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SOME COMMON FIXED POINT THEOREM IN METRIC SPACES OF FISHER AND SESSA

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Abstract. In this paper we have proved for two mappings that they have a unique common fixed point on a compact subset of a metric space. Where one of the mapping is non-expansive and the pair of mappings is weakly commuting. Fisher and Sessa [5] proved the same problem with a closed subset. We replaces the main results by change the closed subsets to compact subsets. Baboli and Ghaemi [1] proved the same type of theorem. We have extended the results of Baboli and Ghaemi [1]

1. Introduction

In 1922, Banach first introduced the concept of fixed point theorem in his Ph.D. thesis. He introduced the notion of contraction mapping, which is known as Banach contraction Principle. Later on Schauder proved a fixed point theorem as: If A is a compact, convex subset of a Banach space X and $T : A \to A$ is a continuous function, then T has a fixed point. The compactness condition on A is a stronger condition and most of the problems in analysis do not have compact setting.
In 1986, Fisher and Sessa [7] proved a fixed point theorem for two self maps on a subset of a Banach Space which is closed convex. Sessa [7] further generalized a results of Das and Naik [2]. In 2015, Baboli and Ghaemi [1] generalized the results of Fisher and Sessa [5].

2. Preliminaries

Schauder further proved the following theorem without using the notion of compactness as:
Theorem 2.1. If A is a closed bounded convex subset of a Banach space X and $T : A \to A$ is a continuous map such that T(A) is a compact, then T has a fixed point.

In this paper we generalize the results of Baboli and Ghaemi [1].

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Definition 2.2. (Weakly Commuting) Two maps T and R on a metric space (X,d) into itself are weakly commuting iff

$$d(TRx, RTx) \leq d(Rx, Tx) \tag{1}$$

for all x in X.

Definition 2.3. (Non-expansive mapping) A self map T on a metric space X is said to be non-expansive if $d(Tx, Ty) \leq d(x, y)$ for all x, y in X. Two commuting maps clearly satisfy (1) but the converse is not true as is shown in the following example.

Example 2.4. [**1**] Let $X = [0, 1]$, and suppose X is endowed with the Euclidean metric. Define T and R by $Tx = \frac{x}{x+4}$ and $Rx = \frac{x}{2}$ for any x in X. Then

$$d(TRx, RTx) = \frac{x}{x+8} - \frac{x}{2x+8} = \frac{x^2}{2(x+8)(x+4)} \leq \frac{x^2+2x}{2(x+4)} = \frac{x}{2} - \frac{x}{x+4} = d(Rx, Tx)$$

But, for any $x \neq 0$, TRx $=\frac{x}{x+8} > \frac{x}{2x+8}=$ RTx.

Fisher and Sessa proved the following theorem:

Theorem 2.5. [**1**], [5] Let, T and I be two weakly commuting mappings from C into itself satisfying the inequality

$$d(Tx, Ty) \leq ad(Ix, Iy) + (1 - a)max[d(Tx, Ix), d(Ty, Iy)] \tag{2}$$

for all x, y in C where 0<a<1 and C is a closed convex subset of a Banach Space X. If I is linear and non-expansive on C and further I(C) contains T(C), then T and I have a unique common fixed point in C.

3. Main results

Our theorem runs as follows:

Theorem 3.1. Let T and I be two weakly commuting self maps on C satisfying the condition:

$$d(Tx, Ty) \leq ad(Tx, Ix)+bd(Ty, Iy)+cd(Ix, Iy)+(1-a-b-c)max\{d(Tx, Ix), d(Ty, Iy)\} \tag{3}$$

a,b,c>0, 0<a+b+c<1, where C is a compact subset of the Metric Space X.
If I is non-expansive on C and I(C) contains T(C), then T and I have a unique Common fixed point in C.

Proof: Let, x=x_0 be an arbitrary point in C and for any n∈ N choose x_{n+1} such that Tx_n=Ix_{n+1}.
Since C is compact so {x_n} has a convergence subsequence {y_k},$k = 1, 2, \dots \infty$.
Also, let it converges to x^* for some x^*∈ C.
In the following we show each y_k with ${y_n}^k$ where it represents k-th element of {y_n} and n-th element of {x_n}.
Now, we will show that d(Tx^*,Ix^*)=0.
Now, d(Tx^*,Ix^*)
$\leq$ d(Tx^*,T${y_n}^k$)+d(T${y_n}^k$, I${y_n}^k$)+ d(I${y_n}^k$,Ix^*)
$\leq$ ad(Tx^*,Ix^*) + bd(T${y_n}^k$,I${y_n}^k$) +c d(Ix^*,I${y_n}^k$)

+(1-a-b-c) max{d(Tx*,Ix*), d(Ty$_n{}^k$,Iy$_n{}^k$)}+ d(Ty$_n{}^k$,Iy$_n{}^k$)+ d(Iy$_n{}^k$,Ix*); (4)

Case I: If d(Tx*,Ix*) $>$ d(Ty$_n{}^k$,Iy$_n{}^k$), from (4) we get,
(1-a-1+a+b+c)d(Tx*,Ix*) $\leq$ (b+1) d(Ty$_n{}^k$,Iy$_n{}^k$)+(c+1) d(Ix*,Iy$_n{}^k$)
Or, (b+c) d(Tx*,Ix*) $\leq$ (b+1){d(Ty$_n{}^k$,Ix$_{n+1}$)+d(Ix$_{n+1,}$,Iy$_n{}^k$)}+(c+1)d(Ix*,Iy$_n{}^k$)
$\leq$ (b+1) {d(Ty$_n{}^k$,Tx$_n$)+d(x$_{n+1,}$y$_n{}^k$)}+(c+1)d(x*,y$_n{}^k$) (as I is non-expansive mapping)
Taking $\lim_{n\to\infty}$ on both sides of the above inequality we get,
(b+c) d(Tx*,Ix*) $\leq$ (b+1) {d(x*, x*)+d(x*, x*)}+(c+1) d(x*,x*)
i.e., d(Tx*,Ix*) = 0, as b+c $\neq$ 0. (5)

Case II: If d(Ty$_n{}^k$,Iy$_n{}^k$)$>$ d(Tx*,Ix*), from (4) we get,
(1-a)d(Tx*,Ix*) $\leq$ (b+1-a-b-c+1) d(Ty$_n{}^k$,Iy$_n{}^k$)+(c+1)d(Ix*,Iy$_n{}^k$)
Or, (1-a)d(Tx*,Ix*)
$\leq$ (2-a-c){d(Ty$_n{}^k$,Ix$_{n+1}$)+d(Ix$_{n+1,}$,Iy$_n{}^k$)}+(c+1)d(Ix*,Iy$_n{}^k$)
$\leq$ (2-a-c){d(Ty$_n{}^k$,Tx$_n$)+d(x$_{n+1,}$y$_n{}^k$)}+(c+1)d(x*,y$_n{}^k$) (as I is non-expansive mapping)
Taking $\lim_{n\to\infty}$ on both sides of the above inequality we get,
(1-a)d(Tx*,Ix*)
$\leq$ (2-a-c){d(x*, x*)+d(x*, x*)}+(c+1)d(x*,x*)
So, d(Tx*,Ix*) = 0 as 1-a $\neq$ 0. (6)

So from (5) and (6) we get d(Tx*,Ix*) = 0.
Set, K$_n$={x $\in$ C: d(Tx,Ix) $\leq \frac{1}{n}$} and H$_n$={x$\in$ C: d(Tx,Ix) $\leq \frac{2-a-b}{(1-a)n}$.}
Clearly, for each n, K$_n \neq \theta$ and K$_1 \supseteq$ K$_2 \supseteq \ldots \supseteq$K$_n \supseteq \ldots$
Thus each of the set $\overline{TK_n}$, where $\overline{TK_n}$ denotes the closure of TK$_n$, must be nonempty, for arbitrary x, y $\in$ K$_n$.
d(Tx,Ty)$\leq$ ad(Tx,Ix) + bd(Ty,Iy) + c d(Ix,Iy)+(1-a-b-c) max{d(Tx,Ix),d(Ty,Iy)}
Then we have,
$\leq$ a. $\frac{1}{n}$ + b.$\frac{1}{n}$ + c {d(Ix,Tx)+d(Tx,Ty)+d(Ty,Iy)}+(1-a-b-c)$\frac{1}{n}$
$\leq \frac{1}{n}$ {a+b+1-a-b-c} +c {$\frac{1}{n}+\frac{1}{n}$+d(Tx,Ty)}
Or, d(Tx,Ty)$\leq$ (1-c) $\frac{1}{n}$ +$\frac{2}{n}$.c+c.d(Tx,Ty)
Or, d(Tx,Ty) $\leq \frac{1}{(1-c)n}${1-c+2c} =$\frac{1+c}{(1-c).n}$
Thus, $\lim_{n\to\infty}$ diam(TK$_n$)= $\lim_{n\to\infty}$ diam($\overline{TK_n}$)=0.
From the result of Cantor, it follows that $\cap_{n=1}{}^{\infty}\overline{TK_n}$ contains exactly one point w.
Now, let y be an arbitrary in $\overline{TK_n}$.
Then for arbitrary ϵ >0 there is a point y$^/$ in K$_n$ such that d(Ty$^/$,y)< ϵ (7)

Using the weak commutativity of T and I, non-expansiveness of I and applying on (1),(3) and (7) we get,
d(Ty,Iy) $\leq$ d(Ty,TIy$^/$) +d(TIy$^/$, ITy$^/$)+ d(ITy$^/$,Iy)
$\leq$ ad(Ty,Iy) + bd(TIy$^/$,IIy$^/$) +c d(Iy,IIy$^/$)+(1-a-b-c)max{ d(Ty,Iy), d(TIy$^/$,IIy$^/$)}
+d(Iy$^/$, Ty$^/$)+ d(Ty$^/$,y)
Or, (1-a)d(Ty,Iy)$\leq$

bd(Iy$^{/}$,Iy$^{/}$) +c d(y,Iy$^{/}$)+(1-a-b-c)max{d(Ty,Iy),d(Iy$^{/}$,Iy$^{/}$)} +d(Iy$^{/}$,Ty$^{/}$)+ d(Ty$^{/}$,y)
$\leq$ b.0+c.{d(y,Ty$^{/}$)+d(Ty$^{/}$,Iy$^{/}$)}+(1-a-b-c)max{$\frac{1}{n}$,0}+ $\frac{1}{n}$ + ϵ.
Or, (1-a)d(Ty,Iy)$\leq$ c.{$\frac{1}{n}$ +ϵ}+(1-a-b-c)max{$\frac{1}{n}$,0}+ $\frac{1}{n}$ +ϵ
Since, ϵ is arbitrary it follows that
(1-a)d(Ty,Iy)
$\leq$ c.$\frac{1}{n}$ + (1-a-b-c)$\frac{1}{n}$ + $\frac{1}{n}$
Or, d(Ty,Iy)$\leq \frac{(2-a-b)}{(1-a).n} \in$ H$_{n.}$
So, y lies in H$_{n..}$
Thus, $\overline{TK_n} \subseteq$ H$_{n.}$and so the point w must be in H$_{n.}$for n=1,2,3,...
So, d(Tw,Iw)$\leq \frac{(2-a-b)}{(1-a).n}$ for n=1,2,3,...
And so,

$$Tw = Iw. \tag{8}$$

Since, T and I are weakly compatible so,

$$ITw = TIw = T^2w. \tag{9}$$

Thus, d(T^{2}w,Tw)=d(TTw,Tw)
$\leq$ ad(TTw,ITw) + bd(Tw,Iw) +c d(ITw,Iw)+(1-a-b-c) max{d(TTw,ITw),d(Tw,Iw)}
= a.0+b.0+c. d(T^{2}w,Tw)+(1-a-b-c).0 (by (8) &(9)).
Or, (1-c)d(T^{2}w,Tw)$\leq$ 0.
Or, d(T^{2}w,Tw)=0 as 1-c$\neq$0.
So,

$$T^2w = Tw. \tag{10}$$

And Tw=w$^{/}$ is a fixed point of T as Tw$^{/}$= w$^{/}$(from (10)),for 0<a+b+c<1.

(11)

Further Iw$^{/}$=ITw$^{/}$=TIw$^{/}$=TTw$^{/}$=Tw$^{/}$= w$^{/}$ (by(8),(9)) &(10)).
So,w$^{/}$ is also a fixed point of I.

(12)

To prove uniqueness, we let if possible there exists another fixed point z, such as

$$Tz = Iz = z. \tag{13}$$

Then,
d(w$^{/}$,z)=d(Tw$^{/}$,Tz)$\leq$ ad(Tw$^{/}$,Iw$^{/}$) + bd(Tz,Iz) +c d(Iw$^{/}$,Iw)+(1-a-b-c) max{d(Tw$^{/}$,Iw$^{/}$),d(Tz,Iz)}
= ad(w$^{/}$,w$^{/}$) + bd(z,z) +c d(w$^{/}$,z)+(1-a-b-c)max{d(w$^{/}$,w$^{/}$), d(z,z)} (by (11),(12)&(13)).
= a.0 + b.0 +c.d(w$^{/}$,z)+ (1-a-b-c).0
Or, (1-c)d(w$^{/}$,z)$\leq$ 0,
So, d(w$^{/}$,z)=0 as 1-c $\neq$ 0.
So, w$^{/}$=z.
So, the common fixed point of T and I is unique.

Corollary 3.2. Let T and I be two weakly commuting self maps on C satisfying the condition:
d(Tx,Ty)$\leq$ bd(Ty,Iy) +c d(Ix,Iy)+(1-b-c) max{d(Tx,Ix), d(Ty,Iy)}; b,c>0, 0<b+c<1,
where C is a compact subset of the Metric Space X.
If I is non-expansive on C and I(C) contains T(C), then T and I have a unique Common fixed point in C.

Proof: Put a=0 in the main theorem and get the results.

Corollary 3.3. Let T and I be two weakly commuting self maps on C satisfying the condition:
$d(Tx,Ty)\leq ad(Tx,Ix) + c\, d(Ix,Iy)+(1-a-c) \max\{d(Tx,Ix), d(Ty,Iy)\}$; $a,c>0$, $0<a+c<1$, where C is a compact subset of the Metric Space X.
If I is non-expansive on C and I(C) contains T(C), then T and I have a unique Common fixed point in C.

Proof: Put b=0 in the main theorem, and get the result.

Corollary 3.4. Let T and I be two weakly commuting self maps on C satisfying the condition:

$d(Tx,Ty)\leq ad(Tx,Ix) + bd(Ty,Iy)+(1-a-b) \max\{d(Tx,Ix), d(Ty,Iy)\}$; $a,b>0$, $0<a+b<1$, where C is a compact subset of the Metric Space X.
If I is non-expansive on C and I(C) contains T(C), yhen T and I have a unique Common fixed point in C.

Proof: put c=0 in the main theorem and get the result.

Corollary 3.5. Let T and I be two weakly commuting self maps on C satisfying the condition:

$d(Tx,Ty)\leq c\, d(Ix,Iy)+(1-c) \max\{d(Tx,Ix), d(Ty,Iy)\}$;
$0<c<1$, where C is a compact subset of the Metric Space X.
If I is non-expansive on C and I(C) contains T(C), then T and I have a unique Common fixed point in C.

Proof: Put a=b=0 in the main theorem and get the result.

Corollary 3.6. Let T and I be two weakly commuting self maps on C satisfying the condition:
$d(Tx,Ty) \leq ad(Tx,Ix) +(1-a) \max\{d(Tx,Ix), d(Ty,Iy)\}$; $0<a<1$, where C is a compact subset of the Metric Space X.
If I is non-expansive on C and I(C) contains T(C), then T and I have a unique Common fixed point in C.

Proof: Put b=c=0 in the main theorem and get the result.

Corollary 3.7. Let T and I be two weakly commuting self maps on C satisfying the condition:

$d(Tx,Ty)\leq bd(Ty,Iy) +(1-b) \max\{d(Tx,Ix),d(Ty,Iy)\}$;

$0<b<1$, where C is a compact subset of the Metric Space X.
If I is non-expansive on C and I(C) contains T(C), then T and I have a unique

Common fixed point in C.

Proof: Put a=c=0 in the main theorem and get the result.

4. Conclusion

If we put a=b=0, and c=a in our main theorem then we get the result of Baboli and Ghaemi [1]. Our main result is a generalization of many existing results in this literature.

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