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A QUASISTATIC ANTIPLANE THERMAL CONTACT PROBLEM

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Abstract. This paper is devoted to the study of a mechanical problem modelling the antiplane shear deformation of a cylindrical body in frictionnal contact with a rigid obstacl the so-called foundation. The material is assumed to be thermo-viscoelastic, and the friction is modeled with Tresca's law. We derive a variational formulation of the model whitch is in the form of a system coupling a variational elliptic equality for the displacements and a differential heat equation for the temperature. A weak formulation is presented where the existence and uniqueness result estabilished by using general results on evolution equations with monotone operator and fixed point arguments.

1. Introduction

The contact between a deformable body and a foundation is a very frequent and important phenomenon, in industry and in our daily life. For this reason the literature covering this phenomenon is extensive, both in applied mathematics and in engineering or geophysics. The mathematical theory of contact mechanics is concerned with the mathematical structures which underline general problems of contact with different constitutive laws and different contact conditions (see, e.g.[6]). A number of recent publications is dedicated to the study of quasistatic contact models involving thermoviscoelastic materials.

Antiplane shear deformation problems arise naturaly from many real world applications such as rectilinear steady flow of simple fluids, interface stress effects of nanostructured materials, structures with cracks, layered composite functioning materials and phase transitions in solids. In antiplane shear of cylindrical body, the displacement is parallel to generators of cylinder and is dependent of the axial coordinate. The model of a thermo-viscoelastic body is very complex, in addition to elastic and temperature properties, it takes into account viscous characteristics, see [1, 2, 3, 6].

The theory of variational inequalities has not been an exception. Indeed, the cross fertilization between modeling and applications on the one hand and nonlinear mathematical analysis on the other hand was an important aspect which contributed to its development in the last four decades. Currently, the theory of

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variational inequalities became a fully mature discipline which deals with existence, uniqueness or nonuniqueness, regularity and continuous depend results, together with numerical approximations and optimal control of the solutions. It provides results which are of considerable theoretical and applied interest.

The aim of this paper is to recall the attention to the great potential of inequalities in mechanics and physics. In the spirit of the classical book of G. Duvaut and J. L. Lions (see, e.g.[7]), we show how a concrete viscoelastic contact problem leads to a mathematical model which can be solved by using methods of variational inequalities theory.

In this paper we study the frictional contact between a deformable cylinder and a rigid foundation. We consider the case of antiplane shear deformation i.e., the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. Such kind of problems were studied in a number of papers, in the context of various constitutive laws and contact conditions (see, e.g.[4,9-13]).

The novelty in our work consists in the fact that we model the friction with Tresca's law and the material's behavior with a thermoviscoelastic constitutive law.

The paper is organized as follows. In Sect.2, we describe the mechanical problem, specify the assumptions on the data to derive the variational formulation, and then we state our main existence and uniqueness result. In Sect.3, we give the proof of the claimed result.

2. The variational formulation

We consider a body B identified with a region in $\mathbb{R}^3$, it occupies in a fixed and undistorted reference configuration. We assume that B is a cylinder with generators parallel to the x_3-axis with a cross-section which is a regular region in the x_2x_3-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that end effects in the axial direction are negligible. Thus, $B = \Omega \times \ (-\infty, +\infty)$. Let $\partial\Omega = \Gamma$ we assume that Γ is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that the one-dimensional measure of Γ_1, denoted $\operatorname{mes}\Gamma_1$, is strictly positive. Let $T > 0$ and let $[0;T]$ denote the time interval of interest. The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and is in contact with a rigid foundation on $\Gamma_3 \times (-\infty, +\infty)$ during the process. Moreover, the cylinder is subjected to time-dependent volume forces of density f_0 on B and to time-dependent surface tractions of density f_2 on $\Gamma_2 \times (-\infty, +\infty)$.

We assume that

$$f_0 = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2, t) : \Omega \times [0;T] \to \mathbb{R}, \tag{1}$$

$$f_2 = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2, t) : \Gamma \times [0;T] \to \mathbb{R}. \tag{2}$$

The body forces (1) and the surface tractions (2) would be expected to give rise to a deformation of the cylinder whose displacement, denoted by u, is independent of x_3 and has the form

$$u = (0, 0, u) \quad \text{with} \quad u = u(x_1, x_2, t) : \Omega \times [0;T] \to \mathbb{R}. \tag{3}$$

Such kind of deformation is called an antiplane shear.

The infnitesimal strain tensor is denoted by $\varepsilon(u) = \varepsilon_{ij}(u)$ and the stress field by $\sigma = (\sigma_{ij})$. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2 or t.

The material is modeled by the following thermal viscoelastic constitutive law

$$\sigma = \lambda(tr\varepsilon(u))I + 2\mu\varepsilon(u) - M_e\theta, \tag{4}$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé coefficents, $tr(\varepsilon(u)) = \sum_{i=1}^{3} \varepsilon_{ii}(u)$, I is the unit tensor in $\mathbb{R}^3$, θ is the temperature field and $M_e := (m_{ij})$ represents the thermal expansion tensor and has the form

$$M_e = \begin{pmatrix} 0 & 0 & \mathcal{M}_{e_1} \\ 0 & 0 & \mathcal{M}_{e_2} \\ \mathcal{M}_{e_1} & \mathcal{M}_{e_2} & 0 \end{pmatrix}.$$

We assume that $\quad \mathcal{M}_{e_i}(x_1, x_2) : \Omega \to \mathbb{R}$.

In the antiplane context (3), keeping in mind (4), the stress field becomes

$$\sigma = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}, \tag{5}$$

where

$$\sigma_{13} = \sigma_{31} = \mu\partial_{x_1}u - \mathcal{M}_{e_1}.\theta, \quad \sigma_{23} = \sigma_{32} = \mu\partial_{x_2}u - \mathcal{M}_{e_2}.\theta.$$

Neglecting the inertial term in the equation of motion we obtain the quasistatic approximation for the process. Thus, taking in to account (5), (1) and the previous equalities, the equation of equilibrium reduces to the following scalar equation

$$\mu\Delta u + f_0 - \text{div}\theta\,\mathcal{M}_e = 0 \quad \text{in } \Omega \times (0, T),$$

whith

$$\mathcal{M}_e = \begin{pmatrix} \mathcal{M}_{e_1} \\ \mathcal{M}_{e_2} \\ 0 \end{pmatrix}.$$

As the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty) \times (0, T)$, the displacement field vanishes there. Thus, (3) implies

$$u = 0 \qquad \text{on } \Gamma_1 \times (0, T).$$

Let ν denote the unit normal on $\Gamma \times (-\infty, +\infty)$. We have

$$\nu = (\nu_1, \nu_2, 0) \quad \text{with} \quad \nu_i = \nu_i(x_1, x_2) : \Gamma \to \mathbb{R},\ i = 1, 2. \tag{6}$$

For a vector v, we denote by v_ν and v_τ its normal and tangential components on the boundary, given by

$$v_\nu = v.\nu \ , \ v_\tau = v - v_\nu\nu. \tag{7}$$

In (7) and every where in this paper " . " represents the inner product on the space $\mathbb{R}^d$, $d = 2$ or 3. Moreover, throughout this paper the notation $|.|$ will designate the Euclidean norm on $\mathbb{R}^d$, and a dot above a function will represent its derivative with respect to the time variable. For a given stress field σ we denote by σ_ν and σ_τ the normal and the tangential components on the boundary, that is

$$\sigma_\nu = (\sigma\nu).\nu \qquad \sigma_\tau = \sigma\nu - \sigma_\nu\nu. \tag{8}$$

From (5) and (6), we deduce that the Cauchy stress vector is given by

$$\sigma\nu = (0, 0, \mu\partial_\nu u - \theta\mathcal{M}_e.\nu). \tag{9}$$

From now we use the notation $\partial_\nu u = \partial_{x_1}u\nu_1 + \partial_{x_2}u\nu_2$.

Taking into account the traction boundary condition, $\sigma\nu = f_2$ on $\Gamma_2 \times (0,T)$, it follows from (2) and (9) that

$$\mu\partial_\nu u - \theta\mathcal{M}_e.\nu = f_2 \qquad \text{on } \Gamma_2 \times (0,T).$$

Now, we describe the contact condition on $B = \Omega \times (-\infty, +\infty)$. First, from (3) and (6) we infer that $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during all the process. Using now (3), (6)-(8), we conclude that

$$u_\tau = (0,0,u), \qquad \sigma_\tau = (0,0,\mu\partial_\nu u - \theta\mathcal{M}_e.\nu). \tag{10}$$

We assume that the friction is invariant with respect to the x_3 axis and is modeled with Tresca's friction law, that is

$$\begin{cases} |\sigma_\tau| \le g \\ |\sigma_\tau| < g \Rightarrow \dot{u} = 0 \\ |\sigma_\tau| = g \Rightarrow \exists\beta \ge 0, \quad \text{such that } \sigma_\tau = -\beta\dot{u}_\tau \end{cases} \qquad \text{on } \Gamma_3 \times (0,T). \tag{11}$$

Here $g : \Gamma_3 \to \mathbb{R}_+$ is a given function, the friction bound, and $\dot{u}_\tau$ represents the tangential velocity on the contact boundary. The strict inequality holds in the stick zone and the equality in the slip zone. Using now (10) it is straight or ward to see that the conditions (11) imply

$$\begin{cases} |\mu\partial_\nu u - \theta\mathcal{M}_e.\nu| \le g \\ |\mu\partial_\nu u - \theta\mathcal{M}_e.\nu| < g \Rightarrow \dot{u} = 0 \\ |\mu\partial_\nu u - \theta\mathcal{M}_e.\nu| = g \Rightarrow \exists\beta \ge 0, \\ \text{such that } \mu\partial_\nu u - \theta\mathcal{M}_e.\nu = -\beta\dot{u} \end{cases} \qquad \text{on } \Gamma_3 \times (0,T).$$

Finally, we prescribe the initial displacement,

$$u(0) = u_0 \qquad \text{in } \Omega,$$

where u_0 is the given function on Ω.

We collect the above equations and conditions to obtain the classical formulation of the antiplane problem for thermo-viscoelastic materials with longterm memory, in frictional contact with a foundation.

Problem *P*: Find the displacement field $u : \Omega \times (0,T) \to S^d$ and a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}_+$,

$$\mu\Delta u + f_0 - \text{div}\theta\,\mathcal{M}_e = 0 \qquad \text{in } \Omega \times (0,T), \tag{12}$$

$$\dot{\theta} - \text{div}(K\nabla\theta) = -\mathcal{M}_e\nabla\dot{u} + q(t) \qquad \text{in } \Omega \times (0,T), \tag{13}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{14}$$

$$\mu\partial_\nu u - \theta\mathcal{M}_e.\nu = f_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{15}$$

$$\begin{cases} |\mu\partial_\nu u - \theta\mathcal{M}_e.\nu| \le g \\ |\mu\partial_\nu u - \theta\mathcal{M}_e.\nu| < g \Rightarrow \dot{u} = 0 \\ |\mu\partial_\nu u - \theta\mathcal{M}_e.\nu| = g \Rightarrow \exists\beta \ge 0 \\ \text{such that } \mu\partial_\nu u - \theta\mathcal{M}_e.\nu = -\beta\dot{u} \end{cases} \qquad \text{on } \Gamma_3 \times (0,T), \tag{16}$$

$$\theta_0 = 0 \qquad \text{on } \Gamma_1 \cup \Gamma_2 \times (0,T), \tag{17}$$

$$-K_{ij}\frac{\partial\theta}{\partial x_j}\upsilon_i = K_e(\theta - \theta_R) \qquad \text{on } \Gamma_2 \times (0,T), \tag{18}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{19}$$

The differential equation (13) describes the evolution of the temperature field, where $K := (k_{ij})$ represents the thermal conductivity tensor, $q(t)$ the density of volume heat sources. The associated temperature boundary condition is given by (18), where θ_R is the temperature of the foundation, and k is the heat exchange coefficient between the body and the obstacle. Finally, u_0 , θ_0 represent the initial displacement and temperature, respectively.

We derive now the variational formulation of P . To this end, we introduce the functional space

$$V = \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_2 \right\},$$

and we assume that

$$E = \left\{ \eta \in H^1(\Omega) \mid \eta = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \right\}.$$

Since $meas\, \Gamma_1 > 0$, the Friedrichs - Poincare inequality holds, i.e. there exists a positive constant C_P depends only on Ω and Γ_1), such that

$$\| u \|_{H^1(\Omega)} \leq C_P \| \nabla u \|_{L^2(\Omega)} \qquad \forall\, u \in V.$$

We consider on V the innerproduct given by

$$(u, v)_V = \int_\Omega \nabla u . \nabla v dx \qquad \forall\, u, v \in V,$$

and let $\| \,.\, \|_V$ be the associated norm, i.e.

$$\| v \|_V = \| \nabla v \|_{L^2(\Omega)} \qquad \forall\, v \in V.$$

It follows that $\| \,.\, \|_{H^1(\Omega)}$ and $\| \,.\, \|_V$ are equivalent norms on V and therefore $(V, \| \,.\, \|)$ is a real Hilbert space. By Sobolev's trace theorem we deduce that there exists $C_0 > 0$ (depending only on Ω, Γ_1 and Γ_3) such that

$$\| v \|_{H^1(\Omega)} \leq C_0 \| v \|_V \qquad \forall\, v \in V. \tag{20}$$

If $(X, \| \,.\, \|_X)$ represents a real Banach space, we denote by $C([0, T]; X)$, the space of continuous functions from $[0, T]$ to X, with the norm

$$\| v \|_{C([0;T];X)} = \max_{t \in [0,T]} \| x(t) \|_X$$

and we use standard notations for the Lebesgue space $L^2(0, T; X)$ as well as for the Sobolev space $W^{1,2}(0, T; X)$. In particular, recall that the norm on the space $L^2(0, T; X)$ is given by

$$\| u \|_{L^2(0,T;X)} = \left(\int_0^T \| u(t) \|_X^2 \, dt \right)^{\frac{1}{2}}$$

and the norm on the space $W^{1,2}(0, T; X)$ is given by

$$\| u \|_{W^{1,2}(0,T;X)} = \left(\int_0^T \| u(t) \|_X^2 \, dt + \int_0^T \| \dot{u}(t) \|_X^2 \, dt \right)^{\frac{1}{2}}$$

.

Finally, we suppress the argument X when $X = \mathbb{R}$ thus, for example, we use the notation $W^{1,2}(0, T)$ for the space $W^{1,2}(0, T; \mathbb{R})$ and the notation $\| \,.\, \|_{W^{1,2}(0,T)}$ for

the norm $\| \,.\, \|_{W^{1,2}(0,T;\mathbb{R})}$.
In the study P, we assume that the friction bound function g satisfies

$$g \in L^{\infty}(\Gamma_3) \quad \text{and} \quad g(x) \geq 0 \qquad \text{a.e. } x \in \Gamma_3. \tag{21}$$

The forces and tractions are assumed to have the regularity

$$f_0 \in W^{1,2}(0,T;L^2(\Omega)), \qquad f_2 \in W^{1,2}(0,T;L^2(\Gamma_2)). \tag{22}$$

We consider the functional $j : V \to \mathbb{R}_+$ given by

$$j(v) = \int_{\Gamma^3} g|v|da \qquad \forall v \in V \tag{23}$$

and let $f : [0,T] \to V$ be defined by

$$(f(t), v)_V = \int_{\Omega} f_0(t)vdx + \int_{\Gamma_2} f_2(t)v\, da \qquad \forall v \in V,\ \forall t \in [0,T] \tag{24}$$

The definition of f is based on Riesz's representation theorem and by (22) and (24), we infer that

$$f \in L^2(0,T;V). \tag{25}$$

For the thermal tensors and the heat sources density, we suppose that

$$M_e = (m_{ij}), \quad m_{ij} = m_{ji} \in L^{\infty}(\Omega). \tag{26}$$

The boundary thermal data satisfy

$$q \in W^{1,2}(0,T;L^2(\Omega)),\ \ \theta_R \in W^{1,2}(0,T;L^2(\Gamma_3)),\ \ k_e \in L^{\infty}(\Omega,\mathbb{R}_+). \tag{27}$$

The thermal conductivity tensor verifies the usual symmetry and ellipticity: for some $c_k > 0$ and for all $\xi_i \in \mathbb{R}^d$

$$K = (k_{ij}), \quad k_{ij} = k_{ji} \in L^2(\Omega),\ \ \forall c_k > 0,\ \xi_i \in\ \mathbb{R}^d; \quad k_{ij}\xi_i.\xi_j \leq c_k\xi_i.\xi_j. \tag{28}$$

Finally, we assume that the initial data verfies

$$u_0 \in V, \quad \theta_0 \in L^2(\Omega), \tag{29}$$

and moreover,

$$\mu(u_0, v)_V + j(v) \geq (f(0), v)_V. \tag{30}$$

Using Green's formula it is straight forward to derive the following variational formulation of P. We denote by $\langle\, ,\rangle_{V' \times V}$ the duality pairing between $V^{'}$ and V.

Problem P_V**:** Find a displacement field $u : [0;T] \to V$ and a temperature field $\theta : (0;T) \to E$ such that

$$\mu(u(t), v - \dot{u}(t))_V + (M\theta(t), \varepsilon(v - \dot{u}(t)))_{\mathcal{H}}$$

$$+j(v) - j(\dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \qquad \forall v \in V,\ t \in (0,T),$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t) \qquad \text{in } E',$$

$$u(0) = u_0, \qquad \theta(0) = \theta_0 \qquad \text{in } \Omega.$$

Here, the function $Q : [0,T] \to E'$ and the operators $K : E \to E'$, $R : V \to E'$; $M : E \to V'$ are defined by $\forall v \in V$, $\forall \tau \in E$, $\forall \mu \in E$:

$$\langle Q(t), \mu\rangle_{E'\times E} = \int_{\Gamma_3} k_e \theta_R \mu \, ds + \int_\Omega q\mu \, dx,$$

$$\langle K\tau, \mu\rangle_{E'\times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \mu}{\partial x_j} \frac{\partial \mu}{\partial x_i} \, dx + \int_{\Gamma_3} k_e \tau \mu \, ds,$$

$$\langle Rv, \mu\rangle_{E'\times E} = \int_{\Gamma_3} h_\tau(|v_\tau|)\mu \, ds - \int_\Omega (M_e \, \nabla v)\mu \, dx,$$

$$\langle M\tau, v\rangle_{V'\times V} = (-\tau M_e, \varepsilon(v))_{\mathcal{H}}.$$

Our main existence and uniqueness result is stated as follow.

Theorem 1. *Assume that* (21)-(22), (25) *and* (26) *hold. Then there exists a unique solution* u, θ *of problem* P_V. *Moreover, the solution satisfies*

$$u \in W^{1,2}(0,T;V) \; ; \; \theta \in W^{1,2}(0,T;E^{'}) \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)).$$

An element (u, θ) which solves P_V is called a weak solution of the mechanical problem P. We conclude by Theorem 1. that the antiplane contact problem P has a unique weak solution, provided that (21)-(22), (25) and (30) hold.

3. Existence and uniqueness

The proof of Theorem 1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 1 hold and we denote by $c > 0$ a generic constant, which value may change from lines to lines.

In the first step of the proof, we introduce the set

$$\mathcal{W} = \left\{\eta \in W^{1,2}(0,T;X) \mid \eta(0) = 0_X\right\}, \tag{31}$$

and we prove the following existence and uniqueness result.

Lemma 2. *For all* $\eta \in \mathcal{W}$, *there exists a unique element* $u \in W^{1,2}(0,T;X)$ *such that*

$$\begin{aligned} &a(u_\eta(t), v - \dot{u}_\eta(t))_X + (\eta(t), v - \dot{u}_\eta(t))_X + j(v) - j(\dot{u}(t)) \\ &\geq (f(t), v - \dot{u}_\eta(t))_X \qquad \forall v \in X, \;\; a.e. \; t \in (0,T), \end{aligned} \tag{32}$$

$$u_\eta(0) = u_0. \tag{33}$$

Here X is a real Hilbert space endowed with the inner product $(.\,,.)_X$ and the data a is a bilinear continuous coercive and symmetric form.

Proof. We use an abstract existence and uniqueness result which may be found in [12]. □

In the second step, we use the displacement field u_η obtained in Lemma 2 and we consider the following Lemma.

Lemma 3. *For all $\eta \in \mathcal{W}$, there exists a unique*

$$\theta_\eta \in W^{1,2}(0,T;E^{'}) \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)),\ c>0 \qquad \forall \eta \in L^2([0,T],V^{'}),$$

satisfying

$$\begin{cases} \dot{\theta}_\eta(t)K = R\dot{u}_\eta(t) + Q(t) & \text{in } E^{'} \quad a.e.\ t \in (0,T) \\ \theta_\eta(0) = \theta_0, \end{cases} \tag{34}$$

$$|\theta_{\eta_1} - \theta_{\eta_2}|^2_{L^2(\Omega)} \leq c \int_0^T |\dot{u}_1(s) - \dot{u}_2(s)|^2_V \, ds \quad \forall t \in (0,T), \tag{35}$$

and

$$|\dot{\theta}_{\eta_1} - \dot{\theta}_{\eta_2}|^2_{L^2(\Omega)} \leq c \int_0^T |u_1(s) - u_2(s)|^2_V \, ds \quad \forall t \in (0,T). \tag{36}$$

Proof. The existence and uniqueness result verifying (34) follows from classical result on first order evolution equation, applied to the Gelfand evolution triple

$$E \subset F \equiv F^{'} \subset E^{'}$$

We verify that the operator K is linear continuous and strongly monotone. Now from the expression of the operator R, $v_\eta \in W^{1,2}(0,T;V) \Rightarrow Rv_\eta \in W^{1,2}(0,T;F)$, as $Q \in W^{1,2}(0,T;E)$ then $Rv_\eta + Q \in W^{1,2}(0,T;E)$, we deduce (35) and (36), (see [1]). □

In the next step, we consider the operator $\Lambda : \mathcal{W} \to \mathcal{W}$ defined by

$$\langle \Lambda\eta(t), u \rangle_{V' \times V} = -(M_e \theta_\eta, \varepsilon(u))_{\mathcal{H}} \qquad \forall \eta \in \mathcal{W},\ t \in (0,T). \tag{37}$$

It follows from (34) that the operator Λ is well defined. Since $u \in \mathcal{W}$, implies $\Lambda\eta \in \mathcal{W}$. We have the following result.

Lemma 4. *The operator Λ has a unique fixed point $\eta^* \in \mathcal{W}$.*

Proof. Let $\eta_1, \eta_2 \in \mathcal{W}$ and, for the sake of simplicity, denote $u_1 = u_{\eta_1}$ and $u_2 = u_{\eta_2}$. Using (37) and (26), it follows that

$$\|\Lambda\eta_1(s) - \Lambda\eta_2(s)\|^2_X \leq \|\theta_1(s) - \theta_2(s)\|^2_X \qquad \forall t \in (0,T). \tag{38}$$

Here and in what follows c represents a generic positive constant which may depend on a and T, whose value may change from line to line. Moreover, from (38) we infer that

$$\| \left(\frac{d}{dt}\Lambda\eta_1 \right)(t) - \left(\frac{d}{dt}\Lambda\eta_2 \right)(t) \|_X \leq \|\dot{\theta}_1(s) - \dot{\theta}_2(s)\|_X, \quad \text{a.e. } t \in (0,T),$$

which yields

$$\| \left(\frac{d}{dt}\Lambda\eta_1 \right)(t) - \left(\frac{d}{dt}\Lambda\eta_2 \right)(t) \|^2_X \leq \|\dot{\theta}_1(s) - \dot{\theta}_2(s)\|^2_X,$$

using (36) we have the inequation

$$\| \left(\frac{d}{dt}\Lambda\eta_1 \right)(t) - \left(\frac{d}{dt}\Lambda\eta_2 \right)(t) \|^2_X \leq c\,(\int_0^t \|u_1(s) - u_2(s)\|^2_X \, ds\,).$$

On the other hand, taking into account(32), we have the inequalities

$$\begin{aligned} a(u_1(s), v - \dot{u}_1(s)) + (\eta_1(s), v - \dot{u}_1(s))_X \quad &+ \quad j(v) - j(\dot{u}_1(s)) \\ &\geq \quad (f(s), v - \dot{u}_1(s))_X \end{aligned}$$

$$\begin{aligned} a(u_2(s), v - \dot{u}_2(s)) + (\eta_2(s), v - \dot{u}_2(s))_X \quad &+ \quad j(v) - j(\dot{u}_2(s)) \\ &\geq \quad (f(s), v - \dot{u}_2(s))_X \end{aligned}$$

for all $v \in X$, a.e. $t \in (0,T)$.

The data a is a bilinear, continuous, coercive and symmetric form.

We choose in the first inequality, $v = \dot{u}_1(s)$ in the second inequality, add the results to obtain

$$\frac{1}{2}\frac{\partial}{\partial s}\|u_1(t) - u_2(t)\|_a^2 \leq ((\eta_1(t) - \eta_2(t), \dot{u}_1(t) - \dot{u}_2(t))_X, \quad \text{a.e. } t \in (0,T).$$

Integrating the previous inequality from 0 to t and using (33), we get

$$\begin{aligned} \frac{1}{2}\|u_1(t) - u_2(t)\|_a^2 \quad &\leq \quad (\eta_1(t) - \eta_2(t), u_1(t) - u_2(t))_X \\ &+ \quad \int_0^t (\dot{\eta}_1(s) - \dot{\eta}_2(s), u_1(s) - u_2(s))_X \, ds. \end{aligned}$$

It follows that

$$\begin{aligned} c\|u_1(t) - u_2(t)\|_X^2 \quad &\leq \quad \|\eta_1(t) - \eta_2(t)\|_X \|u_1(t) - u_2(t)\|_X \\ &+ \quad \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X \|u_1(s) - u_2(s)\|_X \, ds, \end{aligned}$$

and, using the inequality $ab \leq \dfrac{a^2}{2\alpha} + 2\alpha b^2$ for a, α, $b > 0$, we find

$$\begin{aligned} \|u_1(t) - u_2(t)\|_X^2 \quad &\leq \quad c(\|\eta_1(t) - \eta_2(t)\|_X^2 + \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2 \, ds \\ &+ \quad \int_0^t \|u_1(s) - u_2(s)\|_X^2 \, ds). \end{aligned} \tag{39}$$

As

$$\eta_1(t) - \eta_2(t) = \int_0^t (\dot{\eta}_1(s) - \dot{\eta}_2(s)) \, ds,$$

we deduce that

$$\|\eta_1(t) - \eta_2(t)\|_X \leq \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X \, ds.$$

Using this inequality in (39), we obtain

$$\|u_1(t) - u_2(t)\|_X^2 \leq c\left(\int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2 \, ds + \int_0^t \|u_1(s) - u_2(s)\|_X^2 \, ds\right).$$

Applying now Gronwall's inequality we deduce

$$\|u_1(t) - u_2(t)\|_X^2 \leq c\,(\int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2 \, ds), \tag{40}$$

which yields

$$\int_0^t \|u_1(s) - u_2(s)\|_X^2 \leq c\,(\int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2 \, ds). \tag{41}$$

Combining now (38), (39), (42)and (43) , we obtain

$$\begin{aligned}\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_X^2 \quad &+ \quad \|\left(\frac{d}{dt}\Lambda\eta_1\right)(t)-\left(\frac{d}{dt}\Lambda\eta_2\right)(t)\|_X^2 \\ &\leq \quad c\,(\int_0^t \|\dot\eta_2(s)-\dot\eta_2(s)\|_X^2\,ds).\end{aligned}$$

Iterating the last inequality $p-$times we infer

$$\begin{aligned}\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_X^2 \quad &+ \quad \|\left(\frac{d}{dt}\Lambda\eta_1\right)(t)-\left(\frac{d}{dt}\Lambda\eta_2\right)(t)\|_X^2 \\ &\leq \quad c^p\,(\int_0^t\int_0^{s_1}...\int_0^{s_{p-1}} \|\dot\eta_2(s_p)-\dot\eta_2(s_p)\|_X^2\,ds_p...ds_1),\end{aligned}$$

where Λ^p denotes the power of the operator Λ . The last inequality implie

$$\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_{W^{1,2}(0,T;X)} \leq \|\eta_1(t)-\eta_2(t)\|_{W^{1,2}(0,T;X)}.$$

Since $\lim_{p\to+\infty}\dfrac{c^pT^p}{p!}=0$, the previous inequality implies that a power Λ^p of Λ, is a contraction in W for p large enough. It follows now from Banach's fixed point theorem that, there exists a unique element $\eta* \in W$ such that $\Lambda^p\eta* = \eta*$. Moreover, since $\Lambda^p(\Lambda\eta*) = \Lambda(\Lambda^p\eta*) = \Lambda\eta*$, we deduce that$\Lambda\eta*$ is also a fixed point of the operator Λ^p . By the uniqueness of the fixed point, we conclude that $\Lambda\eta* = \eta*$, which shows that $\eta*$ is a fixed point of Λ. The uniqueness of the fixed point of the operator Λ follows from the uniqueness of the fixed point of the operator Λ^p. □

We have now all the ingredients to prove the theorem.

Proof of Theorem 1.

Existence. Let $\eta* \in \mathcal{W}$ be the fixed point of Λ and let $u_{\eta*}$ be the function defined by Lemma 1. for $\eta = \eta*$. Since, it follows from (37) that $u_{\eta*}$ is a solution to the problem (32)-(33). Moreover, the regularity $u_{\eta*} \in W^{1,2}(0,T;X)$ is obtained from Lemma 1, θ_{η^*} be the solution to problems (34).

Consider the form $a : V \times V \to \mathbb{R}$, defined by

$$a(u,v) = \mu\int_\Omega \nabla u.\nabla v \qquad \forall u,v \in V.$$

Clearly this form is bilinear, continuous, coercive and symmetric; moreover, using (19) and (20) it follows that the functional j defined by (23) is convex, lower semicontinuous and proper. Taking into account (22) and (24)-(29) then, the u_{η^*} is also solution of (32)-(33); θ_{η^*} be the solution to problems (34).

Then $(u_{\eta*}, \theta_{\eta^*})$ be the solution to Problems P_V

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator defined by (37) and from uniqueness in Lemmas 2., 3. and 4.

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