INCLUSION PROPERTIES FOR CERTAIN $k$–UNIFORMLY SUBCLASSES OF $p$–VALENT FUNCTIONS ASSOCIATED WITH THE LIU-OWA OPERATOR

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ABSTRACT. In this paper, we introduce several $k$–uniformly subclasses of $p$–valent functions defined by the Liu-Owa integral operator and investigate various inclusion relationships for these subclasses. Some interesting applications involving certain classes of integral operators are also considered.

1. INTRODUCTION

Let $A_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega$, analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ ($z \in U$), such that $f(z) = g(\omega(z))$ ($z \in U$). In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [1] and [2]).

For $0 \leq \gamma, \eta < p, k \geq 0$ and $z \in U$, we define $US_p^*(k; \gamma)$, $UC_p(k; \gamma)$, $UK_p(k; \gamma, \eta)$ and $UK_p^*(k; \gamma, \eta)$ the $k$–uniformly subclasses of $A_p$ consisting of all analytic functions which are, respectively, $p$–valent starlike of order $\gamma$, $p$–valent convex of order $\gamma$, $p$–valent close-to-convex of order $\gamma$, and type $\eta$ and $p$–valent quasi-convex of order $\gamma$, and type $\eta$ as follows:

$$US_p^*(k; \gamma) = \left\{ f \in A_p : \mathbb{R} \left( \frac{zf'(z)}{f(z)} - \gamma \right) > k \left| \frac{zf'(z)}{f(z)} - p \right| \right\}, \quad (2)$$

$$UC_p(k; \gamma) = \left\{ f \in A_p : \mathbb{R} \left( 1 + \frac{zf''(z)}{f'(z)} - \gamma \right) > k \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \right\}, \quad (3)$$


\[ UK_p (k; \gamma, \eta) = \left\{ f \in A_p : \exists g \in US'_p (k; \eta), \Re \left( \frac{zf'(z)}{g(z)} - \gamma \right) > k \left| \frac{zf'(z)}{g(z)} - p \right| \right\}, \tag{4} \]

\[ UK'_p (k; \gamma, \eta) = \left\{ f \in A_p : \exists g \in UC'_p (k; \eta), \Re \left( \frac{zf'(z)}{g(z)} - \gamma \right) > k \left| \frac{zf'(z)}{g(z)} - p \right| \right\}. \tag{5} \]

These subclasses were introduced and studied by Al-Kharsani [3]. We note that
(i) \( US'_p (k; \gamma) = US^* (k; \gamma) \) and \( UC'_p (k; \gamma) = UC^* (k; \gamma) \) \((0 \leq \gamma < 1)\) (see [4] and [5]);
(ii) \( US'_p (0; \gamma) = S'_p (\gamma) \) \((0 \leq \gamma < p)\) (see [6] and [7]);
(iii) \( UC'_p (0; \gamma) = C'_p (\gamma) \) \((0 \leq \gamma < p)\) (see [6]);
(iv) \( UK'_p (0; \gamma, \eta) = K'_p (\gamma, \eta) \) \((0 \leq \gamma, \eta < p)\) (see [8]);
(v) \( UK'_p (0; \gamma, \eta) = K'_p (\gamma, \eta) \) \((0 \leq \gamma, \eta < p)\) (see [9]).

Corresponding to a conic domain \( \Omega_{p,k,\gamma} \) defined by
\[ \Omega_{p,k,\gamma} = \left\{ u + iv : u > k \sqrt{(u - p)^2 + v^2 + \gamma} \right\}, \tag{6} \]
we define the function \( q_{p,k,\gamma} (z) \) which maps \( \mathbb{U} \) onto the conic domain \( \Omega_{p,k,\gamma} \) such that \( 1 \in \Omega_{p,k,\gamma} \) as the following:
\[ q_{k,\gamma} (z) = \begin{cases} \frac{p + (p - 2\gamma)z}{1 - z} & (k = 0), \\ \frac{p - \gamma}{1 - k^2} \cos \left\{ \frac{2}{n} \left( \cos^{-1} k \right) i \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right\} - k^2 p - \gamma & (0 < k < 1), \\ \frac{p + 2(p - \gamma)}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 & (k = 1), \\ \frac{p - \gamma}{k^2 - 1} \sin \left\{ \frac{\pi}{2 \zeta (k)} \int_0^{\sqrt{z}} \frac{dt}{\sqrt{1 - t^2}} \right\} + k^2 p - \gamma & (k > 1), \end{cases} \tag{7} \]

where \( u(z) = \frac{z - \sqrt{z}}{1 - \sqrt{z}}, x \in (0, 1) \) and \( \zeta (k) \) is such that \( k = \cosh \frac{\pi \zeta (k)}{4}. \) By virtue of the properties of the conic domain \( \Omega_{p,k,\gamma} \), we have
\[ \Re \{ q_{p,k,\gamma} (z) \} > \frac{kp + \gamma}{k + 1}. \tag{8} \]

Making use of the principal of subordination between analytic functions and the definition of \( q_{p,k,\gamma} (z) \), we may rewrite the subclasses \( US'_p (k; \gamma) \), \( UC'_p (k; \gamma) \), \( UK'_p (k; \gamma, \beta) \) and \( UK'_p (k; \gamma, \beta) \) as the following:
\[ US'_p (k; \gamma) = \left\{ f \in A_p : \frac{zf'(z)}{f(z)} < q_{p,k,\gamma} (z) \right\}, \tag{9} \]
\[ UC'_p (k; \gamma) = \left\{ f \in A_p : 1 + \frac{zf''(z)}{f'(z)} < q_{p,k,\gamma} (z) \right\}, \tag{10} \]
\[ UK'_p (k; \gamma, \eta) = \left\{ f \in A_p : \exists g \in US'_p (k; \eta), \frac{zf'(z)}{g(z)} < q_{p,k,\gamma} (z) \right\}. \tag{11} \]
\[ UK_p^* (k; \gamma, \eta) = \left\{ f \in A_p : \exists g \in UC_p (k; \eta), \frac{(zf'(z))'}{g'(z)} < q_{p,k,\gamma} (z) \right\} . \] (12)

Motivated essentially by Jung et al. [10], Liu and Owa [11] introduced the integral operator \( Q_{\beta,p} : A_p \to A_p (\alpha \geq 0, \beta > -p, p \in \mathbb{N}) \) as follows (see also [12]):

\[
Q_{\beta,p}^\alpha f(z) = \left\{ \left( \frac{p+\alpha+\beta-1}{p+\beta-1} \right) \frac{\alpha}{z} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) \, dt \right\} (\alpha > 0),
\]
\[
f(z) \quad (\alpha = 0). \] (13)

For \( f \in A_p \) given by (1), then from (13), we deduce that

\[
Q_{\beta,p}^\alpha f(z) = z^p + \frac{\Gamma (\alpha + \beta + p)}{\Gamma (\beta + p)} \sum_{n=p+1}^{\infty} \frac{\Gamma (\beta + n)}{\Gamma (\alpha + \beta + n)} a_n z^n (\alpha \geq 0; \beta > -p; p \in \mathbb{N}). \] (14)

It is easily verified from the definition (14) that

\[
z \left( Q_{\beta,p}^{\alpha+1} f(z) \right)' = (\alpha + \beta + p) Q_{\beta,p}^\alpha f(z) - (\alpha + \beta) Q_{\beta,p}^{\alpha+1} f(z). \] (15)

We note that

\[
Q_{c,p}^1 f(z) = F_{c,p} (f) (z) = \frac{c + p}{ze} \int z^{c-1} f(z) \, dt \quad (c > -p), \] (16)

where the operator \( F_{c,p} \) is the generalized Bernardi–Libera–Livingston integral operator (see [13] and [14]). Also, we note that the one-parameter family of integral operator \( Q_{\beta,1}^\alpha = Q_{\beta}^\alpha \) was defined by Jung et al. [10] and studied by Aouf [15] and Gao et al. [16].

Next, using the operator \( Q_{\beta,p}^\alpha \), we introduce the following \( k \)--uniformly classes of \( p \)--valent functions for \( \alpha \geq 0, \beta > -p, p \in \mathbb{N}, k \geq 0 \) and \( 0 \leq \gamma, \eta < p \):

\[
US_p^\alpha (\alpha; k; \gamma) = \left\{ f \in A_p : Q_{\beta,p}^\alpha f(z) \in US_p^\alpha (k; \gamma) ; z \in U \right\}, \] (17)

\[
UC_p (\alpha; k; \gamma) = \left\{ f \in A_p : Q_{\beta,p}^\alpha f(z) \in UC_p (k; \gamma) ; z \in U \right\}, \] (18)

\[
UK_p (\alpha; k; \gamma, \eta) = \left\{ f \in A_p : Q_{\beta,p}^\alpha f(z) \in UK_p (k; \gamma, \eta) ; z \in U \right\}, \] (19)

\[
UK_p^* (\alpha; k; \gamma, \eta) = \left\{ f \in A_p : Q_{\beta,p}^\alpha f(z) \in UK_p^* (k; \gamma, \eta) ; z \in U \right\}. \] (20)

We also note that

\[
f \in US_p^\alpha (\alpha; k; \gamma) \Leftrightarrow \frac{zf'}{p} \in UC_p (\alpha; k; \gamma), \] (21)

and

\[
f \in UK_p (\alpha; k; \gamma, \eta) \Leftrightarrow \frac{zf'}{p} \in UK_p^* (\alpha; k; \gamma, \eta). \] (22)

In this paper, we investigate several inclusion properties of the classes \( US_p^\alpha (\alpha; k; \gamma), \) \( UC_p (\alpha; k; \gamma), \) \( UK_p (\alpha; k; \gamma, \eta), \) and \( UK_p^* (\alpha; k; \gamma, \eta) \) associated with the operator \( Q_{\beta,p}^\alpha \). Some applications involving integral operators are also considered.
2. Inclusion properties involving the operator $Q_{\beta,p}^\alpha$  

In order to prove the main results, we shall need the following lemmas.

**Lemma 1** [17]. Let $h(z)$ be convex univalent in $\mathbb{U}$ with $\Re \{h(z) + \gamma\} > 0 \ (\eta, \gamma \in \mathbb{C})$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = h(0)$, then

\[
p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} \prec h(z)
\]

implies

\[
p(z) \prec h(z).
\]

**Lemma 2** [1]. Let $h(z)$ be convex univalent in $\mathbb{U}$ and let $w$ be analytic in $\mathbb{U}$ with $\Re \{w(z)\} \geq 0$. If $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = h(0)$, then

\[
p(z) + w(z)p'(z) \prec h(z)
\]

implies

\[
p(z) \prec h(z).
\]

**Theorem 1.** Let $(\alpha + \beta)(k + 1) + kp + \gamma > 0$. Then,

\[
US_p^* (\alpha; k; \gamma) \subset US_p^* (\alpha + 1; k; \gamma).
\]

**Proof.** Let $f \in US_p^* (\alpha; k; \gamma)$ and set

\[
p(z) = \frac{z \left( Q_{\beta,p}^{\alpha+1} f(z) \right)'}{Q_{\beta,p}^{\alpha+1} f(z)} \quad (z \in \mathbb{U}),
\]

where the function $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = p$. Using (15), (27) and (28), we have

\[
\frac{z \left( Q_{\beta,p}^{\alpha} f(z) \right)'}{Q_{\beta,p}^{\alpha} f(z)} = p(z) + \frac{zp'(z)}{p(z) + \alpha + \beta} \prec q_{p,k,\gamma}(z).
\]

Since $(\alpha + \beta)(k + 1) + kp + \gamma > 0$, we see that

\[
\Re \{q_{p,k,\gamma}(z) + \alpha + \beta\} > 0 \quad (z \in \mathbb{U}).
\]

Applying Lemma 1 to (29), it follows that $p(z) \prec q_{p,k,\gamma}(z)$, that is, $f \in US_p^* (\alpha + 1; k; \gamma)$. Therefore, we complete the proof of Theorem 1. \hfill \Box

**Theorem 2.** Let $(\alpha + \beta)(k + 1) + kp + \gamma > 0$. Then,

\[
UC_p (\alpha; k; \gamma) \subset UC_p (\alpha + 1; k; \gamma).
\]

**Proof.** Applying (21) and Theorem 1, we observe that

\[
f \in UC_p (\alpha; k; \gamma) \iff \frac{zf'}{p} \in US_p^* (\alpha; k; \gamma)
\]

\[
\iff \frac{zf'}{p} \in US_p^* (\alpha + 1; k; \gamma) \quad \text{(by Theorem 1)},
\]

which evidently proves Theorem 2. \hfill \Box

Next, by using Lemma 2, we obtain the following inclusion relation for $UK_p (\alpha; k; \gamma, \eta)$.

**Theorem 3.** Let $(\alpha + \beta)(k + 1) + kp + \eta > 0$. Then,

\[
UK_p (\alpha; k; \gamma, \eta) \subset UK_p (\alpha + 1; k; \gamma, \eta).
\]
Proof. Let \( f \in UK_p (\alpha; k; \gamma, \eta) \). Then, from the definition of \( UK_p (\alpha; k; \gamma, \eta) \), there exists a function \( r (z) \in US_p^* (k; \eta) \) such that
\[
\frac{z \left( Q_{\beta,p}^\alpha f (z) \right)'}{r (z)} \prec q_{p,k,\gamma} (z). \tag{33}
\]
Choose the function \( g \) such that \( Q_{\beta,p}^\alpha g (z) = r (z) \). Then, \( g \in US_p^* (\alpha; k; \eta) \) and
\[
\frac{z \left( Q_{\beta,p}^\alpha f (z) \right)'}{Q_{\beta,p}^\alpha g (z)} \prec q_{p,k,\gamma} (z). \tag{34}
\]
Now let
\[
p(z) = \frac{z \left( Q_{\beta,p}^{\alpha+1} f (z) \right)'}{Q_{\beta,p}^{\alpha+1} g (z)} \quad (z \in \mathbb{U}), \tag{35}
\]
where \( p (z) \) is analytic in \( \mathbb{U} \) with \( p (0) = p \). Since \( g \in US_p^* (\alpha; k; \eta) \), by Theorem 1, we know that \( g \in US_p^* (\alpha + 1; k; \eta) \). Let
\[
t(z) = \frac{z \left( Q_{\beta,p}^{\alpha+1} g (z) \right)'}{Q_{\beta,p}^{\alpha+1} g (z)} \quad (z \in \mathbb{U}), \tag{36}
\]
where \( t (z) \) is analytic in \( \mathbb{U} \) with \( \Re \{ t (z) \} > \frac{kp + \eta}{k + 1} \). Also, from (35), we note that
\[
Q_{\beta,p}^{\alpha+1} z f' (z) = Q_{\beta,p}^{\alpha+1} g (z) p (z). \tag{37}
\]
Differentiating both sides of (37) with respect to \( z \), we obtain
\[
\frac{z \left( Q_{\beta,p}^{\alpha+1} z f' (z) \right)'}{Q_{\beta,p}^{\alpha+1} g (z)} = \frac{z \left( Q_{\beta,p}^{\alpha+1} g (z) \right)'}{Q_{\beta,p}^{\alpha+1} g (z)} p (z) + z p' (z)
= t (z) p (z) + z p' (z). \tag{38}
\]
Now using the identity (15) and (36), we obtain
\[
\frac{z \left( Q_{\beta,p}^\alpha f (z) \right)'}{Q_{\beta,p}^\alpha g (z)} = \frac{Q_{\beta,p}^\alpha z f' (z)}{Q_{\beta,p}^\alpha g (z)} = \frac{z \left( Q_{\beta,p}^{\alpha+1} z f' (z) \right)'}{z \left( Q_{\beta,p}^{\alpha+1} g (z) \right)'} + (\alpha + \beta) \frac{Q_{\beta,p}^{\alpha+1} z f' (z)}{Q_{\beta,p}^{\alpha+1} g (z)}
= \frac{z \left( Q_{\beta,p}^{\alpha+1} z f' (z) \right)'}{Q_{\beta,p}^{\alpha+1} g (z)} + (\alpha + \beta) \frac{z \left( Q_{\beta,p}^{\alpha+1} f (z) \right)'}{Q_{\beta,p}^{\alpha+1} g (z)}
= \frac{z \left( Q_{\beta,p}^{\alpha+1} g (z) \right)'}{Q_{\beta,p}^{\alpha+1} g (z)} + \alpha + \beta
= t (z) p (z) + z p' (z) + (\alpha + \beta) p (z)
= \frac{t (z) p (z) + z p' (z)}{t (z) + \alpha + \beta}.
\]
Since \((\alpha + \beta)(k+1) + kp + \gamma > 0\) and \(\Re \{t(z)\} > \frac{kp + \eta}{k+1}\), we see that \\
\[\Re \{t(z) + \alpha + \beta\} > 0 \quad (z \in \mathbb{U}).\]

Hence, applying Lemma 2, we can show that \(p(z) < q_{p,k,\gamma}(z)\) so that \(f \in UK_p(\alpha; k; \gamma, \eta)\). Therefore, we complete the proof of Theorem 3.

**Theorem 4.** Let \((\alpha + \beta)(k+1) + kp + \eta > 0\). Then,
\[UK^*_p(\alpha; k; \gamma, \eta) \subset UK^*_{p+1}(\alpha + 1; k; \gamma, \eta).\]  

**Proof.** Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (21), we can also prove Theorem 4 by using Theorem 3 and the equivalence (22).

3. **Inclusion properties involving the integral operator \(F_{c,p}\)**

In this section, we consider the generalized Libera integral operator \(F_{c,p}\) defined by (16).

**Theorem 5.** Let \(c > -p\) and \(0 \leq \gamma < p\). If \(f \in US^*_p(\alpha; k; \gamma)\), then \(F_{c,p}(f) \in US^*_p(\alpha; k; \gamma)\).

**Proof.** Let \(f \in US^*_p(\alpha; k; \gamma)\) and set
\[p(z) = z \left(\frac{Q^\alpha_{\beta,p} F_{c,p}(f)(z)}{Q^\alpha_{\beta,p} F_{c,p}(f)(z)}\right)' \quad (z \in \mathbb{U}),\]  
where \(p(z)\) is analytic in \(\mathbb{U}\) with \(p(0) = p\). From (16), we have
\[z \left(\frac{Q^\alpha_{\beta,p} F_{c,p}(f)(z)}{Q^\alpha_{\beta,p} F_{c,p}(f)(z)}\right)' = (c + p) Q^\alpha_{\beta,p} f(z) - c Q^\alpha_{\beta,p} F_{c,p}(f)(z).\]  

Then, by using (40) and (41), we obtain
\[(c + p) \frac{Q^\alpha_{\beta,p} f(z)}{Q^\alpha_{\beta,p} F_{c}(f)(z)} = p(z) + c.\]  

Taking the logarithmic differentiation on both sides of (42) and multiplying by \(z\), we have
\[z \left(\frac{Q^\alpha_{\beta,p} f(z)}{Q^\alpha_{\beta,p} F_{c}(f)(z)}\right)' = p(z) + \frac{zp'(z)}{p(z)} < q_{k,\gamma}(z) \quad (z \in \mathbb{U}).\]

Hence, by virtue of Lemma 1, we conclude that \(p(z) < q_{k,\gamma}(z)\) in \(\mathbb{U}\), which implies that \(F_{c,p}(f) \in US^*_p(\alpha; k; \gamma)\). \(\square\)

Next, we derive an inclusion property involving \(F_{c,p}(f)\), which is given by the following.

**Theorem 6.** Let \(c > -p\) and \(0 \leq \gamma < p\). If \(f \in UC_p(\alpha; k; \gamma)\), then \(F_{c,p}(f) \in UC_p(\alpha; k; \gamma)\).
Proof. By applying Theorem 5, it follows that

\[
f \in UC_p (\alpha; k; \gamma) \iff \frac{zf'}{p} \in US_p^* (\alpha; k; \gamma)
\]

\[
\implies F_{c,p} \left( \frac{zf'}{p} \right) \in US_p^* (\alpha; k; \gamma) \quad \text{(by Theorem 5)}
\]

\[
\iff \left( F_{c,p} (f) \right)' \in US_p^* (\alpha; k; \gamma)
\]

\[
\iff F_{c,p} (f) \in UC_p (\alpha; k; \gamma),
\]

which proves Theorem 6. \qed

Theorem 7. Let \( c > -p \) and \( 0 \leq \gamma, \eta < p \). If \( f \in UK_p (\alpha; k; \gamma, \eta) \), then \( F_{c,p} (f) \in UK_p (\alpha; k; \gamma, \eta) \).

Proof. Let \( f \in UK_p (\alpha; k; \gamma, \eta) \). Then, in view of the definition of the class \( UK_p (\alpha; k; \gamma, \eta) \), there exists a function \( g \in US_p^* (\alpha; k; \eta) \) such that

\[
\frac{z \left( Q_{\beta,p} f(z) \right)'}{Q_{\beta,p} (z)} < q_{k,\gamma}(z).
\]

Thus, we set

\[
p(z) = \frac{z \left( Q_{\beta,p} f(z) \right)'}{Q_{\beta,p} (z)} \quad (z \in \mathbb{U}),
\]

where \( p(z) \) is analytic in \( \mathbb{U} \) with \( p(0) = p \). Since \( g \in US_p^* (\alpha; k; \gamma) \), we see from Theorem 5 that \( F_{c,p} (f) \in US_p^* (\alpha; k; \gamma) \). Let

\[
t(z) = \frac{z \left( Q_{\beta,p} g(z) \right)'}{Q_{\beta,p} (z)} \quad (z \in \mathbb{U}),
\]

where \( t(z) \) is analytic in \( \mathbb{U} \) with \( \Re \{ t(z) \} > \frac{k\eta + \eta}{k + 1} \). Also, from (46), we note that

\[
Q_{\beta,p} z F'_{c,p} (f) (z) = Q_{\beta,p} F_{c,p} (g) (z) \cdot p (z).
\]

Differentiating both sides of (48) with respect to \( z \), we obtain

\[
\frac{z \left( Q_{\beta,p} z F'_{c,p} (f) (z) \right)'}{Q_{\beta,p} F_{c,p} (g) (z)} = \frac{z \left( Q_{\beta,p} F_{c,p} (g) (z) \right)'}{Q_{\beta,p} F_{c,p} (g) (z)} p (z) + z p' (z)
\]

\[
= t(z) p(z) + zp'(z).
\]
Now using the identity (41) and (49), we obtain
\[
\frac{z \left( Q^\alpha_{\beta,p} f (z) \right)'}{Q^\alpha_{\beta,p} g (z)} = \frac{z \left( Q^\alpha_{\beta,p} z^F'_{c,p} (f) (z) \right)'}{z \left( Q^\alpha_{\beta,p} F_{c,p} (g) (z) \right)} + cQ^\alpha_{\beta,p} z^F'_{c,p} (f) (z)
\]
\[
= \frac{z \left( Q^\alpha_{\beta,p} F_{c,p} (g) (z) \right)'}{Q^\alpha_{\beta,p} F_{c,p} (g) (z)} + c
\]
\[
= \frac{t (z) p (z) + z p' (z) + c p (z)}{t (z) + c}
\]
\[
= p (z) + \frac{z p' (z) + c p (z)}{t (z) + c}.
\]  

Since \( c(k + 1) + kp + \eta \geq 0 \) and \( \Re \{ t (z) \} > \frac{kp + \eta}{k + 1} \), we see that
\[
\Re \{ t (z) + c \} > 0 \quad (z \in \mathbb{U}).
\]  

Hence, applying Lemma 2 to (50), we can show that \( p (z) \prec q_{p,k,\gamma} (z) \) so that
\[
f \in UK^*_p (\alpha; k; \gamma, \eta).
\]

**Theorem 8.** Let \( c > -p \) and \( 0 \leq \gamma, \eta < p \). If \( f \in UK^*_p (\alpha; k; \gamma, \eta) \), then \( F_{c,p} (f) \) \( UK^*_p (\alpha; k; \gamma, \eta) \).

**Proof.** Just as we derived Theorem 6 as consequence of Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7. \qed

**References**


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