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A NOTE ON THE GLOBAL BEHAVIOR OF A SYSTEM OF SECOND-ORDER RATIONAL DIFFERENCE EQUATIONS

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Abstract. In this paper, we consider the following system of rational difference equations

$$x_{n+1} = \frac{\alpha_1 y_n}{\beta_1 + \gamma_1 x_{n-1}},\ y_{n+1} = \frac{\alpha_2 x_n}{\beta_2 + \gamma_2 y_{n-1}},\ n = 0, 1, 2, ...$$

where $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in (0, \infty)$ and the initial values $x_{-1}, x_0, y_{-1}, y_0 \in (0, \infty)$. Our main aim is to investigate the local asymptotic stability and global stability of equilibrium points, and the rate of convergence of positive solutions of the system. Moreover, some numerical examples are established to illustrate our theoretical results.

1. Introduction

Difference equations arise naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics. There are plenty of papers and books that can be found regarding to the theory and applications of difference equations, see [1, 10, 11, 15, 16, 17, 14, 22] and the references cited therein. Recently, there has been a lot of works concerning the global behaviors of positive solutions of rational difference equations and positive solutions of systems of rational difference equations [2, 3, 5, 6, 7, 8, 9, 12, 13, 18, 21, 24]. It is extremely difficult to understand thoroughly the global behaviors of solutions of rational difference equations and solutions of systems of rational difference equations, although they have very simple forms. Therefore, the study of rational difference equations and systems of rational difference equations is worth further consideration.

In [15] the authors studied the global asymptotic behavior of solutions of the discrete delay logistic model

$$x_{n+1} = \frac{\alpha x_n}{1 + \beta x_{n-k}},\ n = 0, 1, 2, \dots,$$

where $\alpha \in (1, \infty), \beta \in (0, \infty),\ k \in \{0, 1, 2, \dots\}$ and the initial conditions $x_{-k}, \dots, x_{-1}, x_0$ are arbitrary nonnegative numbers.

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In [16] the authors studied the global asymptotic behavior of solutions of Pielou's difference equation

$$y_{n+1} = \frac{py_n}{1 + y_{n-1}},\ n = 0, 1, 2, \ldots,$$

where $p > 0$ and the initial conditions y_{-1}, y_0 are arbitrary nonnegative numbers.

Motivated by these above papers, in this paper we will consider the following system of difference equations

$$x_{n+1} = \frac{\alpha_1 y_n}{\beta_1 + \gamma_1 x_{n-1}},\ y_{n+1} = \frac{\alpha_2 x_n}{\beta_2 + \gamma_2 y_{n-1}},\ n = 0, 1, 2, ... \tag{1}$$

where $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in (0, \infty)$ and the initial values $x_{-1}, x_0, y_{-1}, y_0 \in (0, \infty)$. More precisely, we investigate the local asymptotic stability and global stability of equilibrium points, and the rate of convergence of positive solutions of the system (1) which converge to its equilibrium point.

2. Boundedness of the positive solutions of (1)

In the first result we will establish condition for the boundedness of every positive solution of the system (1).

Theorem 2.1. *Assume that $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ are positive constants such that*

$$\alpha_1\alpha_2 \le \beta_1\beta_2, \tag{2}$$

then every positive solution $\{(x_n, y_n)\}$ of the system (1) is bounded.

Proof. For any positive solution $\{(x_n, y_n)\}$ of the system (1), it is clear that $x_n > 0$ and $y_n > 0$ forall $n = 1, 2, \ldots$, in which the initial values x_{-1}, x_0, y_{-1}, y_0 are positive real numbers.

From (1), we have

$$x_{n+1} \le \frac{\alpha_1}{\beta_1} y_n \le \frac{\alpha_1\alpha_2}{\beta_1\beta_2} x_{n-1} \le \frac{\alpha_1\alpha_2}{\beta_1\beta_2} . \frac{\alpha_1}{\beta_1} y_{n-2} \le (\frac{\alpha_1\alpha_2}{\beta_1\beta_2})^2 x_{n-3} \le \ldots \le (\frac{\alpha_1\alpha_2}{\beta_1\beta_2})^k x_{n-2k+1}, \tag{3}$$

Therefore, we have

$$x_{n+1} \le (\frac{\alpha_1\alpha_2}{\beta_1\beta_2})^k x_0,\ \text{where } n = 2k-1, k = 1, 2, 3, \ldots, \tag{4}$$

and

$$x_{n+1} \le (\frac{\alpha_1\alpha_2}{\beta_1\beta_2})^k x_{-1},\ \text{where } n = 2k-2, k = 1, 2, 3, \ldots. \tag{5}$$

Similarly, we have

$$y_{n+1} \le (\frac{\alpha_1\alpha_2}{\beta_1\beta_2})^k y_0,\ \text{where } n = 2k-1, k = 1, 2, 3, \ldots, \tag{6}$$

and

$$y_{n+1} \le (\frac{\alpha_1\alpha_2}{\beta_1\beta_2})^k y_{-1},\ \text{where } n = 2k-2, k = 1, 2, 3, \ldots. \tag{7}$$

Hence, from (33) the proof is completed. □

3. Global behavior

In the following, we state some main definitions used in this paper.

Let I, J be some intervals of real numbers and let

$$f : I^2 \times J^2 \longrightarrow I \text{ and } g : I^2 \times J^2 \longrightarrow J \tag{8}$$

are continuously differentiable functions. Then, for all initial values $(x_{-1}, x_0, y_{-1}, y_0) \in I^2 \times J^2$, the system of difference equations

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}),\ y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}),\ n = 0, 1, 2, \ldots, \tag{9}$$

has a unique solution $\{(x_n, y_n)\}_{n=1}^{\infty}$.

Definition 3.1. *A point $(\bar{x}, \bar{y})$ is called an equilibrium point of the system* (9) *if*

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{y}, \bar{y}),\ \bar{y} = g(\bar{x}, \bar{x}, \bar{y}, \bar{y}). \tag{10}$$

Definition 3.2. [9, 17] *Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system* (9).

i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every initial point $(x_i, y_i), i \in \{-1, 0\}$ if $\sum_{i=-1}^{0} \|(x_i, y_i) - (\bar{x}, \bar{y})\| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \epsilon$ for all $n > 0$. An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable (the Euclidean norm in R^2 given by $\|(x, y)\| = \sqrt{x^2 + y^2}$ is denoted by $\|.\|$).

ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\sum_{i=-1}^{0} \|(x_i, y_i) - (\bar{x}, \bar{y})\| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

iii) An equilibrium point $(\bar{x}, \bar{y})$ is called a global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

iv) An equilibrium point $(\bar{x}, \bar{y})$ is called an asymptotic global attractor if it is global attractor and stable.

Definition 3.3. [9, 17] *Let $(\bar{x}, \bar{y})$ be an equilibrium point of a map $F = (f, x_n, g, y_n)$, where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of* (9) *about the equilibrium point $(\bar{x}, \bar{y})$ is given by*

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ and F_J is a Jacobian matrix of the system (9) *about the equilibrium point $(\bar{x}, \bar{y})$.*

In order to corresponding linearized form of system (1) we consider the following transformation:

$$(x_n, y_n, x_{n-1}, y_{n-1}) \longrightarrow (f, g, f_1, g_1), \tag{11}$$

where $f = x_{n+1}, g = y_{n+1}, f_1 = x_n, g_1 = y_n$. The linearized system of (1) about $(\bar{x}, \bar{y})$ is given by

$$Y_{n+1} = F_J(\bar{x}, \bar{y}) Y_n, \tag{12}$$

where $Y_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ and the Jacobian matrix of the system (1) about the equilibrium point $(\bar{x}, \bar{y})$ is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \frac{\alpha_1}{\beta_1+\gamma_1\bar{x}} & \frac{-\gamma_1\bar{x}}{\beta_1+\gamma_1\bar{x}} & 0 \\ \frac{\alpha_2}{\beta_2+\gamma_2\bar{y}} & 0 & 0 & \frac{-\gamma_2\bar{y}}{\beta_2+\gamma_2\bar{y}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{13}$$

The following results will be useful in the sequel.

Lemma 3.1. [9] *Assume that $X_{n+1} = F(X_n), n = 0, 1, 2, \ldots,$ is a system of difference equations such that $\bar{X}$ is a fixed point of F. If all eigenvalues of Jacobian matrix F_J about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\bar{X}$ is unstable.*

Lemma 3.2. [4] *Assume that $q_0, q_1, \ldots, q_k$ are real numbers such that*

$$|q_0| + |q_1| + \ldots + |q_k| < 1.$$

Then all roots of the equation

$$\lambda^{k+1} + q_0\lambda^k + \ldots + q_{k-1}\lambda + q_k = 0$$

lie inside the unit disk.

The next theorem will show the existence of zero equilibrium and establish the condition for existence of positive equilibrium of the system (1).

Theorem 3.3. *Suppose that $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ are positive constants which satisfy*

$$\alpha_1\alpha_2 > \beta_1\beta_2. \tag{14}$$

Then the system (1) has a zero equilibrium $(0, 0)$ and a positive equilibrium $(\bar{x}, \bar{y})$, otherwise, the system (1) has a unique zero equilibrium $(0, 0)$.

Proof. We consider the following system of algebraic equations:

$$x = \frac{\alpha_1 y}{\beta_1 + \gamma_1 x}, \ y = \frac{\alpha_2 x}{\beta_2 + \gamma_2 y}. \tag{15}$$

From (15), we obtain:

$$x = 0, \tag{16}$$

and

$$ax^3 + bx^2 + cx + d = 0, \tag{17}$$

where $a = \gamma_1^2\gamma_2, b = 2\beta_1\gamma_1\gamma_2, c = \alpha_1\beta_2\gamma_1 + \beta_1^2\gamma_2, d = \alpha_1(\beta_1\beta_2 - \alpha_1\alpha_2)$.
From (16), $x = 0$ follows $y = 0$. Hence, $(0, 0)$ is a solution of (15), so $(0, 0)$ is an equilibrium of (1).

We consider the function

$$F(x) = ax^3 + bx^2 + cx + d. \tag{18}$$

Since (14) holds, then from (18) we have:

$$F(0) = d = \alpha_1(\beta_1\beta_2 - \alpha_1\alpha_2) < 0, \tag{19}$$

and

$$\lim_{x\to+\infty} F(x) = +\infty. \tag{20}$$

From (18) we have:

$$F'(x) = 3ax^2 + 2bx + c > 0 \text{ for all } x \geq 0. \tag{21}$$

From (21) implies that $F(x)$ is increasing function on $[0, \infty)$. Moreover, from (19) and (21) we can conclude that the equation (17) has a unique positive solution.
If (14) does not hold, it means that $d > 0$ or $d = 0$, then $F(x) > 0$ for all $x \geq 0$ as $d > 0$ or $F(x) = 0$ has only one root $x = 0$ as $d = 0$. The proof of the theorem is completed. □

In the next theorem we will study the asymptotic behavior of the positive solutions of (1). The next lemma is a slight modification of Theorem 1.16 of [11] and for readers convenience we state it without its proof.

Lemma 3.4. *Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+,\ g : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous function, $\mathbb{R}^+ = (0,\ \infty)$ and $a_1,\ b_1,\ a_2,\ b_2$ be positive numbers such that $a_1 < b_1,\ a_2 < b_2$. Suppose that*

$$f : [a_1,\ b_1] \times [a_2,\ b_2] \longrightarrow [a_1,\ b_1],\ g : [a_1,\ b_1] \times [a_2,\ b_2] \longrightarrow [a_2,\ b_2]. \tag{22}$$

In addition, assume that $f(x,\ y)$ (res. $g(x,\ y)$) is decreasing with respect to x (res. y) for every y (res. x) and increasing with respect to y (res. x) for every x (res. y). Finally suppose that if $m_1,\ M_1,\ m_2,\ M_2$ are real numbers such that

$$m_1 = f(M_1, m_2), M_1 = f(m_1, M_2), m_2 = g(m_1, M_2), M_2 = g(M_1, m_2), \tag{23}$$

then $m_1 = M_1$ and $m_2 = M_2$. Then the following system of difference equations

$$x_{n+1} = f(x_{n-1}, y_n),\ y_{n+1} = g(x_n, y_{n-1}) \tag{24}$$

has a unique positive equilibrium $(\bar{x},\ \bar{y})$ such that $\lim_{n\to\infty}(x_n, y_n) = (\bar{x},\ \bar{y})$.

Theorem 3.5. *If* (14) *holds, then the system* (1) *has a unique positive equilibrium $(\bar{x}, \bar{y})$ which is global attractor.*

Proof. From (1) we have

$$x_{n+1} = \frac{\alpha_1 y_n}{\beta_1 + \gamma_1 x_{n-1}} \leq \frac{\alpha_1 y_n}{\gamma_1 x_{n-1}}, \tag{25}$$

and

$$y_{n+1} = \frac{\alpha_2 x_n}{\beta_2 + \gamma_2 y_{n-1}} \leq \frac{\alpha_2}{\beta_2} x_n. \tag{26}$$

Then, from (25) and (26) we imply

$$x_{n+1} \leq \frac{\alpha_1 y_n}{\gamma_1 x_{n-1}} \leq \frac{\alpha_1}{\gamma_1 x_{n-1}} \frac{\alpha_2}{\beta_2} x_{n-1} = \frac{\alpha_1\alpha_2}{\beta_2\gamma_1}. \tag{27}$$

Hence,

$$x_n \in [0, \frac{\alpha_1\alpha_2}{\beta_2\gamma_1}], n = 2, 3, 4, \ldots \tag{28}$$

Argue similarly, we have

$$y_n \in [0, \frac{\alpha_1\alpha_2}{\beta_1\gamma_2}], n = 2, 3, 4, \ldots \tag{29}$$

Now we consider two following functions

$$f(x, y) = \frac{\alpha_1 y}{\beta_1 + \gamma_1 x},\ g(x, y) = \frac{\alpha_2 x}{\beta_2 + \gamma_2 y}. \tag{30}$$

It is easy to see that $f(x,y)$ is decreasing with respect to x for every y and increasing with respect to y for every x; $g(x,y)$ is decreasing with respect to y for every x and increasing with respect to x for every y. Moreover, $f: I\times J \longrightarrow I, g: I\times J \longrightarrow J$, where $I=[0,\dfrac{\alpha_1\alpha_2}{\beta_2\gamma_1}]$ and $J=[0,\dfrac{\alpha_1\alpha_2}{\beta_1\gamma_2}]$.
Let $m_1,\ M_1,\ m_2,\ M_2$ be positive real numbers such that

$$\begin{aligned} m_1 = f(M_1,m_2), M_1 = f(m_1,M_2),\\ m_2 = g(m_1,M_2), M_2 = g(M_1,m_2). \end{aligned} \tag{31}$$

It means that

$$m_1 = \frac{\alpha_1 m_2}{\beta_1+\gamma_1 M_1},\ M_1 = \frac{\alpha_1 M_2}{\beta_1+\gamma_1 m_1}, \tag{32}$$

and

$$m_2 = \frac{\alpha_2 m_1}{\beta_2+\gamma_2 M_2},\ M_2 = \frac{\alpha_2 M_1}{\beta_2+\gamma_2 m_2}. \tag{33}$$

From (32) we have

$$\beta_1 m_1 + \gamma_1 m_1 M_1 = \alpha_1 m_2, \tag{34}$$

and

$$\beta_1 M_1 + \gamma_1 m_1 M_1 = \alpha_1 M_2. \tag{35}$$

By subtracting (35) from (34) we obtain

$$\beta_1(m_1 - M_1) = \alpha_1(m_2 - M_2). \tag{36}$$

Similarly, from (33) we have

$$\beta_2(m_2 - M_2) = \alpha_2(m_1 - M_1). \tag{37}$$

Moreover, from (36) and (37) we imply

$$\frac{\alpha_1\alpha_2 - \beta_1\beta_2}{\alpha_1}(m_1 - M_1) = 0. \tag{38}$$

Because (14) holds, then from (38) we have $m_1 = M_1$, and as consequence, from (37) follows $m_2 = M_2$. By using Lemma 3.4, we complete the proof of this theorem. $\square$

Theorem 3.6. *Assume that*

$$\alpha_1\alpha_2 < \beta_1\beta_2, \tag{39}$$

then the zero equilibrium $(0,0)$ *is globally asymptotically stable, and if* (14) *holds then the zero equilibrium* $(0,0)$ *is unstable.*

Proof. The characteristic polynomial of Jacobian matrix $F_J(\bar{x},\bar{y})$ about $(\bar{x},\bar{y})$ is given by

$$P(\lambda) = \lambda^4 + \left[\frac{\gamma_2\bar{y}}{\beta_2+\gamma_2\bar{y}} - \frac{\gamma_1\bar{x}}{\beta_1+\gamma_1\bar{x}} - \frac{\alpha_1\alpha_2}{(\beta_1+\gamma_1\bar{x})(\beta_2+\gamma_2\bar{y})}\right]\lambda^2 + \frac{\gamma_1\gamma_2\bar{x}\bar{y}}{(\beta_1+\gamma_1\bar{x})(\beta_2+\gamma_2\bar{y})}. \tag{40}$$

When $(\bar{x},\bar{y}) = (0,0)$, we have

$$P_{(0,0)}(\lambda) = \lambda^4 - \frac{\alpha_1\alpha_2}{\beta_1\beta_2}\lambda^2. \tag{41}$$

The equation $P_{(0,0)}(\lambda) = 0$ has four roots which are $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$ and $\lambda_4 = -\sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$.
If (39) holds, we can see that $|\lambda_1| = |\lambda_2| = 0 < 1$ and $|\lambda_3| = |\lambda_4| = \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}} < 1$,

and it follows from Lemma 3.1 that the zero equilibrium point $(0,0)$ of the system (1) is locally asymptotically stable. Moreover, because (39) holds and from (4)–(7) we have $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$. Hence, the zero equilibrium point $(0,0)$ is globally asymptotically stable. If (14) holds, then $|\lambda_3| > 1$, it follows that the zero equilibrium point $(0,0)$ is unstable. This completes the proof. □

4. Rate of convergence

In this section, firstly we will establish the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of the systems (1), and then we imply the result for the case in which the solution of (1) converges to the equilibrium $O = (0,0)$. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [19] and [20].

The following results give the rate of convergence of solutions of a system of difference equations

$$\mathbf{x}_{n+1} = [A + B(n)]\mathbf{x}_n \tag{42}$$

where $\mathbf{x}_n$ is a k-dimensional vector, $A \in C^{k\times k}$ is a constant matrix, and $B: Z^+ \longrightarrow C^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to \infty, \tag{43}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm; $\|.\|$ also denotes the Euclidean norm in R^2 given by

$$\|\mathbf{x}\| = \|(x,\ y)\| = \sqrt{x^2 + y^2}. \tag{44}$$

Theorem 4.1. *([23]) Assume that condition (43) holds. If $\mathbf{x}_n$ is a solution of system (42), then either $\mathbf{x}_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 4.2. *([23]) Assume that condition (43) holds. If $\mathbf{x}_n$ is a solution of system (42), then either $\mathbf{x}_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Now we will state and prove the main result of this section.

Theorem 4.3. *Assume that $\{(x_n, y_n)\}$ is a positive solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}, \lim_{n\to\infty} y_n = \bar{y}$. Then the error vector*

$$\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \\ x_{n-1} - \bar{x} \\ y_{n-1} - \bar{y} \end{pmatrix}$$

of every solution $(x_n, y_n) \neq (\bar{x}, \bar{y})$ of (1) satisfies both of the following asymptotic relations:

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = |\lambda_i(J_F(\bar{x}, \bar{y}))| \text{ for some } i \in \{1,\ 2,\ 3,\ 4\}$$

and

$$\lim_{n\to\infty}\frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|}=|\lambda_i(J_F(\bar{x},\bar{y}))| \text{ for some } i\in\{1,\ 2,\ 3,\ 4\}$$

where $|\lambda_i(J_F(\bar{x},\bar{y}))|$ *is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium* $(\bar{x},\bar{y})$.

Proof. Let $\{(x_n,y_n)\}$ be an arbitrary positive solution of the system (1) such that $\lim_{n\to\infty}x_n=\bar{x},\lim_{n\to\infty}y_n=\bar{y}$. Firstly, we will find a system satisfied by the error terms, which are given as

$$\begin{aligned}x_{n+1}-\bar{x}=&\frac{\alpha_1 y_n}{\beta_1+\gamma_1 x_{n-1}}-\frac{\alpha_1\bar{y}}{\beta_1+\gamma_1\bar{x}}=\frac{\alpha_1 y_n(\beta_1+\gamma_1\bar{x})-\alpha_1\bar{y}(\beta_1+\gamma_1 x_{n-1})}{(\beta_1+\gamma_1 x_{n-1})(\beta_1+\gamma_1\bar{x})}\\=&\frac{\alpha_1(\beta_1+\gamma_1\bar{x})}{(\beta_1+\gamma_1 x_{n-1})(\beta_1+\gamma_1\bar{x})}(y_n-\bar{y})-\frac{\alpha_1\gamma_1\bar{y}}{(\beta_1+\gamma_1 x_{n-1})(\beta_1+\gamma_1\bar{x})}(x_{n-1}-\bar{x}),\end{aligned}\tag{45}$$

and

$$\begin{aligned}y_{n+1}-\bar{y}=&\frac{\alpha_2 x_n}{\beta_2+\gamma_2 y_{n-1}}-\frac{\alpha_2\bar{x}}{\beta_2+\gamma_2\bar{y}}=\frac{\alpha_2 x_n(\beta_2+\gamma_2\bar{y})-\alpha_2\bar{x}(\beta_2+\gamma_2 y_{n-1})}{(\beta_2+\gamma_2 y_{n-1})(\beta_2+\gamma_2\bar{y})}\\=&\frac{\alpha_2(\beta_2+\gamma_2\bar{y})}{(\beta_2+\gamma_2 y_{n-1})(\beta_2+\gamma_2\bar{y})}(x_n-\bar{x})-\frac{\alpha_2\gamma_2\bar{x}}{(\beta_2+\gamma_2 y_{n-1})(\beta_2+\gamma_2\bar{y})}(y_{n-1}-\bar{y}).\end{aligned}\tag{46}$$

Let $e_n^1=x_n-\bar{x}$ and $e_n^2=y_n-\bar{y}$, then from (45) and (46) we have:

$$\begin{aligned}e_{n+1}^1&=p_n e_n^2+q_n e_{n-1}^1,\\e_{n+1}^2&=r_n e_n^1+s_n e_{n-1}^2,\end{aligned}$$

where

$$\begin{aligned}p_n&=\frac{\alpha_1(\beta_1+\gamma_1\bar{x})}{(\beta_1+\gamma_1 x_{n-1})(\beta_1+\gamma_1\bar{x})},\\q_n&=-\frac{\alpha_1\gamma_1\bar{y}}{(\beta_1+\gamma_1 x_{n-1})(\beta_1+\gamma_1\bar{x})},\\r_n&=\frac{\alpha_2(\beta_2+\gamma_2\bar{y})}{(\beta_2+\gamma_2 y_{n-1})(\beta_2+\gamma_2\bar{y})},\\s_n&=-\frac{\alpha_2\gamma_2\bar{x}}{(\beta_2+\gamma_2 y_{n-1})(\beta_2+\gamma_2\bar{y})}.\end{aligned}$$

Taking the limmits of $p_n,\ q_n,\ r_n$ and s_n as $n\to\infty$, we obtain

$$\begin{aligned}\lim_{n\to\infty}p_n&=\frac{\alpha_1}{(\beta_1+\gamma_1\bar{x})},\ \lim_{n\to\infty}q_n=-\frac{\alpha_1\gamma_1\bar{y}}{(\beta_1+\gamma_1\bar{x})^2},\\\lim_{n\to\infty}r_n&=\frac{\alpha_2}{(\beta_2+\gamma_2\bar{y})},\ \lim_{n\to\infty}s_n=-\frac{\alpha_2\gamma_2\bar{x}}{(\beta_2+\gamma_2\bar{y})^2}.\end{aligned}$$

that is

$$\begin{aligned}p_n&=\frac{\alpha_1}{(\beta_1+\gamma_1\bar{x})}+\delta_n,\ q_n=-\frac{\alpha_1\gamma_1\bar{y}}{(\beta_1+\gamma_1\bar{x})^2}+\eta_n,\\r_n&=\frac{\alpha_2}{(\beta_2+\gamma_2\bar{y})}+\theta_n,\ s_n=-\frac{\alpha_2\gamma_2\bar{x}}{(\beta_2+\gamma_2\bar{y})^2}+\zeta_n.\end{aligned}$$

where $\delta_n\to 0,\ \eta_n\to 0,\ \theta_n\to 0$ and $\zeta_n\to 0$ as $n\to\infty$.

Now, we have system of the form (42):

$$\mathbf{e}_{n+1}=(A+B(n))\mathbf{e}_n,$$

where

$$A = \begin{pmatrix} 0 & \frac{\alpha_1}{(\beta_1+\gamma_1\bar{x})} & -\frac{\alpha_1\gamma_1\bar{y}}{(\beta_1+\gamma_1\bar{x})^2} & 0 \\ \frac{\alpha_2}{(\beta_2+\gamma_2\bar{y})} & 0 & 0 & -\frac{\alpha_2\gamma_2\bar{x}}{(\beta_2+\gamma_2\bar{y})^2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B(n) = \begin{pmatrix} 0 & \delta_n & \eta_n & 0 \\ \theta_n & 0 & 0 & \zeta_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\|B(n)\| \to 0 \text{ as } n \to \ \infty.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \\ e^1_n \\ e^2_n \end{pmatrix} = A \begin{pmatrix} e^1_n \\ e^2_n \\ e^1_{n-1} \\ e^2_{n-1} \end{pmatrix}.$$

The system is exactly linearized system of (1) evaluated at the equilibrium $E = (\bar{x},\ \bar{y})$. Then from Theorem 4.1 and Theorem 4.2 we can imply the result. □

In case the equilibrium is $O = (0,0)$, we have the following result.

Theorem 4.4. *Assume that* (39) *holds. Then the error vector of every solution* $(x_n, y_n) \neq (0,0)$ *of* (1) *satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = \lim_{n\to\infty} \sqrt{x_n^2 + y_n^2 + x_{n-1}^2 + y_{n-1}^2} = |\lambda_i(J_F(0,0))| \text{ for some } i \in \{1,\ 2,\ 3,\ 4\}$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = \lim_{n\to\infty} \sqrt{\frac{x_{n+1}^2 + y_{n+1}^2 + x_n^2 + y_n^2}{x_n^2 + y_n^2 + x_{n-1}^2 + y_{n-1}^2}} = |\lambda_i(J_F(0,0))| \text{ for some } i \in \{1,\ 2,\ 3,\ 4\}$$

where $|\lambda_i(J_F(0,0))|$ *is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium* $J_F(0,0)$ *i.e.* $\lambda_i \in \{-\sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}, 0, \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}\}$.

5. Periodic nature of solutions of (1)

Theorem 5.1. *Assume that* (33) *holds, then system* (1) *has no positive solutions of prime period two.*

Proof. We suppose contrarily that the system (1) has a distinctive prime period-two positive solutions:

$$\ldots, (p_1, q_1), (p_2, q_2), (p_1, q_1), (p_2, q_2), \ldots, \tag{47}$$

where p_1, p_2, q_1, q_2 are positive real numbers and $p_1 \neq p_2, q_1 \neq q_2$. Then, from system (1), we have:

$$p_1 = \frac{\alpha_1 q_2}{\beta_1 + \gamma_1 p_1},\ p_2 = \frac{\alpha_1 q_1}{\beta_1 + \gamma_1 p_2}, \tag{48}$$

and

$$q_1 = \frac{\alpha_2 p_2}{\beta_2 + \gamma_2 q_1},\ q_2 = \frac{\alpha_2 p_1}{\beta_2 + \gamma_2 q_2}. \tag{49}$$

From (48) and (49), we get:

$$\beta_1 p_1 + \gamma_1 p_1^2 = \alpha_1 q_2, \tag{50}$$

$$\beta_1 p_2 + \gamma_1 p_2^2 = \alpha_1 q_1, \tag{51}$$

$$\beta_2 q_1 + \gamma_2 q_1^2 = \alpha_2 p_2, \tag{52}$$

$$\beta_2 q_2 + \gamma_2 q_2^2 = \alpha_2 p_1. \tag{53}$$

On subtracting (50) and (51) we obtain:

$$[\beta_1 + \gamma_1(p_1 + p_2)](p_1 - p_2) = \alpha_1(q_2 - q_1). \tag{54}$$

Similarly, by subtracting (52) and (53) we have:

$$[\beta_2 + \gamma_2(q_1 + q_2)](q_1 - q_2) = \alpha_2(p_2 - p_1). \tag{55}$$

From (54) and (55) we can imply:

$$\{[\beta_1 + \gamma_1(p_1 + p_2)][\beta_2 + \gamma_2(q_1 + q_2)] - \alpha_1\alpha_2\}(q_1 - q_2) = 0. \tag{56}$$

Because of (33), we can see that

$$[\beta_1 + \gamma_1(p_1 + p_2)][\beta_2 + \gamma_2(q_1 + q_2)] - \alpha_1\alpha_2 > 0. \tag{57}$$

Hence, from (56) follows $q_1 = q_2$, and $p_1 = p_2$ as a consequence from (55), which is a contradiction. This completes the proof of the theorem. □

6. Examples

In order to verify our theoretical results and to support our theoretical discussion, we consider several interesting numerical examples. These examples represent different types of qualitative behavior of solutions of the systems (1). All plots in this section are drawn with Matlab.

Example 6.1. *Let $\alpha_1 = 1.7, \beta_1 = 1.6, \gamma_1 = 0.06, \alpha_2 = 26, \beta_2 = 21, \gamma_2 = 0.03$. We can see that $\alpha_1\alpha_2 > \beta_1\beta_2$, so the condition* (14) *holds, according to Theorem 3.5 we follow the system* (1) *has a unique positive equilibrium $(\bar{x}, \bar{y})$, which is global attractor. The system* (1) *can be written as*

$$x_{n+1} = \frac{1.7y_n}{1.6 + 0.06x_{n-1}},\ y_{n+1} = \frac{26x_n}{21 + 0.03y_{n-1}} \tag{58}$$

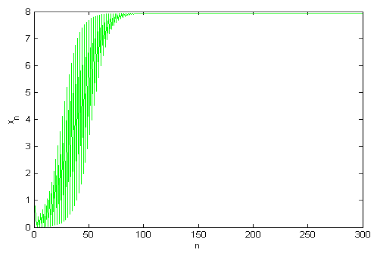
with initial conditions $x_{-1} = 0.8$, $x_0 = 0.3$, $y_{-1} = 0.004$ and $y_0 = 0.008$.

In this case, the unique positive equilibrium point of the system (1) is $(\bar{x}, \bar{y}) = (7.9327, 9.6871)$, which is global attractor. In Figure 1, the plot of x_n is shown in Figure 1 (a), the plot of y_n is shown in Figure 1 (b), and a phase portrait of the system (58) is shown in Figure 1 (c).

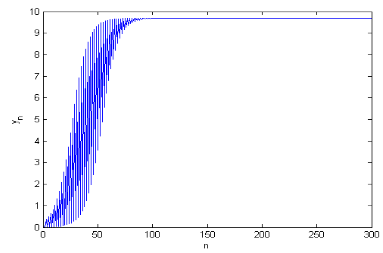
Example 6.2. *Let $\alpha_1 = 14, \beta_1 = 17, \gamma_1 = 0.5, \alpha_2 = 18, \beta_2 = 21, \gamma_2 = 0.8$. We can see that the condition* (39) *of Theorem 3.6 is satisfied, i.e, $\alpha_1\alpha_2 < \beta_1\beta_2$. The system* (1) *can be written as*

$$x_{n+1} = \frac{14y_n}{17 + 0.5x_{n-1}},\ y_{n+1} = \frac{18x_n}{21 + 0.8y_{n-1}} \tag{59}$$

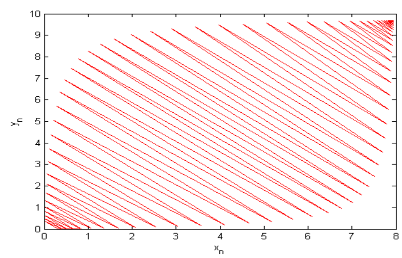
with initial conditions $x_{-1} = 0.18$, $x_0 = 0.2$, $y_{-1} = 0.16$ and $y_0 = 0.03$.



(a) Plot of x_n for the system (58)

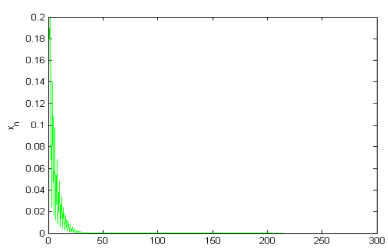


(b) Plot of y_n for the system (58)

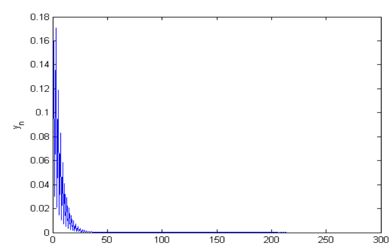


(c) An attractor of the system (58)

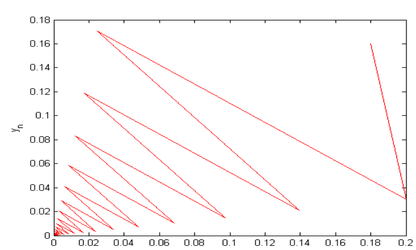
Figure 1. Plots for the system (58)



(a) Plot of x_n for the system (59)



(b) Plot of y_n for the system (59)



(c) An attractor of the system (59)

Figure 2. Plots for the system (59)

In this case, the zero equilibrium point $O(0,0)$ is globally asymptotically stable. Moreover, in Figure 2, the plot of x_n is shown in Figure 2 (a), the plot of y_n is shown in Figure 2 (b), and an attractor of the system (59) is shown in Figure 2 (c).

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Second Ed. Dekker, New York, 1992, 2000.

[2] L. Berg, S. Stević, On some systems of difference equations, Appl. Math. Comput., Vol. 218, 1713-1718, 2011.

[3] DŽ. Burgić and Z. Nurkanović, An example of globally asymptotically stable anti-monotonic system of rational difference equations in the plane, Sarajevo Journal of Mathematics, Vol. 5, No. 18, 235-245, 2009.

[4] E. Camouzis, G. Ladas, Dynamics of Third Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/ CRC, Boca Raton, London, 2008.

[5] E. Camouzis, G. Papaschinopoluos, Global asymptotic behavior of positive solutions on the system of rational difference equations, Appl. Math. Lett.,Vol. 17, 733-737, 2004.

[6] D. Clark, M. R. S Kulenovic, A coupled system of rational difference equations, Comput. Math. Appl.,Vol. 43, 849-867, 2002.

[7] Q. Din, Dynamics of a discrete Lotka-Volterra model, Adv. Difference Equ., Vol. 2013, doi:10.1186/1687-1847-2013-95.

[8] Q. Din, Asymptotic behavior of an anti-competitive system of second-order difference equations, J. Egyptian Math. Soc., Vol. 24, 37- 43, 2016.

[9] Q. Din, T. F. Ibrahim, K. A. Khan, Behavior of a competitive system of second-order difference equations, Scientific World J., Volume 2014, doi:10.1155/2014/283982.

[10] Y. Enatsu, Y. Nakata, and Y. Muroya, Global stability for a class of discrete SIR epidemic models, Mathematical Biosciences and Engineering, Vol. 7, No. 2, 347-61. doi: 10.3934/mbe.2010.7.347, 2010.

[11] E.A. Grove, G.Ladas, Periodicities in Nonlinear Difference Equations, Chapman and Hall, CRC, 2005.

[12] S. Kalabusić, M.R.S. Kulenović, Dynamics of certain anti-competitive systems of rational difference equations in the plane, J. Difference Equ. Appl., Vol. 17, No. 11, 1599-1615, 2011.

[13] V.V. Khuong, M.N. Phong, A note on global behavior of solutions and positive nonoscillatory solution of a difference equation, Comm. App. Nonlinear Anal.,Vol. 18, No. 4, 77-88,2011.

[14] Vu Van Khuong and Mai Nam Phong, Global behavior and the rate of convergence of a system of two difference equations of exponential form, PanAmer. Math. J., Vol. 27, No. 1, 67-78, 2017.

[15] V.L. Kocic, G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic, Dordrecht, 1993.

[16] M.R.S. Kulenović, G.Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/ CRC, Boca Raton, London, 2001.

[17] M.R.S. Kulenović and O. Merino, Discrete Dynamical Systems and Difference Equations with Mathematica, Chapman& Hall/CRC, Boca Raton, Fla, USA, 2002.

[18] M.R.S. Kulenović, M. Nurkanović, Global asymptotic behavior of a two dimensional system of difference equations modeling cooperation, J. Differ. Equations Appl., Vol. 9, No. 1, 149-159, 2003.

[19] M. R. S. Kulenović and Z. Nurkanović, The rate of convergence of solution of a three dimensional linear fractional systems of difference equations, Zbornik radova PMF Tuzla - Svezak Matematika, Vol. 2, 1-6, 2005.

[20] M. R. S. Kulenović and M. Nurkanović, Asymptotic behavior of a competitive system of linear fractional difference equations, Adv. Difference Equ., Volume 2006, Art. ID 19756, 13pp.

[21] M.N. Phong, Global behavior of a system of rational difference equations, Comm. App. Nonlinear Anal., Vol. 23, No. 3, 93-107, 2016.

[22] M.N. Phong, V.V. Khuong, Asymptotic behavior of a system of two difference equations of exponential form, Int. J. Nonlinear Anal. Appl., Vol. 7, No. 2, 319-329, 2016.

[23] M. Pituk, More on Poincare's and Peron's theorems for difference equations, J. Difference Equ. Appl., Vol. 8, 201-216, 2002.

[24] S. Stević, New class of solvable systems of difference equations, Appl. Math. Lett., Vol. 63, 137-144, 2017.

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