1\le i,j\le n} c_i\bar{c}_j \langle \pi(k^{-1}.x_i)u, \pi(k^{-1}.x_j)u\rangle \langle \mu_\delta(\tilde{k}_i^{-1}k)v, \mu_\delta(\tilde{k}_j^{-1}k)v\rangle dk\\
&= \int_K \langle \sum_{1\le i\le n} c_i\pi(k^{-1}.x_i)u \otimes \mu_\delta(\tilde{k}_i^{-1}k)v, \sum_{1\le j\le n} c_j\pi(k^{-1}.x_j)u \otimes \mu_\delta(\tilde{k}_j^{-1}k)v\rangle dk\\
&= \int_K \parallel \sum_{1\le i\le n} c_i\pi(k^{-1}.x_i)u \otimes \mu_\delta(\tilde{k}_i^{-1}k)v \parallel^2 dk \ge 0
\end{aligned}$$

Remark 3.3. From the proof of Theorem 3.1., we can deduce the expression of the spherical Fourier transform of type δ. In fact,

$$\begin{aligned}\mathfrak{F}f(\phi) &= \int_G\int_K f(k,x)\phi_\delta(k,x)dkdx \\ &= \int_G\int_K \chi_\delta * f_K(k,k.x)\pi(x)\otimes\mu_{\check{\delta}}(k^{-1})dkdx \\ &= \int_G\int_K f(k,k.x)\pi(x)\otimes\mu_{\check{\delta}}(k^{-1})dkdx = \int_G \pi(x)\otimes F_f^\delta(x)dx.\end{aligned}$$

This formula generalizes that of Theorem 3.2. in [11] where the representation π is one dimensional, say a unitary character of N.

Since the representations of Heisenberg groups are well-known, we express in the following result the spherical functions of type δ for semidirect product with two-step nilpotent factor by means of those of semidirect product with Heisenberg groups as factor. We conserve the notations of the introduction and designate by p_l the orthogonal projection of N on $N/Z_l \simeq A_l \times H_l$.

Theorem 3.4. Let N be a connected, simply connected two step nilpotent Lie group and K a compact subgroup of $Aut(N)$ such that (N,K) is not a Gelfand pair. Then there exists $l\in\eta^*$, $\delta_l\in\widehat{K_l}$ and a spherical function ϕ_l of type δ_l on $K_l\propto H_l$ such that the map $\psi_l=\chi_l\otimes\phi_l\circ p_l$ is a spherical function of type δ_l on $K_l\propto N$, where χ_l is a unitary character of A_l.

Proof. Since (N,K) is not a Gelfand pair, there exists $l\in\eta^*$ such that (H_l,K_l) is not a Gelfand pair (see [1]). This implies that H_l is a Heisenberg group. Let π_l be the unitary irreducible representation of N associated with l by the Kirillov method. We know thanks to the proof of Lemma 2.2 in [1] that $\pi_l=\chi_l\otimes\pi_{H_l}\circ p_l$ where π_{H_l} is a unitary irreducible representation of H_l, χ_l is a unitary character of A_l defined by $\chi_l=e^{il^*(a)}$ with $l^*=l\mid A_l$. The representations π_l, $\chi_l\otimes\pi_{H_l}$ and π_{H_l} are realized in the same space that we shall suppose to be $\mathcal{F}_\lambda$ ($\lambda\in\mathbb{R}^*$), the space of Fock-Bargman. For all $k\in K_l$, there exists an operator $\tau_{\lambda,l}(k)$ which intertwines π_{H_l} and $\pi_{H_l}^k$. The map $\tau_{\lambda,l}$ from K_l to $\mathcal{F}_\lambda$ defines a unitary representation of K_l. Let $\mathcal{F}_\lambda=\oplus m_qP_q$ the decomposition in irreducible components, where P_q is the space of homogeneous polynomial of degree q and m_q is the multiplicity of $\tau_{\lambda,l}\mid P_q$ in $\tau_{\lambda,l}$. Since (H_l,K_l) is not a Gelfand pair then $\mathcal{F}_\lambda$ contains an irreducible K_l-module, say P_q, with m_q strictly larger than 1. We set $\delta_l=\tau_{\lambda,l}\mid P_r$ and consider the spherical function of type δ_l on H_l associated to l or $\delta_l\otimes\pi_l\tau_{\lambda,l}$ which we notice ϕ_l. We will then show that the map $\psi_l=(\chi_l\otimes\phi_l)\circ p_l$ is a spherical function of type δ_l on N. First, ψ_l is K_l-central. In fact,

$$\begin{aligned}\psi_l(k(a,z,t,z')k^{-1}) &= (\chi_l\otimes\phi_l)\circ p_l(k.a,k.z,t,k.z') \\ &= \chi_l(k.a)\phi_l(k.z,t) \\ &= e^{il^*(k.a)}\phi_l(k(z,t)k^{-1}) \\ &= e^{i(k^{-1}.l^*)(a)}\phi_l(k(z,t)k^{-1}) \\ &= e^{il^*(a)}\phi_l(z,t)\end{aligned}$$

where the last equality is due to the fact that ϕ_l is K_l-central and $k^{-1}.l^* = l^*$ since $k \in K_l$. Then, we have

$$\begin{aligned}\chi_{\delta_l} * \psi_l(x) =& \int_{K_l} \chi_{\delta_l}(k)\psi_{\lambda,l}(k^{-1}.x)dk\\ =& \int_{K_l} \chi_{\delta_l}(k)(\chi_l(k^{-1}.a)\phi_l(k^{-1}, k^{-1}.z, t))dk\\ =& \int_{K_l} \chi_{\delta_l}(k)(\chi_l(a)\phi_l(k^{-1}, k^{-1}.z, t))dk\\ =&\chi_l(a)\int_{K_l} \chi_{\delta_l}(k)\phi_l(k^{-1}, k^{-1}.z, t)dk\\ =&\chi_l(a)(\chi_{\delta_l} * \phi_l(z,t))\\ =&\chi_l(a)\phi_l(z,t)\\ =&\psi_l(x)\end{aligned}$$

Also, ψ_l is continuous and quasi-bounded. In fact, since ϕ_l is quasi-bounded there exists a semi-norm ρ such that $sup\frac{\|\phi_l(z,t)\|}{\rho(z,t)} < \infty$. Then, setting $\varrho = (1_{A_l} \otimes \rho) \circ p_l$, which is a semi- norm on N, we have $sup\frac{\|\psi_l(x)\|}{\varrho(x)} = sup\frac{\|\phi_l(z,t)\|}{\rho(z,t)} < \infty$. Now for $k, \tilde{k}_1, \tilde{k}_2 \in K_l$, $x = (a, z, t, z') \in N$ and $y = (a_1, z_1, t_1, z'_1) \in N$ we have

$$\begin{aligned}\int_{K_l} \psi_l(k(\tilde{k}_1, x)k(\tilde{k}_2, y))dk =& \int_{K_l} \chi_l((k.a)(k\tilde{k}_1k.a_1))\phi_l(k\tilde{k}_1k\tilde{k}_2, (k.z,t)(k\tilde{k}_1k.z_1, t_1))dk\\ =&\chi_l(a)\chi_l(a_1)\int_{K_l} \phi_l(k(\tilde{k}_1, z, t)k(\tilde{k}_2, z_1, t_1))dk\\ =&\chi_l(a)\chi_l(a_1)\phi_l(\tilde{k}_1, z, t)\phi_l(\tilde{k}_2, z_1, t_1)\\ =&\chi_l(a)\phi_l(\tilde{k}_1, z, t)\chi_l(a_1)\phi_l(\tilde{k}_2, z_1, t_1)\\ =&\psi_l(\tilde{k}_1, x)\psi_l(\tilde{k}_2, y)\end{aligned}$$

which proves that $\psi_{\lambda,l}$ verifies the functional equation in ([12], page 16) and achieves the proof.

Corollary 3.5. For p_r, p_s respectively in P_r and P_s and for $\lambda > 0$
$\psi_{\lambda,l}(\tilde{k}, a, z, t, z')p_s \otimes p_r(w \otimes w') = e^{il^*(a)}e^{\lambda(-it-|z|^2}\int_{K_l} e^{2\lambda(w,k^{-1}.z)}P_s(w - k^{-1}.z) \otimes P_r(k\tilde{k}^{-1}k^{-1}.w')dk$

Proof. According to Theorem 3.1., we have
$\phi_l(\tilde{k}, z, t) = \int_{K_l} \pi_\lambda(k^{-1}.(z,t)) \otimes \mu_\delta(k^{-1}\tilde{k}^{-1}k)dk$. But for $\lambda > 0$, $\pi_\lambda(z,t)f(w) = e^{\lambda(-it+2(w,z)-|z|^2)}f(w - z)$ and $\tau_{\lambda,l}(k)f(w) = f(k^{-1}.w)$ for all $(z,t) \in H_l$, $w \in \mathbb{C}^n$, $k \in K_l$. So for p_r, p_s respectively in P_r and P_s we have, $\phi_l(\tilde{k}, z, t)p_s \otimes p_r(w \otimes w') = e^{\lambda(-it-|z|^2)}\int_{K_l} e^{2\lambda(w,k^{-1}.z)}p_s(w - k^{-1}.z) \otimes p_r(k\tilde{k}^{-1}k^{-1}.w')dk$ and finally
$\psi_{\lambda,l}(\tilde{k}, a, z, t, z')p_s \otimes p_r(w \otimes w') = e^{il^*(a)}e^{\lambda(-it-|z|^2)}\int_{K_l} e^{2\lambda(w,k^{-1}.z)}p_s(w - k^{-1}.z) \otimes p_r(k\tilde{k}^{-1}k^{-1}.w')dk$.

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