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COMMON FIXED POINT THEOREMS USING T-HARDY ROGERS TYPE CONTRACTIVE CONDITION AND F-CONTRACTION ON A COMPLETE 2-METRIC SPACE

DINANATH BARMAN, KRISHNADHAN SARKAR AND KALISHANKAR TIWARY

Abstract. In this paper we have proved some common fixed point theorems using T-Hardy Rogers Type Contraction condition and F-Contraction on a complete 2-metric space and generalized many existing results in this literature.

1. Introduction

Fixed point is an important part of mathematics. It is used in various branches of mathematics such as Numerical Analysis, Differential Equation, Functional Analysis, Topology etc. Not only in Mathematics, it is also used in Biology, Chemestry, Physics and also in many other branches.

In 1922, Banach first investigate a fixed point theorem in metric space and it is well known as Banach Fixed Point Theorem. After that many researchers of this field have generalized that theorem in various ways. They have restricted the conditions or have changed the spaces. After Banach, Kannan [9] generarized that theorem. After Kannan, Chaterjea [4] generalized that fixed point theorem. Reich gave a generalization of Chaterjea's[4] fixed point theorem. In 1973, Hardy and Rogers [7] have also generalized the fixed point theorem of Reich. After that, many researchers have been using different type of Hardy Rogers contractive condition to obtain a new fixed point results.

In this paper we have generalized many existing results which are in 2-metric spaces. 2-metric space is a generalization of metric space, which has been introducd by Gähler [6]. In this space, the mapping goes $X \times X \times X \to \mathbf{R}^+$. If T is a mapping in 2-metric space then $T : X \times X \times X \to \mathbf{R}^+$. Basically, 2-metric means the area of a triangle in $\mathbf{R}$.

In this paper we have introduced T-Hardy Rogers contractive condition and F-contraction in 2-metric spaces.

M. Abbas et. al [1], Altum et. al [2] uses T- Hardy Rogers condition to prove fixed point theorem in various spaces other than 2-metric spaces.

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Purpose of this paper is to prove some common fixed point theorem in 2-metric spaces. We also gave some corollaries as T-Reich, T-Chaterjea and T-Kannan.

In our next theorem we use F-contraction mapping in 2-metric spaces, which is also a new concept in 2-metric spaces. We use F-contraction mapping on Hardy Rogers type mapping. Bhutia [3], Minak et. al [11], Piri and Kumam [13], Sgroi and Vetro[16], Udo-utun[17], uses F-contraction in complete metric space. And Vetro [18] uses F-contraction of Hardy Rogers type in multistage decision processes. Our theorems are generalization of many existing results in this literature.

In our last theorem, we have used F-contraction in 2-metric spaces to prove the existing theorem in complete metric spaces and also have generalized them.

To read about T-Hardy-Rogers type condition, please see [1],[2],[14],[18], [19] and to read about F-contraction, please see [3], [10], [11],[13], [15],[16]-[18].

2. Prelimimaries and Definitions

Definition 2.1. (2-Metric space)[6]
Let X be a non-empty set and $d : X \times X \times X \to [0, \infty)$ is a real valued function which satisfied the following conditions:

i) for every distinct points x, y there is a point z in X such that $d(x, y, z) \neq 0$;

ii) $d(x, y, z) = 0$ if any two of three of x, y, z are equal;

iii) $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$ and for all permutations $p(x, y, z)$ of x, y, z;

iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

Then, d is called a 2-metric and (X, d) is called a 2-metric space.
In this paper we write X as a 2-metric space unless otherwise stated.

Definition 2.2. (Cauchy sequence in 2-Metric Space)
A sequence $\{x_n\}$ is said to be a Cauchy sequence if $d(x_n, x_m, a) \to 0$ as $n, m \to \infty$.

Definition 2.3.(Complete 2-Metric Space)
A space X is said to be complete if every Cauchy sequence converges in X.

Definition 2.4.(Function τ)[[18],[19]]
Consider the function $\tau : \mathbf{R}^+ \to \mathbf{R}^+$ with the following property:

$$\lim inf_{t \to s^+} \tau(t) > 0; \forall s \geq 0.$$

Definition 2.5.(T-Hardy Rogers Contractive condition in 2-Metric space)
Let (X, d) be a 2-metric space and $T, f : X \to X$ be self maps. Then f is said to satisfy T-Hardy Rogers contractive condition, if
$d(Tfx, Tfy, a) \leq a_1 d(Tx, Ty, a) + a_2 d(Tx, Tfx, a) + a_3 d(Ty, Tfy, a)$
$+ a_4 d(Tx, Tfy, a) + a_5 d(Ty, Tfx, a)$
for all $a_i \geq 0$, $\sum_{i=1}^{5} a_i < 1$, $\forall\, x, y, a \in X$.

Definition 2.6.(F-Contraction)
Wardowski[19] defined F- contraction on complete metric space.
We have defined F-contraction on 2-metric space as follows:

Let(X, d) be a 2-metric space. A mapping $T : X \to X$ is said to be an $F-$ *contraction* if there exists $\tau > 0$ such that for all $x, y, a \in X$, with

$$d(Tx, Ty, a) > 0 \Rightarrow \tau + F(d(Tx, Ty, a)) < F(d(x, y, a)),$$

where F satisfies following conditions:

(F1) F is strictly increasing;
(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exist $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
Note: Wardowski [[19], Remark 2.1] says that every $F-$ *contraction* mapping is a contractive mapping and is continuous. This result also holds in 2-Metric Space.

J.D. Bhutia [3] used weakly compatible mappings in metric space. We generalized it in 2-Metric space as following:
Definition 2.7. (Weakly Compatible mappings in 2-Metric Space)
Two self mappings f and g are said to be Weakly Compatible in 2-Metric Space if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

3. Main Results

Theorem 3.1. Let(X,d) be a complete 2- metric space and $\{T_i^{m_i}\}$ be a family of self maps on X satisfying
$d(T_i^{m_i}x, T_j^{m_j}y, a)$
$\leq a_1 d(x,y,a) + a_2 d(x, T_i^{m_i}x, a) + a_3 d(y, T_j^{m_j}y, a) + a_4 d(x, T_j^{m_j}y, a) + a_5 d(y, T_i^{m_i}x, a)$
for all $x, y, a \in X$ and for all $a_i \geq 0$ and $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$.
Then the family of maps $\{T_i^{m_i}\}_{i=1}^{\infty}$ have unique common fixed point in X.

Proof. Putting $f_i = T_i^{m_i}$, we have from given condition
$d(f_i x, f_j y, a) \leq a_1 d(x,y,a) + a_2 d(x, f_i x, a) + a_3 d(y, f_j y, a) + a_4 d(x, f_j y, a) + a_5 d(y, f_i x, a)$.
Let us choose a sequence $\{x_n\}$ for a fixed $x_0 \in X$ by $x_n = f_n x_{n-1}$ where $n = 1, 2, 3, ...$
Now, $d(x_{n+1}, x_n, a) = d(f_{n+1}x_n, f_n x_{n-1}, a)$
$\leq a_1 d(x_n, x_{n-1}, a) + a_2 d(x_n, f_{n+1}x_n, a) + a_3 d(x_{n-1}, f_n x_{n-1}, a)$
$+ a_4 d(x_n, f_n x_{n-1}, a) + a_5 d(x_{n-1}, f_{n+1}x_n, a)$
$= a_1 d(x_n, x_{n-1}, a) + a_2 d(x_n, x_{n+1}, a) + a_3 d(x_{n-1}, x_n, a)$
$+ a_4 d(x_n, x_n, a) + a_5 d(x_{n-1}, x_{n+1}, a)$
$\leq (a_1 + a_3) d(x_n, x_{n-1}, a) + a_2 d(x_n, x_{n+1}, a) + a_5 [d(x_{n-1}, x_{n+1}, x_n)$
$+ d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a)]$,

$$(1 - a_2 - a_5) d(x_{n+1}, x_n, a) \leq (a_1 + a_3 + a_5) d(x_n, x_{n-1}, a) + a_5 d(x_{n-1}, x_n, x_{n+1}). \quad (1)$$

Putting $a = x_{n-1}$ in (1) we get
$(1 - a_2 - a_5) d(x_{n+1}, x_n, x_{n-1}) \leq (a_1 + a_3 + a_5) d(x_n, x_{n-1}, x_{n-1}) + a_5 d(x_{n-1}, x_n, x_{n+1})$
Or, $(1 - a_2 - 2a_5) d(x_{n+1}, x_n, x_{n-1}) \leq 0$
Or, $d(x_{n+1}, x_n, x_{n-1}) = 0$ [since $(1 - a_2 - 2a_5) \neq 0$].
Therefore,
$d(x_{n+1}, x_n, a)$
$\leq k d(x_n, x_{n-1}, a), k = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} < 1$ [since $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$]
$\leq k^2 d(x_{n-1}, x_{n-2}, a)$
.
.
.
$\leq k^n d(x_1, x_0, a)$
$\Rightarrow \lim_{n\to\infty} d(x_{n+1}, x_n, a) = 0$.
For $n > m$,
$d(x_n, x_m, a) \leq d(x_n, x_m, x_{n-1}) + d(x_n, x_{n-1}, a) + d(x_{n-1}, x_m, a)$.

Taking $\lim_{n,m\to\infty}$ on both sides we have,
$\lim_{n,m\to\infty} d(x_n, x_m, a)$
$\leq \lim_{n,m\to\infty} d(x_n, x_m, x_{n-1}) + \lim_{n,m\to\infty} d(x_n, x_{n-1}, a) + \lim_{n,m\to\infty} d(x_{n-1}, x_m, a)$
$= \lim_{n,m\to\infty} d(x_{n-1}, x_m, a)$
.
.
.
$= \lim_{n,m\to\infty} d(x_m, x_m, a)$
$= 0.$

This shows that $\{x_n\}$ is a Cauchy sequence in X.
Since X is complete, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$.
Now we have to show x is a fixed point of the family $\{T_i^{m_i}\}$.
We have,

$\lim_{n\to\infty} d(T_n^{m_n}x, x, a) = \lim_{n\to\infty} d(f_n x, x, a)$
$\leq \lim_{n\to\infty}[d(f_n x, x, x_n) + d(f_n x, x_n, a) + d(x_n, x, a)]$
$= \lim_{n\to\infty} d(f_n x, f_n x_{n-1}, a)$
$\leq \lim_{n\to\infty}[a_1 d(x, x_{n-1}, a) + a_2 d(x, f_n x, a) + a_3 d(x_{n-1}, f_n x_{n-1}, a) + a_4 d(x, f_n x_{n-1}, a) + a_5 d(x_{n-1}, f_n x, a)]$
$= \lim_{n\to\infty}[a_2 d(x, f_n x, a) + a_3 d(x_{n-1}, f_n x_{n-1}, a) + a_4 d(x, f_n x_{n-1}, a) + a_5 d(x_{n-1}, f_n x, a)]$
$= \lim_{n\to\infty}[a_2 d(x, f_n x, a) + a_3 d(x_{n-1}, x_n, a) + a_4 d(x, x_n, a) + a_5 d(x_{n-1}, f_n x, a)]$
$\leq \lim_{n\to\infty}[a_2[d(x, f_n x, x_n) + d(x, x_n, a) + d(x_n, f_n x, a)] + a_5[d(x_{n-1}, f_n x, x_n) + d(x_{n-1}, x_n, a) + d(x_n, f_n x, a)]]$
$= \lim_{n\to\infty}[a_2 d(x_n, f_n x, a) + a_5 d(x_n, f_n x, a)]$
$= \lim_{n\to\infty}(a_2 + a_5) d(x_n, f_n x, a)$
$\leq \lim_{n\to\infty}(a_2 + a_5)[d(x_n, f_n x, x) + d(x_n, x, a) + d(x, f_n x, a)]$
$= \lim_{n\to\infty}(a_2 + a_5) d(f_n x, x, a)$
$= \lim_{n\to\infty}(a_2 + a_5) d(T_n^{m_n}x, x, a).$
$\Rightarrow \lim_{n\to\infty}(1 - a_2 - a_5) d(T_n^{m_n}x, x, a) = 0$
$\Rightarrow \lim_{n\to\infty} d(T_n^{m_n}x, x, a) = 0$ [since $1 - a_2 - a_5 \neq 0$]
$\Rightarrow T_n^{m_n}x = x.$
So, x is a unique common fixed point of the family $\{T_i^{m_i}\}$.
To show the uniqueness, let y be an another fixed point of $T_n^{m_n}$.

Then $d(x, y, a)$
$= d(T_i^{m_i}x, T_j^{m_j}y, a)$
$\leq a_1 d(x, y, a) + a_2 d(x, T_i^{m_i}x, a) + a_3 d(y, T_j^{m_j}y, a) + a_4 d(x, T_j^{m_j}y, a) + a_5 d(y, T_i^{m_i}x, a)$
$= a_1 d(x, y, a) + a_2 d(x, x, a) + a_3 d(y, y, a) + a_4 d(x, y, a) + a_5 d(y, x, a)$
$\Rightarrow (1 - a_1 - a_4 - a_5) d(x, y, a) \leq 0$
$\Rightarrow d(x, y, a) = 0$ [since$(1 - a_1 - a_4 - a_5) \neq 0$]
$\Rightarrow x = y \ \forall a \in X.$
Therefore, x is a common fixed point of $\{T_i^{m_i}\}_{i=1}^{\infty}$.

Example 3.1. Let $X = [0, 1)$ and $d : X \times X \times X \to [0, \infty)$ defined by

$$d(x, y, z) = min\{|x - y|, |y - z|, |z - x|\}$$

and $T_n : X \to X$ be a self maps where $T_n x = x^n$.
Then clearly (X, d) is a 2-metric space.

Again,
$d(T_n x, x, a) = d(x^n, x, a) = min\{|x^n - x|, |x - a|, |a - x^n|\}$.

Therefore,
$\lim_{n\to\infty} d(T_n x, x, a) = \lim_{n\to\infty} \min\{x, |x-a|, a\} = 0$ iff $x = 0$ *or*, $x = a$.

Thus for all $a \in X$, $\lim_{n\to\infty} \min\{|x^n - x|, |x-a|, |a-x^n|\} = 0$, iff $x = 0$.
If $x = 0$, then $T_n x = 0^n = 0$.
Thus 0 is the only fixed point of $\{T_n\}$.

Theorem 3.2. Let, (X, d) be a complete 2-metric space and $T : X \to X$ be a injective mapping and $f_1, f_2 : X \to X$ are non-decreasing satisfying T-Hardy-Rogers condition
$d(Tf_1x, Tf_2y, a)$
$\leq a_1 d(Tx, Ty, a) + a_2 d(Tx, Tf_1x, a) + a_3 d(Ty, Tf_2y, a) + a_4 d(Tx, Tf_2y, a) + a_5 d(Ty, Tf_1x, a)$
where $a_i \geq 0$ and $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$ for all $x, y, a \in X$.
Then, f_1 and f_2 have a unique common fixed point in X.
Proof. Let, x_0 be an arbitrary but fixed in X and $\{x_n\}$ be a sequence in X such that $x_n = f_1 x_{n-1}, x_{n-1} = f_2 x_{n-2}$.
Therefore by the given condition we have
$d(Tx_n, Tx_{n-1}, a) = d(Tf_1x_{n-1}, Tf_2x_{n-2}, a)$
$\leq a_1 d(Tx_{n-1}, Tx_{n-2}, a) + a_2 d(Tx_{n-1}, Tf_1x_{n-1}, a) + a_3 d(Tx_{n-2}, Tf_2x_{n-2}, a)$
$+ a_4 d(Tx_{n-1}, Tf_2x_{n-2}, a) + a_5 d(Tx_{n-2}, Tf_1x_{n-1}, a)$
$= a_1 d(Tx_{n-1}, Tx_{n-2}, a) + a_2 d(Tx_{n-1}, Tx_n, a) + a_3 d(Tx_{n-2}, Tx_{n-1}, a)$
$+ a_4 d(Tx_{n-1}, Tx_{n-1}, a) + a_5 d(Tx_{n-2}, Tx_n, a)$
which implies,

$(1 - a_2) d(Tx_n, Tx_{n-1}, a)$
$\leq (a_1 + a_3) d(Tx_{n-1}, Tx_{n-2}, a) + a_5 d(Tx_{n-2}, Tx_n, a)$
$\leq (a_1 + a_3) d(Tx_{n-1}, Tx_{n-2}, a) + a_5 [d(Tx_{n-2}, Tx_n, Tx_{n-1}) + d(Tx_{n-2}, Tx_{n-1}, a) +$
$d(Tx_{n-1}, Tx_n, a)]$

Or,

$$(1-a_2-a_5) d(Tx_n, Tx_{n-1}, a) \leq (a_1+a_3+a_5) d(Tx_{n-1}, Tx_{n-2}, a) + a_5 d(Tx_{n-2}, Tx_n, Tx_{n-1}). \quad (2)$$

Now,
$d(Tx_{n-2}, Tx_n, Tx_{n-1})$
$= d(Tx_n, Tx_{n-1}, Tx_{n-2})$
$= d(Tf_1x_{n-1}, Tf_2x_{n-2}, Tx_{n-2})$
$\leq a_1 d(Tx_{n-1}, Tx_{n-2}, Tx_{n-2}) + a_2 d(Tx_{n-1}, Tf_1x_{n-1}, Tx_{n-2}) + a_3 d(Tx_{n-2}, Tf_2x_{n-2},$
$Tx_{n-2}) + a_4 d(Tx_{n-1}, Tf_2x_{n-2}, Tx_{n-2}) + a_5 d(Tx_{n-2}, Tf_1x_{n-1}, Tx_{n-2})$
$= a_2 d(Tx_{n-1}, Tx_n, Tx_{n-2}) + a_3 d(Tx_{n-2}, Tx_{n-1}, Tx_{n-2})$
$+ a_4 d(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) + a_5 d(Tx_{n-2}, Tx_n, Tx_{n-2})$
$= 0$
Or, $(1 - a_2) d(Tx_{n-2}, Tx_n, Tx_{n-1}) = 0$
Or, $d(Tx_{n-2}, Tx_n, Tx_{n-1}) = 0$ [since $(1 - a_2) \neq 0$]

From (2) we have $(1-a_2-a_5) d(Tx_n, Tx_{n-1}, a) \leq (a_1+a_3+a_5) d(Tx_{n-1}, Tx_{n-2}, a)$

$\Rightarrow d(Tx_n, Tx_{n-1}, a) \leq \frac{a_1+a_3+a_5}{1-a_2-a_5} d(Tx_{n-1}, Tx_{n-2}, a)$
$= \alpha d(Tx_{n-1}, Tx_{n-2}, a)$ [where $\alpha = \frac{a_1+a_3+a_5}{1-a_2-a_5}$]
$= \alpha^2 d(Tx_{n-2}, Tx_{n-3}, a)$
.
.
.

$= \alpha^n d(Tx_1, Tx_0, a)$
$\to 0$ as $n \to \infty$ [since $\alpha = \frac{a_1+a_3+a_5}{1-a_2-a_5} < 1$]
Or, $d(Tx_n, Tx_{n-1}, a) = 0$ as $n \to \infty$.

Since, T is a continuous (being $F-contraction$), we have $d(x_n, x_{n-1}, a) = 0$ as $n \to \infty$.

Let $n > m$,
$\lim_{n,n\to\infty} d(Tx_n, Tx_m, a)$
$\leq \lim_{n,n\to\infty}[d(Tx_n, Tx_m, Tx_{n-1}) + d(Tx_n, Tx_{n-1}, a) + d(Tx_{n-1}, Tx_m, a)]$
$= \lim_{n,n\to\infty} d(Tx_{n-1}, Tx_m, a)$
.
.
.
$\leq \lim_{n,n\to\infty} d(Tx_m, Tx_m, a) = 0$
i.e., $\lim_{n,n\to\infty} d(Tx_n, Tx_m, a) = 0$
$\Rightarrow \lim_{n,n\to\infty} d(x_n, x_m, a) = 0$.
Therefore, $\{x_n\}$ is a cauchy sequence in X.

Since X is complete, there exists $x \in X$ such that $d(x_n, x, a) = 0$ as $n \to \infty$, i.e.,

$\lim_{n\to\infty} x_n = x; x \in X$.

Now

$\lim_{n\to\infty} d(f_1x, x, a)$
$\leq \lim_{n\to\infty}[d(f_1x, x, x_n) + d(f_1x, x_n, a) + d(x_n, x, a)] = \lim_{n\to\infty} d(f_1x, x_n, a)$
$= \lim_{n\to\infty} d(Tf_1x, Tx_n, a)$ [Since, T is injective]
$= \lim_{n\to\infty} d(Tf_1x, Tf_2x_{n-1}, a)$
$\leq \lim_{n\to\infty}[a_1d(Tx, Tx_{n-1}, a) + a_2d(Tx, Tf_1x, a) + a_3d(Tx_{n-1}, Tf_2x_{n-1}, a)$
$+ a_4d(Tx, Tf_2x_{n-1}, a) + a_5d(Tx_{n-1}, Tf_1x, a)]$
$= \lim_{n\to\infty}[a_1d(Tx, Tx_{n-1}, a) + a_2d(Tx, Tf_1x, a) + a_3d(Tx_{n-1}, Tx_n, a)$
$+ a_4d(Tx, Tx_n, a) + a_5d(Tx_{n-1}, Tf_1x, a)]$
$= \lim_{n\to\infty}[a_1d(x, x_{n-1}, a) + a_2d(x, f_1x, a) + a_3d(x_{n-1}, x_n, a)$
$+ a_4d(x, x_n, a) + a_5d(x_{n-1}, f_1x, a)]$
Or, $\lim_{n\to\infty}(1 - a_2)d(f_1x, x, a) \leq \lim_{n\to\infty} a_5d(x_{n-1}, f_1x, a)$
$\leq \lim_{n\to\infty} a_5[d(x_{n-1}, f_1x, x) + d(x_{n-1}, x, a) + d(x, f_1x, a)]$
Or, $\lim_{n\to\infty}(1 - a_2 - a_5)d(f_1x, x, a) \leq 0$
Or,$\lim_{n\to\infty} d(f_1x, x, a) = 0$ [since $1 - a_2 - a_5 \neq 0$]
Or, $f_1x = x$.
So, x is fixed point of f_1.
Again,

$\lim_{n\to\infty} d(x, f_2x, a) \leq \lim_{n\to\infty}[d(x, f_2x, x_n) + d(x, x_n, a) + d(x_n, f_2x, a)]$
$= \lim_{n\to\infty} d(x_n, f_2x, a) = \lim_{n\to\infty} d(Tx_n, Tf_2x, a)$ [since T is injective]
$= \lim_{n\to\infty} d(Tf_1x_{n-1}, Tf_2x, a)$
$\leq \lim_{n\to\infty}[a_1d(Tx_{n-1}, Tx, a) + a_2d(Tx_{n-1}, Tf_1x_{n-1}, a) + a_3d(Tx, Tf_2x, a)$
$+ a_4d(Tx_{n-1}, Tf_2x, a) + a_5d(Tx, Tf_1x_{n-1}, a)]$
$= \lim_{n\to\infty}[a_1d(Tx_{n-1}, Tx, a) + a_2d(Tx_{n-1}, Tx_n, a) + a_3d(Tx, Tf_2x, a)$
$+ a_4d(Tx_{n-1}, Tf_2x, a) + a_5d(Tx, Tx_n, a)]$
$= \lim_{n\to\infty}[a_1d(x_{n-1}, x, a) + a_2d(x_{n-1}, x_n, a) + a_3d(x, f_2x, a) + a_4d(x_{n-1}, f_2x, a) + a_5d(x, x_n, a)]$
Or, $\lim_{n\to\infty}(1 - a_3)d(x, f_2x, a) \leq \lim_{n\to\infty} a_4d(x_{n-1}, f_2x, a)$

$\leq \lim_{n\to\infty} a_4[d(x_{n-1}, f_2x, x) + d(x_{n-1}, x, a) + d(x, f_2x, a)]$
$\Rightarrow (1 - a_3 - a_4)d(x, f_2x, a) \leq 0$
Or, $d(x, f_2x, a) = 0$ [since $1 - a_3 - a_4 \neq 0$]
Or, $f_2x = x$.
Thus x is also a fixed point of f_2 and so x is a common fixed point of f_1 and f_2.
Let, y be an another common fixed point of f_1 and f_2.
Since T is injective and x, y are fixed point of f_1 and f_2, we have
$d(x, y, a) = d(f_1x, f_2y, a) = d(Tf_1x, Tf_2y, a)$
$\leq a_1d(Tx, Ty, a) + a_2d(Tx, Tf_1x, a) + a_3d(Ty, Tf_2y, a) + a_4d(Tx, Tf_2y, a)$
$+ a_5d(Ty, Tf_1x, a)$
$= a_1d(x, y, a) + a_2d(x, x, a) + a_3d(y, y, a) + a_4d(x, y, a) + a_5d(y, x, a)$
i.e., $(1 - a_1 - a_4 - a_5)d(x, y, a) \leq 0$
i.e., $d(x, y, a) = 0$ (as $1 - a_1 - a_4 - a_5 \neq 0$).
So $x = y$.

Therefore, f_1 and f_2 have a unique common fixed point in X.

Corollary 3.1. If $f_1 = f_2 = f$ where $f : X \to X$ be non-decreasing and $T : X \to X$ be a injective mapping in a complete 2-metric space (X, d) satistfying T-Hardy-Rogers condition
$d(Tfx, Tfy, a)$
$\leq a_1d(Tx, Ty, a) + a_2d(Tx, Tfx, a) + a_3d(Ty, Tfy, a) + a_4d(Tx, Tfy, a)$
$+ a_5d(Ty, Tfx, a)$
and for all $a_i \geq 0, a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$ for all $x, y, a \in X$, then T has a unique common fixed point in X.

Note 3.1. T-Reich type contractive condition on 2-metric space

Let, (X, d) be a complete 2-metric space and $T : X \to X$ be a continuous, injective mapping and $f_1, f_2 : X \to X$ are non-decreasing satisfying T-Reich condition
$d(Tf_1x, Tf_2y, a)$
$\leq a_1d(Tx, Ty, a) + a_2d(Tx, Tf_1x, a) + a_3d(Ty, Tf_2y, a)$ where $a_i \geq 0$ and $a_1 + a_2 + a_3 < 1$ for all $x, y, a \in X$.
Then, f_1 and f_2 have a unique common fixed point.

Note 3.2. T-Chaterjea type contractive condition on 2-metric space

Let (X, d) be a complete 2-metric space and $T : X \to X$ be a continuous, injective mapping and $f_1, f_2 : X \to X$ are non-decreasing satisfying T-Chaterjea condition
$d(Tf_1x, Tf_2y, a) \leq a_4d(Tx, Tf_2y, a) + a_5d(Ty, Tf_1x, a)$ where $a_i \geq 0$ and $a_4 + 2a_5 < 1$ for all $x, y, a \in X$.
Then, f_1 and f_2 have a unique common fixed point.

Note 3.3. T-Kannan type contractive condition on 2-metric space

Let (X, d) be a complete 2-metric space and $T : X \to X$ be a continuous, injective mapping and $f_1, f_2 : X \to X$ are non-decreasing satisfying T-Kannan condition
$d(Tf_1x, Tf_2y, a) \leq a_2d(Tx, Tf_1x, a) + a_3d(Ty, Tf_2y, a)$ where $a_i \geq 0$ and $a_2 + a_3 < 1$ for all $x, y, a \in X$.
Then, f_1 and f_2 have a unique common fixed point.

Note 3.4. T-Banach type contractive condition on 2-metric space

Let, (X, d) be a complete 2-metric space and $T : X \to X$ be a continuous, injective mapping and $f_1, f_2 : X \to X$ are non-decreasing satisfying T-Banach condition $d(Tf_1x, Tf_2y, a) \leq a_1 d(Tx, Ty, a)$ where $a_1 \geq 0$ and $a_1 < 1$ for all $x, y, a \in X$. Then f_1 and f_2 have a unique common fixed point.

Theorem 3.3. Let (X, d) be a complete 2-metric space and $\{T_i^{m_i}\}_{i=1}^{\infty}$ be a family of self maps on X satisfying $F-contraction$ and the relation

$$\tau(d(x, y, a)) + F(d(T_i^{m_i}x, T_j^{m_j}y, a)) \leq F(M(x, y, a))$$

where F is continuous and $M(x, y, a) = a_1 d(x, y, a) + a_2 d(x, T_i^{m_i}x, a) + a_3 d(y, T_j^{m_j}y, a) + a_4 d(x, T_j^{m_j}y, a) + a_5 d(y, T_i^{m_i}x, a)$, and $a_p \geq 0, p = 1, 2, 3, 4, 5.$, $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$.
Then the family of self maps have a unique common fixed point in X.
Proof. Consider the sequence $\{x_n\}$ for a fixed $x_0 \in X$ such that $x_{n+1} = T_j^{m_j}x_n, x_n = T_i^{m_i}x_{n-1}$.

If $x_{n+1} = x_n$, then $T_i^{m_i}$ and $T_j^{m_j}$ have common fixed point.

Suppose $x_{n+1} \neq x_n \Rightarrow d(x_{n+1}, x_n, a) > 0$ for all $a \in X$.
Let $f_i = T_i^{m_i}$ and $f_j = T_j^{m_j}$.
Since f_i and f_j are $F-contraction$, we have for $\tau(d(x_{n+1}, x_n, a)) > 0$,

$$\tau(d(x_{n+1}, x_n, a)) + F(d(f_i x_{n+1}, f_j x_n, a)) \leq F(M(x_{n+1}, x_n, a)), \tag{3}$$

where,
$M(x_{n+1}, x_n, a)$
$= a_1 d(x_{n+1}, x_n, a) + a_2 d(x_{n+1}, f_i x_{n+1}, a) + a_3 d(x_n, f_j x_n, a) + a_4 d(x_{n+1}, f_j x_n, a)$
$+ a_5 d(x_n, f_i x_{n+1}, a)$
$= a_1 d(x_{n+1}, x_n, a) + a_2 d(x_{n+1}, x_{n+2}, a) + a_3 d(x_n, x_{n+1}, a) + a_4 d(x_{n+1}, x_{n+1}, a) + a_5 d(x_n, x_{n+2}, a)$
$\leq a_1 d_{n+1} + a_2 d_{n+2} + a_3 d_{n+1} + a_5\{d(x_n, x_{n+1}, x_{n+1}) + d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a)\}$

$$= (a_1 + a_3 + a_5)d_{n+1} + (a_2 + a_5)d_{n+2} + a_5 d(x_n, x_{n+2}, x_{n+1}), \tag{4}$$

where,
$d_{n+1} = d(x_{n+1}, x_n, a)$
We claim that $d(x_{n+2}, x_{n+1}, x_n) = 0$.
Suppose $d(x_{n+2}, x_{n+1}, x_n) \neq 0 \ \forall \ n$.

Putting $a = x_n$ in (3) and in (4) we have

$$\tau(d(x_{n+1}, x_n, x_n)) + F(d(f_i x_{n+1}, f_j x_n, x_n)) \leq F(M(x_{n+1}, x_n, x_n))$$

Or, $\tau(0) + F(d(x_{n+2}, x_{n+1}, x_n))$
$\leq F((a_2 + 2a_5)d(x_{n+2}, x_{n+1}, x_n))$
$< F(d(x_{n+2}, x_{n+1}, x_n))$ [since $a_2 + 2a_5 < 1$]
Or, $\tau(0) < 0$,
which is a contradiction as $\tau(s) > 0 \, \forall s \geq 0$.
Thus $d(x_{n+2}, x_{n+1}, x_n) = 0$.
So from (3) we have,

$\tau(d(x_{n+1}, x_n, a)) + F(d(f_i x_{n+1}, f_j x_n, a)) \leq F((a_1 + a_3 + a_5)d_{n+1} + (a_2 + a_5)d_{n+2})$

Now, either $d_{n+1} > d_{n+2}$ or $d_{n+1} < d_{n+2}$.
Let us first suppose that $d_{n+1} < d_{n+2}$.
Then from (3),
$\tau(d(x_{n+1}, x_n, a)) + F(d(f_i x_{n+1}, f_j x_n, a)) \leq F((a_1 + a_2 + a_3 + 2a_5)d_{n+2})$
Or, $\tau(d_{n+1}) + F(d(x_{n+2}, x_{n+1}, a)) \leq F((a_1 + a_2 + a_3 + 2a_5)d_{n+2})$
Or, $\tau(d_{n+1}) + F(d_{n+2}) < F(d_{n+2})$
Or, $\tau(d_{n+1}) < 0$,
which is a contradiction as $\tau(s) > 0 \, \forall s \geq 0$.
Therefore, $d_{n+1} > d_{n+2}$.

Let $\tau(d_n) \geq r > 0 \ \forall n \in \mathbf{N}$.
Then we have,

$$\tau(d_{n+1}) + F(d_{n+2}) < F(d_{n+1})$$

Or, $F(d(x_{n+2}, x_{n+1}, a)) < F(d(x_{n+1}, x_n, a)) - r < F(d(x_n, x_{n-1}, a)) - 2r$
$< F(d(x_{n-1}, x_{n-2}, a)) - 3r < ... < F(d(x_1, x_0, a)) - nr$
Taking limit as $n \to \infty$ and using (F2), we have $\lim_{n\to\infty} F(d(x_{n+2}, x_{n+1}, a)) = -\infty$.
So, from the definition of F-contraction we have,

$$\lim_{n\to\infty} d(x_{n+2}, x_{n+1}, a) = 0. \quad (5)$$

Now we have to show $\{x_n\}$ is a Cauchy sequence.

Let $n > m$, then $d(x_n, x_m, a) \leq d(x_n, x_m, x_{n-1}) + d(x_n, x_{n-1}, a) + d(x_{n-1}, x_m, a)$.
Taking $\lim_{n,m\to\infty}$ on both sides we have,
$\lim_{n,m\to\infty} d(x_n, x_m, a) \leq \lim_{n,m\to\infty} d(x_n, x_m, x_{n-1}) + \lim_{n,m\to\infty} d(x_n, x_{n-1}, a)$
$+ \lim_{n,m\to\infty} d(x_{n-1}, x_m, a)$

$$= \lim_{n,m\to\infty} d(x_{n-1}, x_m, a)[\text{using}(5)]$$

$$\vdots$$

$$= \lim_{n,m\to\infty} d(x_m, x_m, a) = 0.$$

Therefore $\{x_n\}$ is a Cauchy sequence in X.
Since, X is complete, there exists a $x \in X$ such that $d(x_n, x, a) = 0$ as $n \to \infty \forall \, x \in X$.
Clearly,
$\lim_{n,m\to\infty} M(x_n, x_m, a)$
$= \lim_{n,m\to\infty}[a_1 d(x_n, x_m, a) + a_2 d(x_{n+1}, f_i x_{n+1}, a) + a_3 d(x_n, f_j x_n, a)$
$+ a_4 d(x_{n+1}, f_j x_n, a) + a_5 d(x_n, f_i x_{n+1}, a)]$
$= \lim_{n,m\to\infty}[a_1.0 + a_2 d(x_n, x_{n+1}, a) + a_3 d(x_m, x_{m+1}, a) + a_4 d(x_n, x_{m+1}, a)$
$+ a_5 d(x_m, x_{n+1}, a)]$
$= 0.$
Therefore from (3) we have $\lim_{n,m\to\infty}[\tau(d(x_n, x_m, a)) + F(d(f_i x_n, f_j x_m, a))] \leq \lim_{n,m\to\infty} F(M(x_n, x_m, a))$
Or, $\lim_{n,m\to\infty}[\tau(0) + F(d(f_i x_n, f_j x_m, a))] \leq F(0)$.
Since,

$$\lim_{n,m\to\infty} F(d(f_i x_n, f_j x_m, a)) \leq \lim_{n,m\to\infty}[\tau(0) + F(d(f_i x_n, f_j x_m, a))] \leq F(0)$$

Or, $\lim_{n,m\to\infty} d(f_i x_n, f_j x_m, a) = 0$
Or, $\lim_{n\to\infty} f_i x_n = \lim_{m\to\infty} f_j x_m$.
Therefore,

$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f_i x_n = x = \lim_{m\to\infty} f_j x_m = \lim_{m\to\infty} x_{m+1}$.
Thus x is a common fixed point of f_i and f_j i.e., of $T_i^{m_i}$ and $T_j^{m_j}$.
To show x is a unique common fixed point, let x' be another fixed point of $T_i^{m_i}$ and $T_j^{m_j}$.
If possible let $d(x, x', a) \neq 0$. Then,

$$\tau(d(x, x', a)) + F(d(f_i x, f_j x', a)) \leq F(M(x, x', a)), \tag{6}$$

where,
$M(x, x', a) = a_1 d(x, x', a) + a_2 d(x, f_i x, a) + a_3 d(x', f_j x', a)$
$+ a_4 d(x, f_j x', a) + a_5 d(x', f_i x, a)$
$= a_1 d(x, x', a) + a_2 d(x, x, a) + a_3 d(x', x', a) + a_4 d(x, x', a) + a_5 d(x', x, a)$
$= (a_1 + a_4 + a_5) d(x', x, a)$.
So from (6) we get, $\tau(d(x, x', a)) + F(d(f_i x, f_j x', a)) \leq F((a_1 + a_4 + a_5) d(x', x, a))$
Or, $\tau(d(x, x', a)) + F(d(x, x', a)) \leq F((a_1 + a_4 + a_5) d(x', x, a)) < F(d(x, x', a))$
Or, $\tau(d(x, x', a)) < 0$, which is a contradiction.
Thus $d(x, x', a) = 0$ i.e., $x = x'$.
Therefore x is a unique fixed point of $T_i^{m_i}$ and $T_j^{m_j}$.

Corollary 3.2.
Let, (X, d) be a complete 2-metric space and $T_1, T_2 : X \to X$ are $F-contraction$ and the condition $\tau(d(x, y, a)) + F(d(T_1 x, T_2 y, a)) \leq F(M(x, y, a))$
where,

$$M(x, y, a) = a_1 d(x, y, a) + a_2 d(x, T_1 x, a) + a_3 d(y, T_2 y, a) + a_4 d(x, T_2 y, a) + a_5 d(y, T_1 x, a),$$

$a_p \geq 0, p = 1, 2, 3, 4, 5$ and $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$.
Then, T_1 and T_2 have a unique common fixed point in X.
Proof. Take $T_i^{m_i} = T_1$ and $T_j^{m_j} = T_2$, in Theorem 3.3, the result holds.

Theorem 3.4. Let (X, d) be a complete 2-metric space and f_1, f_2, g_1, g_2 are continuous self-maps on X such that $f_1(X) \subseteq g_1(X), f_2(X) \subseteq g_2(X)$ and $g_1(X), g_2(X)$ are closed sets of X. Also (f_1, g_1) and (f_2, g_2) are weakly compatible. F be a continuous function satisfying (F1) and (F2) such that
$\tau + F(d(f_1 x, f_2 y, a)) \leq F(d((g_1 x, g_2 y, a)); \forall\ x, y, a \ \in X; d(f_1 x, f_2 y, a) > 0$.
Then, f_1, f_2, g_1, g_2 have a unique common fixed point in X.
Proof. Let, x_0 be any arbitrary point of X.
Since $f_1(X) \subseteq g_1(X), f_2(X) \subseteq g_2(X)$.
Then there exists points x_1 and x_2 such that

$$f_1 x_0 = g_1 x_1, f_2 x_1 = g_2 x_2.$$

Thus we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$y_n = f_1 x_n = g_1 x_{n+1};\ y_{n+1} = f_2 x_{n+1} = g_2 x_{n+2}.$$

Let us first show that $\{y_n\}$ is a Cauchy sequence.
Since, $f_1 x_n \neq f_2 x_{n+1} \Rightarrow d(f_1 x_n, f_2 x_{n+1}, a) > 0$
Therefore there exists a $\tau > 0$ such that $\tau + F(d(y_n, y_{n+1}, a))$
$= \tau + F(d(f_1 x_n, f_2 x_{n+1}, a)) \leq F(d(g_1 x_n, g_2 x_{n+1}, a)) = F(d(y_{n-1}, y_n, a))$
i.e.,
$F(d(y_n, y_{n+1}, a)) \leq F(d(y_{n-1}, y_n, a)) - \tau \leq F(d(y_{n-2}, y_{n-1}, a)) - 2\tau \leq ... \leq F(d(y_0, y_1, a)) - n\tau$.

Taking $\lim_{n\to\infty}$ and using (F2), we have
$\lim_{n\to\infty} F(d(y_n, y_{n+1}, a)) = -\infty$
$\Rightarrow \lim_{n\to\infty} d(y_n, y_{n+1}, a) = 0$.

Let $n > m$.
Therefore, $\lim_{n,m\to\infty} d(y_n, y_m, a) \leq \lim_{n,m\to\infty}[d(y_n, y_m, y_{n-1}) + d(y_n, y_{n-1}, a) + d(y_{n-1}, y_m, a)] = \lim_{n,m\to\infty} d(y_{n-1}, y_m, a) \leq ... \leq \lim_{n,m\to\infty} d(y_m, y_m, a) = 0$.
Therefore,

$\lim_{n,m\to\infty} d(y_n, y_m, a) = 0 \Rightarrow \{y_n\}$ is a Cauchy sequence.
Since, $g_2(X)$ is closed, there exists $z \in g_2(X)$ such that $\lim_{n\to\infty} y_n = z$.
Thus there exists a $q \in X$ such that $g_2q = z$.
Now,
$\tau+F(d(y_n, f_2q, a)) = \tau+F(d(f_1x_n, f_2q, a)) \leq F(d(g_1x_n, g_2q, a)) = F(d(y_{n-1}, z, a)) \to F(0)$ as $n \to \infty$ [since F is continuous]
Or, $F(d(y_n, f_2q, a)) \leq \tau + F(d(y_n, f_2q, a)) \leq F(0)$ as $n \to \infty$
Since F is increasing and $F(d(y_n, f_2q, a)) \leq F(0)$ as $n \to \infty$
Or, $d(y_n, f_2q, a) \leq 0$
Or, $\lim_{n\to\infty} y_n = f_2q$
Or, $z = f_2q$.
Thus $f_2q = z = g_2q$.

Since, f_2, g_2 are weakly compatible, we have $f_2g_2q = g_2f_2q$ Or, $f_2z = g_2z$.

Again, $\tau + F(d(y_n, f_2z, a)) = \tau + F(d(f_1x_n, f_2z, a)) \leq F(d(g_1x_n, g_2z, a)) = F(d(f_1x_{n-1}, f_2z, a))$
Or, $F(d(y_n, f_2z, a)) \leq F(d(f_1x_{n-1}, f_2z, a)) - \tau$
$\leq F(d(g_1x_{n-1}, g_2z, a)) - 2\tau \leq \ldots \leq F(d(f_1x_0, f_2z, a)) - n\tau$
Or, $F(d(y_n, f_2z, a)) \leq F(d(g_1x_0, g_2z, a)) - (n+1)\tau$.
Taking limit as $n \to \infty$ and applying (F2), we have
$\lim_{n\to\infty} F(d(y_n, f_2z, a)) = -\infty$
Or, $d(y_n, f_2z, a) = 0$
Or, $z = \lim_{n\to\infty} y_n = f_2z$.
Therefore,
$f_2z = z = g_2z$.
Thus z s a common fixed point of f_2 and g_2.
Again, since $f_1(X) \subseteq g_1(X)$ and $g_1(X)$ is closed, there exists a $p \in g_1(X)$ such that $g_1p = z$.

Now,
$\tau+F(d(f_1p, y_{n+1}, a)) = \tau+F(d(f_1p, f_2x_{n+1}, a)) \leq F(d(g_1p, g_2x_{n+1}, a)) = F(d(z, y_n, a))$
Or, $F(d(f_1p, y_{n+1}, a)) \leq \tau + F(d(f_1p, y_{n+1}, a)) \leq F(d(z, y_n, a)) \leq F(0)$ as $n \to \infty$
$\Rightarrow \lim_{n\to\infty} d(f_1p, y_{n+1}, a) = 0$
Or, $f_1p = \lim_{n\to\infty} y_{n+1} = z$
Or, $g_1p = z = f_1p$.
Also, f_1 and f_2 are weakly compatible, we have

$f_1g_1p = g_1f_1p$ Or, $f_1z = g_1z$.

Again,

$\tau + F(d(f_1z, y_{n+1}, a)) = \tau + F(d(f_1z, f_2x_{n+1}, a)) \leq F(d(g_1z, g_2x_{n+1}, a))$
i.e., $F(d(f_1z, y_{n+1}, a)) \leq F(d(g_1z, g_2x_{n+1}, a)) - \tau = F(d(f_1z, f_2x_n, a)) - \tau \leq F(d(g_1z, g_2x_n, a)) - 2\tau \leq \ldots \leq F(d(g_1z, g_2x_0, a)) - (n+2)\tau$.
Again applying (F2), we have
$\lim_{n\to\infty} F(d(f_1z, y_{n+1}, a)) = -\infty$

Or, $\lim_{n\to\infty} d(f_1z, y_{n+1}, a) = 0$
Or, $f_1z = \lim_{n\to\infty} y_{n+1} = z$
Or, $f_1z = z = g_1z = g_2z = f_2z$
Or, z is a common fixed point of f_1, f_2, g_1 and g_2.
Now, we have to prove the uniqueness of the fixed point.

Let y be an another common fixd point.
Then, $F(d(y_n, y, a)) = F(d(f_1x_n, f_2y, a))$
$\leq F(d(g_1x_n, g_2y, a)) - \tau$
$= F(d(f_1x_{n-1}, f_2y, a)) - \tau$
$\leq F(d(g_1x_{n-1}, g_2y, a)) - 2\tau$
$\vdots$
$\leq F(d(g_1x_0, g_2y, a)) - (n+1)\tau$.

Again applying (F2) we have
$\lim_{n\to\infty} F(d(y_n, y, a)) = -\infty$.
Or, $\lim_{n\to\infty} d(y_n, y, a) = 0$
Or, $z = \lim_{n\to\infty} y_n = y$.

Therefore, z is a unique common fixed point of f_1, f_2, g_1 and g_2.

Corollary 3.3.
Let, (X, d) be a complete 2-metric space, f, g_1, g_2 are continuous self-maps on X such that $f(X) \subseteq g_1(X), f(X) \subseteq g_2(X)$ and $g_1(X), g_2(X)$ are closed sets of X. (f, g_1) and (f, g_2) are weakly compatible. F be a continuous function satisfying (F1) and (F2) such that
$\tau + F(d(fx, fy, a)) \leq F(d((g_1x, g_2y, a)); \forall x, y, a \in X, d(fx, fy, a) > 0$
Then f, g_1, g_2 have a unique common fixed point in X.
Proof If we put $f_1 = f_2 = f$ in our above theorem, the result hold.

Corollary 3.4.
Let, (X, d) be a complete 2-metric space, f, g are continuous self-maps on X such that $f(X) \subseteq g(X)$ and g are closed sets of X. (f, g) is weakly compatible. F be a continuous function satisfying (F1) and (F2) such that
$\tau + F(d(fx, fy, a)) \leq F(d((gx, gy, a)); \forall x, y, a \in X, d(fx, fy, a) > 0$
Then f, g have a unique common fixed point in X.
Proof. If we put $f_1 = f_2 = f$ and $g_1 = g_2 = g$ in our above theorem, we will get the result.

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Dinanath Barman
Department of Mathematics, Raiganj University, Raiganj, West Bengal, India.
E-mail address: dinanathbarman85@gmail.com

Krishnadhan Sarkar*
Department of Mathematics, Raniganj Girls' College, Raniganj, Paschim Bardhaman, West Bengal, India.
E-mail address: sarkarkrishnadhan@gmail.com

Kalishankar Tiwary
Department of Mathematics, Raiganj University, Raiganj, West Bengal, India.
E-mail address: tiwarykalishankar@yahoo.com