VALUE DISTRIBUTION OF GENERAL $q$-DIFFERENCE POLYNOMIALS

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Abstract. In this article, we mainly study the value distribution of more general $q$-difference polynomials for a transcendental entire function of zero and finite order. These are significant generalization of earlier results. As a very special case, we obtain the results of N. X. Xu and C. P. Zhong and others.

1. Introduction, Definitions and Results

For a meromorphic function $f$ in the complex plane we assume familiarity with the standard notations of Nevanlinna theory such as, $T(r, f)$, $N(r, f)$ and $m(r, f)$ etc., as explained in [7, 19]. We need the following definitions.

Definition 1.1. Let $f(z)$ and $a(z)$ be meromorphic functions in the complex plane. If $T(r, a) = S(r, f)$, then $a(z)$ is called a small function of $f(z)$, where $S(r, f) = o(T(r, f))$ as $r \to \infty$, except possibly on a set of finite linear measure.

Definition 1.2. Let $M_j(f(qz)) = f^{l_{0j}} f^{l_{1j}}(q_1 z) f^{l_{2j}}(q_2 z) \cdots f^{l_{kj}}(q_k z) = \prod_{i=0}^{k} f^{l_i j}(q_i z)$, \hspace{1cm} (1)

where $q_0 = 1$ and $q_1, q_2, \ldots, q_k \in \mathbb{C}\setminus\{0\}$, $l_{0j}, l_{1j}, \ldots, l_{kj}$ are non-negative integers. Let the degree and weight of the monomial be $\gamma_{M_j} = l_{0j} + l_{1j} + \cdots + l_{kj}$ and $\Gamma_{M_j} = l_{0j} + 2l_{1j} + \cdots + (k + 1)l_{kj} = \sum_{i=0}^{k} (i + 1)l_{ij}$, respectively. If

$$P_q(f(qz)) = \sum_{j=1}^{s} a_j M_j(f(qz)),$$ \hspace{1cm} (2)

where $a_j (j = 1, 2, 3, \ldots, s)$ are constants, then $P_q(f(qz))$ is called a difference polynomial in $f$ of degree $\gamma_{P_q}$ and the weight $\Gamma_{P_q}$. We define upper and lower degree of $P_q(f(qz))$ as follows $\gamma_{P_q} = \max_{1 \leq j \leq s} \gamma_{P_q}$, $\gamma_{P_q} = \min_{1 \leq j \leq s} \gamma_{P_q}$ and $\Gamma_{P_q} = \max_{1 \leq j \leq s} \Gamma_{P_q}$, $\Gamma_{P_q} = \min_{1 \leq j \leq s} \Gamma_{P_q}$. If $\tau_{P_q} = \gamma_{P_q}$, then $P_q(f(qz))$ is called homogeneous $q$-difference polynomial in $f(qz)$, otherwise non-homogeneous.
Definition 1.3. [18] For a meromorphic function $f(z)$, the order and exponent of convergence of zeros is defined respectively as follows

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \quad \lambda(f) = \limsup_{r \to \infty} \frac{\log N(r, \frac{1}{f})}{\log r}.$$

A Borel exceptional value of $f(z)$ is any value $a$ satisfying $\lambda(f - a) < \sigma(f)$.

In 1959, W. K. Hayman [8], discussed about Picard values of an entire and meromorphic functions and their derivatives. He obtained the following result.

**Theorem A.** Let $f(z)$ be a transcendental entire function. Then

1. For $n \geq 3$ and $a \neq 0$, $\psi(z) = f'(z) - a(f(z))^n$ assumes all finite values infinitely often.
2. For $n \geq 2$, $\phi(z) = f'(z)(f(z))^n$ assumes all finite values except possibly zero infinitely often.

As we have seen in recent years many researchers [4, 3, 5, 6, 9, 11, 13, 14, 16, 17, 15] are showing interest in the study of difference analogue of the Nevanlinna theory. Many articles [10, 14, 12, 18] have focused on the study of difference version of Hayman conjecture.

In 2007, I. Laine and C. C. Yang [10], considered the difference version of Theorem A and obtained the following result.

**Theorem B.** Let $f(z)$ be a transcendental entire function of finite order, $c$ is a nonzero complex constant and $n \geq 2$, then $f^n(z)f(z + c)$ takes every nonzero value infinitely often.

Again in 2011, K. Liu and X. G. Qi [14] proved the following result by considering $q$-difference polynomials.

**Theorem C.** If $f(z)$ is a transcendental meromorphic function of zero order, $a, q$ are nonzero complex constants. If $n \geq 6$, then $f^n(z)f(qz + c)$ assumes every nonzero value $b \in C$ infinitely often. If $n \geq 8$, then $f^n(z) + a[f(qz + c) - f(z)]$ assumes every nonzero value $b \in C$ infinitely often.

In the same year, K. Liu, X. L. Liu and T. B. Cao [12] obtained extension of above results by considering zero distribution of $q$-difference polynomials.

**Theorem D.** If $f(z)$ is a transcendental meromorphic function of zero order, $a, q$ are nonzero complex constants, $\alpha(z)$ is a nonzero small function with respect to $f$. If $n \geq 6$, then $f^n(z)(f^n - a)f(qz + c) - \alpha(z)$ has infinitely many zeros. If $n \geq 7$, then $f^n(z)(f^n - a)(f(qz + c) - f(z)) - \alpha(z)$ has infinitely many zeros.

In 2016, N. Xu and C. P. Zhong [18] generalized above results to more general case and proved the following results.

**Theorem E.** Let $f(z)$ be a transcendental entire function of zero order, $a$ be a nonzero complex constant, $q \in \mathbb{C} \setminus \{0, 1\}$, $n$ be any positive integer. Considering $q$-difference polynomial $H(z) = f(qz) - a(f(z))^n$,

1. if $n = 3$, then $H(z) - a(z)$ has infinitely many zeros, where $a(z)$ is a nonzero small function with respect to $f(z)$.
2. In particular, if $a(z)$ is a nonzero rational function, then the condition $n = 3$ can be reduced to $n > 1$.

**Theorem F.** Let $f(z)$ be a transcendental entire function of zero order, $q_1, q_2, \ldots, q_m$ be non-zero complex constants such that at least one of them is not equal to 1, $a \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{N}^+$. Considering $q$-difference polynomial $F(z) = f(q_1z)f(q_2z)\cdots f(q_mz) - a(f(z))^n$,
(1) If \( m < \frac{n-1}{2} \). Then \( F(z) - \alpha(z) \) has infinitely many zeros, where \( \alpha(z) \) is a nonzero small function with respect to \( f(z) \).

(2) In particular, if \( \alpha(z) \) is a nonzero rational function, then the condition \( m < \frac{n-1}{2} \) can be reduced to \( n > m \).

(3) If \( m \neq n \), then also \( F(z) - \alpha(z) \) has infinitely many zeros.

**Theorem G.** Let \( f(z) \) be a transcendental entire function of finite and positive order \( \sigma(f) \), \( q_1, q_2, \ldots, q_m \) be non-zero complex constants such that at least one of them is not equal to 1 and \( q_1^\sigma(f) + q_2^\sigma(f) + \cdots + q_m^\sigma(f) \neq n \), \( a \in \mathbb{C} - \{0\}, m, n \in \mathbb{N}^+ \).

If \( f(z) \) has finitely many zeros, then \( F(z) - \alpha(z) \) has infinitely many zeros, where \( \alpha(z) \) is the non-zero small function with respect to \( f(z) \).

If \( G(z) \) be an entire function with order less than one and if \( F(z) - a(f(z))^n = G(z) \), then \( F(z) \) has infinitely many zeros.

In this article we generalize all the above results to more general \( q \)-difference polynomials.

**Theorem 1.1.** Let \( f(z) \) be a zero order transcendental entire function, \( q_1, q_2, \ldots, q_m \) be non-zero complex constants and at least one of them is not equal to 1, \( a \in \mathbb{C} - \{0\}, \gamma_{P_\alpha}, n \in \mathbb{N} \). Let the \( q \)-difference polynomial be \( H(z) = P_q(f(qz)) - aP(f) \), where \( P_q(f(qz)) \) be as defined in (1.2) and \( P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0 \).

1. If \( \gamma_{P_\alpha} < \frac{n-1}{2} \), then \( H(z) - \alpha(z) \) has infinitely many zeros, where \( \alpha(z) \neq 0 \) is a small function of \( f \).

2. If \( \alpha(z) \neq 0 \) is a rational function, then \( \gamma_{P_\alpha} < \frac{n-1}{2} \) reduces to \( n > \gamma_{P_\alpha} \).

**Corollary 1.1.** The \( q \)-difference polynomial \( P_q(f(qz)) - aP(f) - R(z) = 0 \) has no zero order transcendental entire solution when \( n > \gamma_{P_\alpha} \), where \( R(z) \) is a nonzero rational function.

**Remark 1.1.** Substituting \( l_{01} = 0, l_{11} = 1 \) in (1.2) we get \( \gamma_{P_\alpha} = 1 \). Hence we get Theorem E.

**Theorem 1.2.** Let \( f(z) \) be a zero order transcendental entire function, \( q_0 = 1 \) and \( q_1, q_2, \ldots, q_m \) be non-zero complex constants and at least one of them is not equal to 1, \( a \in \mathbb{C} - \{0\}, \gamma_{\alpha_{P_\alpha}}, n \in \mathbb{N} \). If \( (2\gamma_{P_\alpha} - \gamma_{\alpha_{P_\alpha}}) \neq n \), then \( H(z) - \alpha(z) \) has infinitely many zeros, where \( \alpha(z) \neq 0 \) is a small function of \( f \).

**Remark 1.2.** Substituting \( j = 1, l_{01} = 0, l_{11} = l_{11} = \cdots = l_{ik} = 1 \) in (1.2) and considering \( P(f) = f^n \) then Theorem 1.1 and 1.2 reduces to Theorem F.

All the previous results are obtained for the case when \( f(z) \) is a transcendental entire function of zero order. In Theorem 1.3 and 1.4 by considering \( f(z) \) as a finite and positive order transcendental entire function we discuss the value distribution of \( q \)-difference polynomial \( H(z) \).

**Theorem 1.3.** Let \( f(z) \) be a finite and positive order transcendental entire function \( \sigma(f) \), \( q_0 = 1 \) and \( q_1, q_2, \ldots, q_m \) be non-zero complex constants and at least one of them is not equal to 1 and \( l_{01} = l_{11} q_1^\sigma(f) + l_{21} q_2^\sigma(f) + \cdots + l_{m1} q_m^\sigma(f) \neq n \), \( a \in \mathbb{C} - \{0\}, \gamma_{P_\alpha}, n \in \mathbb{N} \). If \( f(z) \) has finitely many zeros. Then \( H(z) - \alpha(z) \) has infinitely many zeros, where \( \alpha(z) \neq 0 \) is a small function of \( f \).

**Theorem 1.4.** Let \( f(z) \) be a finite and positive order transcendental entire function and \( G(z) \) is an entire function with order less than 1, \( q_0 = 1 \) and \( q_1, q_2, \ldots, q_m \) be
non-zero complex constants and at least one of them is not equal to 1 and \( l_{0j} + l_{1j}q_1^j + l_{2j}q_2^j + \cdots + l_{kj}q_k^j \neq n, a \in \mathbb{C} - \{0\}, r \neq \infty \). If
\[
P_q(f(qz)) - aP(f) = G(z),
\]
then \( f(z) \) has infinitely many zeros.

**Remark 1.3.** Substituting \( j = 1 \), \( l_{01} = 0, l_{11} = l_{12} = \cdots = l_{1m} = 1 \) in (1.2) and considering \( P(f) = f^n \) then Theorem 1.3 and 1.4 reduce to Theorem G.

2. Some Lemmas.

**Lemma 2.1.** [20] Let \( f(z) \) be a transcendental meromorphic function of zero order and \( q \) be a non-zero complex constant. Then
\[
T(r, f(qz)) = (1 + o(1))T(r, f(z)) \text{ or } T(r, f(qz)) = T(r, f(z)) + S_1(r, f),
\]
on a set of lower logarithmic density 1.

**Lemma 2.2.** [2] Let \( f(z) \) be a nonconstant zero order meromorphic function and \( q \in \mathbb{C} \setminus \{0\} \). Then
\[
m \left( r, \frac{f(qz)}{f(z)} \right) = S(r, f),
\]
on a set of logarithmic density 1.

**Lemma 2.3.** [1] If an entire function \( f \) has a finite exponent of convergence \( \lambda(f) \) for its zero-sequence, then \( f \) has a representation in the form \( f(z) = Q(z)e^{g(z)} \), satisfying \( \lambda(Q) = \sigma(Q) = \lambda(f) \). Further, if \( f \) is of finite order, then \( g \) in the above form is a polynomial of degree less or equal to the order of \( f \).

**Lemma 2.4.** [19] Suppose that \( f_1(z), f_2(z), \ldots, f_n(z), (n \geq 2) \) are meromorphic functions and \( g_1(z), g_2(z), \ldots, g_n(z) \) are entire functions satisfying the following conditions,
\begin{enumerate}
  \item \( \sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0; \)
  \item \( g_j(z) - g_k(z) \) are not constants for \( 1 \leq j < k \leq n; \)
  \item for \( 1 \leq j \leq n, 1 \leq h < k \leq n, T(r, f_j) = O(T(r, e^{g_k - g_h}))(r \to \infty, r \notin E). \)
\end{enumerate}

Then \( f_j(z) \equiv 0(j = 1, 2, \ldots, n) \).

**Lemma 2.5.** Let \( f(qz) \) be a zero-order meromorphic function and \( P_q(f(qz)) \) be a \( q \)-difference polynomial in \( f \) of degree \( n \geq 1 \) with coefficients \( a_j(z) \), upper degree \( \gamma_{p} \) and lower degree \( \gamma_{m} \), then
\[
m \left( r, \frac{P_q(f(qz))}{f^{\gamma_{p}}} \right) \leq (\gamma_{p} - \gamma_{m})m \left( r, \frac{1}{f} \right) + S_1(r, f),
\]
on a set of logarithmic density 1.

**Proof.** Let \( M_j(f(qz)) \) and \( P_q(f(qz)) \) are defined as in (1.1) and (1.2) respectively, then
\[
\left| \frac{P_q(f(qz))}{f^{\gamma_{p}}} \right| = \sum_{j=1}^{s} |a_j| \left| \frac{M_j(f(qz))}{f^{\gamma_{m}}} \right| \left| \frac{1}{f} \right|^{\gamma_{p} - \gamma_{m}},
\]
where $\gamma_{M_j}$ is the degree of the monomial $M_j(f)$.

**Case 1:** When $|f(qz)| \leq 1$, $|\frac{1}{f(qz)}| \geq 1$ and $|\frac{1}{f(qz)}|^\gamma_{M_j} \geq 1$, and we have
\[
\left| \frac{1}{f(qz)} \right|^\gamma_{M_j} \leq \left| \frac{1}{f(qz)} \right|^\gamma_{M_j}^{\text{min}} \leq s \gamma_{M_j} \leq 1.
\]
Hence we get from (2.1),
\[
\frac{P_q(f(qz))}{f^{\gamma_{P_q}}} \leq \left[ \sum_{j=1}^s |a_j| \left| \frac{f(qz)}{f} \right|^{l_{ij}} \right].
\]
Using the logarithmic derivative lemma, we get
\[
m \left( r, \frac{P_q(f(qz))}{f^{\gamma_{P_q}}} \right) \leq (\gamma_{P_q} - \gamma_{P_q}) m \left( r, \frac{1}{f(qz)} \right) + S_1(r, f(qz)).
\]
Since $f$ is a meromorphic function of zero order, we have
\[
S_1(r, f(qz)) = S_1(r, f).
\]
Hence
\[
m \left( r, \frac{P_q(f(qz))}{f^{\gamma_{P_q}}} \right) \leq (\gamma_{P_q} - \gamma_{P_q}) m \left( r, \frac{1}{f} \right) + S_1(r, f).
\]
Outside of a possible exceptional set with the finite logarithmic measure.

**Case 2:** When $|f(qz)| > 1$ we have $|\frac{1}{f(qz)}| \leq 1$, $|\frac{1}{f(qz)}|^\gamma_{M_j} \leq 1$ and $\log^+ |\frac{1}{f(qz)}|^\gamma_{M_j} = 0$.
Hence from (2.1) and logarithmic derivative lemma we get,
\[
m \left( r, \frac{P_q(f(qz))}{f^{\gamma_{P_q}}} \right) \leq S_1(r, f(qz)).
\]
Proceeding as in Case 1, we get,
\[
m \left( r, \frac{P_q(f(qz))}{f^{\gamma_{P_q}}} \right) \leq S_1(r, f),
\]
\[
\leq (\gamma_{P_q} - \gamma_{P_q}) m \left( r, \frac{1}{f} \right) + S_1(r, f).
\]

### 3. Proofs of the Theorems.

#### Proof of Theorem 1.1.

(1) Let $\Phi(z) = \frac{P_q(f(qz)) - \alpha(z)}{a \Phi(z)}$. From the condition $\gamma_{P_q} < \frac{n-1}{2-\frac{n}{2}}$ we get $n > \gamma_{P_q}$.
Since $f(z)$ is a zero order transcendental entire function, by Lemma 2.1, we get,
\[
T(r, P(z)) = \frac{P_q(f(qz)) - \alpha(z)}{a \Phi(z)},
\]
\[
nT(r, f) \leq T(r, P_q(f(qz))) + T(r, \alpha(z)) + T(r, \Phi(z)) + O(1),
\]
\[
\leq \gamma_{P_q} T(r, f) + T(r, \Phi(z)) + S(r, f).
\]
From the above equation, we obtain
\[
(n - \gamma_{P_q}) T(r, f) \leq T(r, \Phi(z)) + S(r, f),
\]
on a set of logarithmic density 1. Since \( n > \tau_{P_q} \) we can note that \( \Phi(z) \) is transcendental. On the other hand,

\[
T(r, \Phi(z)) = T \left( r, \frac{P_q(f(qz)) - \alpha(z)}{a P(f)} \right) \leq T(r, P_q(f(qz))) + T(r, \alpha(z)) + T(r, P(f)) + O(1)
\]

Therefore

\[
T(r, \Phi(z)) \leq (n + \tau_{P_q}) T(r, f) + S(r, f).
\]

From (3.1), (3.2) and the condition \( n > \tau_{P_q} \), we get \( T(r, \Phi(z)) = O(T(r, f)) \).

Suppose \( H(z) - \alpha(z) \) has finitely many zeros, then \( \Phi(z) \) has only finite 1-points. Hence

\[
N \left( r, \frac{1}{\Phi(z) - 1} \right) = S(r, \Phi(z)) = S(r, f).
\]

We can note from the second fundamental theorem

\[
T(r, \Phi(z)) \leq \mathcal{N}(r, \Phi) + \mathcal{N} \left( r, \frac{1}{\Phi} \right) + \mathcal{N} \left( r, \frac{1}{\Phi - 1} \right) + S(r, \Phi)
\]

\[
\leq \frac{1}{n} \mathcal{N}(r, \Phi) + \tau_{P_q} T(r, f) + S(r, f),
\]

\[
\left( 1 - \frac{1}{n} \right) T(r, \Phi) \leq \tau_{P_q} T(r, f) + S(r, f).
\]

From (3.1), (3.3), we get

\[
\left( 1 - \frac{1}{n} - \frac{\tau_{P_q}}{n - \tau_{P_q}} \right) T(r, \Phi) \leq S(r, \Phi).
\]

which is a contradiction, since \( \tau_{P_q} < \frac{n-1}{2n} \). Hence \( H(Z) - \alpha(z) \) has infinitely many zeros.

(2.) By Lemma 2.1, we have

\[
T(r, H(z)) \leq T(r, P_q(f(qz)) - a P(f)) \leq T(r, P_q(f(qz))) + T(r, P(f)) + S(r, f)
\]

\[
\leq (n + \tau_{P_q}) T(r, f) + S(r, f).
\]

On the other side,

\[
T(r, aP(f)) \leq T(r, P_q(f(qz)) - H(z)) \leq T(r, P_q(f(qz))) + T(r, H(z)),
\]

\[
nT(r, f) \leq \tau_{P_q} T(r, f) + T(r, H(z)) + S(r, f).
\]

From (3.5) and (3.6) we obtain,

\[
(n - \tau_{P_q}) T(r, f) + S(r, f) \leq T(r, H) \leq (n + \tau_{P_q}) T(r, f) + S(r, f).
\]

From the above equation we obtain, \( T(r, H) = O(T(r, f)) \). Since \( n > \tau_{P_q} \) and \( \sigma(f) = 0 \), clearly \( H(z) \) is of zero order.

Let us assume that \( R(z) = H(z) - \alpha(z) \) has finitely many zeros. Then \( R(z) \) becomes a rational function, since \( H(z) \) is a function of zero order and \( \alpha(z) \) is a non-zero rational function. Then we get \( T(r, H) = S(r, f) \), which is a contradiction to our assumption. Hence, \( H(z) - \alpha(z) \) has infinitely many zeros.
Proof of Theorem 1.2.
Let us assume that $H(Z) - \alpha(z)$ has finitely many zeros, by Lemma 2.1, we obtain
\[
T(r, H(z) - \alpha(z)) = T(r, P_q(f(qz))) - aP(f) - \alpha(z)) \\
\leq T(r, P_q(f(qz))) + T(r, P(f)) + T(r, \alpha(z)) + S(r, f) \\
\leq \gamma_{P_q} T(r, f) + nT(r, f) + S(r, f), \\
\leq (n + \gamma_{P_q}) T(r, f) + S(r, f).
\]
From the above inequality we get $\sigma(H(z) - \alpha(z)) = 0$.
From the Hadamard factorization theorem, we obtain
\[
H(z) - \alpha(z) = P_q(f(qz)) - aP(f) - \alpha(z) = P_1(z),
\]
where $P_1(z)$ is a polynomial. Rewriting (3.8), we get
\[
aP(f) = P_q(f(qz)) - P_1(z) - \alpha(z).
\]
When $n > (2\gamma_{P_q} - \gamma_{P_q})$, from (3.9) and Lemma 2.1, we have
\[
T(r, aP(f)) = T(r, P_q(f(qz))) - P_1(z) - \alpha(z)) \\
nT(r, f) \leq \gamma_{P_q} T(r, f) + S(r, f), \\
\leq (2\gamma_{P_q} - \gamma_{P_q}) T(r, f) + S(r, f).
\]
Which is a contradiction to the assumption.
When $n < 2\gamma_{P_q} - \gamma_{P_q}$, from (3.9), Lemma 2.2 and Lemma 2.5, we have
\[
T(r, P_q(f(qz))) = m(r, P_q(f(qz))) = m \left( r, f^{\gamma_{P_q}} \frac{P_q(f(qz))}{P_q} \right) \\
\geq m(r, f^{\gamma_{P_q}}) - m \left( r, \frac{f^{\gamma_{P_q}}}{P_q(f(qz))} \right) \\
\geq \gamma_{P_q} m(r, f) - (\gamma_{P_q} - \gamma_{P_q}) m(r, f) + S(r, f) \\
\geq (2\gamma_{P_q} - \gamma_{P_q}) m(r, f) + S(r, f).
\]
On the other hand by (3.9), we get
\[
T(r, P_q(f(qz))) = T(r, aP(f) + P_1(z) + \alpha(z)) \\
(2\gamma_{P_q} - \gamma_{P_q}) T(r, f) \leq nT(r, f) + S(r, f).
\]
Which is a contradiction to our assumption. Hence $H(z) - \alpha(z)$ has infinitely many zeros.

Proof of Theorem 1.3.
Let $f(z)$ be a finite and positive order transcendental entire function and has finitely many zeros, then from Lemma 2.3, $f(z)$ can be expressed in the form
\[
f(z) = g(z)e^{h(z)},
\]
where $g(z) \not\equiv 0$, $h(z)$ are polynomials. Set
\[
h(z) = a_k z^k + \cdots + a_0,
\]
where $a_k(\not\equiv 0), \ldots, a_0$ are constants. Given that $\sigma(f) \not\equiv 0$, hence $\sigma(f) = \deg h(z) = k \geq 1$. 

From (1.2), (3.10) and (3.11) we have

\[ P_q(f(qz)) = \sum_{j=1}^{s} \prod_{i=0}^{k} a_{ij}f(q_i z)^{l_{ij}} \]

\[ = \sum_{j=1}^{s} \prod_{i=0}^{k} a_{ij}g(q_i z)^{l_{ij}} e^{h(q_i z)} \]

\[ = \sum_{j=1}^{s} \prod_{i=0}^{k} a_{ij}g(q_i z)^{l_{ij}} e^{\alpha_k(l_0 + l_1 q_1^k + l_2 q_2^k + \ldots + l_m q_m^k)z^k e^{a_k - 1(l_0 + l_1 q_1^k + l_2 q_2^k + \ldots + l_m q_m^k)z^k - 1} \ldots e^{a_0(l_0 + l_1 z + l_2 + \ldots + l_m)} .} \]

\[ P_q(f(qz)) = \sum_{j=1}^{s} P_2(z) e^{\alpha_k(l_0 + l_1 q_1^k + l_2 q_2^k + \ldots + l_m q_m^k)z^k} , \quad (17) \]

where \( P_2(z) = \prod_{i=1}^{k} a_{ij}g(q_i z)^{l_{ij}} e^{a_k - 1(l_0 + l_1 q_1^k + l_2 q_2^k + \ldots + l_m q_m^k)z^k - 1} \ldots e^{a_0(l_0 + l_1 z + l_2 + \ldots + l_m)} . \)

Thus \( \sigma(P_2) \leq k - 1 < k . \) On the other side, from (3.10) and (3.11) we have

\[ P(f) = a_n f^n + a_{n-1} f^{n-1} + \ldots + a_0 \]

\[ = a_n g^n e^{a_k z^k} + a_{n-1} g^{n-1} e^{(n-1)k} + \ldots + a_0 \]

\[ = a_n g^n e^{a_k z^k} + a_{n-1} g^{n-1} e^{(n-1)z^k + \ldots + a_0} \]

\[ = e^{\alpha_k z^k} \{ a_n g^n e^{a_k z^k} + a_{n-1} g^{n-1} e^{(n-1)z^k + \ldots + a_0} \} . \quad (18) \]

where \( P_3(z) = a_n g^n e^{a_k z^k} + a_{n-1} g^{n-1} e^{a_k z^k} + \ldots + a_0 . \)

From (3.12) and (3.13), we get

\[ H(z) = \sum_{j=1}^{s} P_2(z) e^{\alpha_k(l_0 + l_1 q_1^k + l_2 q_2^k + \ldots + l_m q_m^k)z^k} - aP_3(z) e^{\alpha_k z^k} \{ 0 \} . \quad (19) \]

Since \( P_2(z) \neq 0, P_3(z) \neq 0, \sigma(P_2) < k, \sigma(P_3) < k, l_0 + l_1 q_1^{\sigma(f)} + l_2 q_2^{\sigma(f)} + \ldots + l_m q_m^{\sigma(f)} \neq n \), it follows that \( H(z) \) is a transcendental entire function and \( \sigma(H) = \sigma(f) = k . \)

Suppose \( H(z) - \alpha(z) \) has finitely many zeros, then \( \sigma(H - \alpha(z)) < \sigma(H) = \sigma(f) \). Hence \( H(z) - \alpha(z) \) can be expressed as

\[ H(z) - \alpha(z) = S(z) e^{\alpha z} , \quad (20) \]

where \( S(z) \) is an entire function with \( \sigma(S) < k, t \neq 0 \) is a constant. From (3.14) and (3.15), we get

\[ \sum_{j=1}^{s} P_2(z) e^{\alpha_k(l_1 q_1^k + l_2 q_2^k + \ldots + l_m q_m^k)z^k} - aP_3(z) e^{\alpha_k z^k} - S(z) e^{\alpha z} - \alpha(z) = 0 . \quad (21) \]

Since \( l_1 q_1^{\sigma(f)} + l_2 q_2^{\sigma(f)} + \ldots + l_m q_m^{\sigma(f)} \neq n \).

**Case (i):** \( a_k(l_0 + l_1 q_1^k + l_2 q_2^k + \ldots + l_m q_m^k)z^k \neq t, na_k z^k \neq t \). By Lemma 2.4, we obtain \( P_2(z) = 0, P_3(z) = 0, S(z) = 0, \alpha(z) = 0 \). This is a contradiction.
Case (ii): $a_k(l_{0j} + l_{1j}q_1^k + l_{2j}q_2^k + \cdots + l_{mj}q_m^k)z^k = t$. Then (3.16) can be written as
\[
\left( \sum_{j=1}^{s} P_2(z) - S(z) \right) e^{a_k(l_{0j} + l_{1j}q_1^k + l_{2j}q_2^k + \cdots + l_{mj}q_m^k)z^k} - aP_3(z)e^{na_kz^k} - \alpha(z) = 0.
\]
By Lemma 2.4, we obtain $P_2(z) - S(z) = 0, P_3(z) = 0, \alpha(z) = 0$. This is a contradiction.

Case (iii): $na_k = t$, following the same procedure as above, we arrive at a contradiction. Hence, $H(z) - \alpha(z)$ has infinitely many zeros.

Proof of Theorem 1.4.

Let us assume that $f(z)$ has finitely many zeros.

Using (3.12) and (3.13) in (1.3), we get
\[
\sum_{j=1}^{s} P_2(z)e^{a_k(l_{0j} + l_{1j}q_1^k + l_{2j}q_2^k + \cdots + l_{mj}q_m^k)z^k} - aP_3(z)e^{na_kz^k} = G(z),
\]
where $P_2(z)$ and $P_3(z)$ are defined as in Theorem 1.3.

Since $P_2(z)(\neq 0), P_3(z)(\neq 0), \sigma(P_2) < k, \sigma(P_3) < k, l_{0j} + l_{1j}q_1^k + l_{2j}q_2^k + \cdots + l_{mj}q_m^k \neq n$ we get $\sigma(G) < 1 < k$. From (3.17) and Lemma 2.4, we get $P_2(z) = 0, P_3(z) = 0, G(z) = 0$, which is a contradiction. Hence $f(z)$ has infinitely many zeros.

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References
