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GENERALIZED FIXED POINT THEOREMS OF PANDHARE AND WAGHMODE IN HILBERT SPACE

N. SESHAGIRI RAO AND K. KALYANI

Abstract. This paper elucidates the existence and uniqueness of a fixed point to a self mapping over a closed subset of Hilbert space with rational expressions in the contraction inequality. This result is developed for a pair of mappings, positive integers powers of a pair of mappings and again further extended to a sequence of mappings in the space. Moreover special cases assure that these results are generalizations of well known proven important results. Our results mainly focus on the generalization of Pandhare and Waghmode result in Hilbert space.

1. Introduction

First the existence and unique fixed point was given by the mathematician Banach in 1922, which was acclaimed as Banach's contraction principle and has an important role in the development of various results connected with Fixed point Theory and Approximation Theory. Around the same time Nadler [10] has considered a convergent sequence of contraction mappings and the convergence of the associated sequence of their fixed points. Some of the results obtained therein, used to obtain necessary and sufficient condition for a separable or a reflexive Banach space to be finite dimensional. Furthermore; some of the results were applied to differential equations also. In this very article he also studied fixed points of contraction mappings on the cartesian product of two complete metric spaces. The extension of Banach contraction principle was also given by Sehgal [18]. Almost during the same time Kannan [6] investigated the extension of Banach fixed point theorem by removing the completeness of the space with different sufficient conditions. Later Singh, Reich [14] also discussed some generalization of Banach's fixed point theorem and some remarks on it. Zamfirescu [24] obtained various results similar to the well-known contraction theorem of Banach and some of these results were sufficient enough to include the theorem of Kannan and Singh. Further, it has been observed that Chatterjee's theorem can be derived from this theorem of Zamfirescu by taking suitable combinations of positive constants. In order to generalize Banach contraction principle and some results of Kannan, Wong [23] obtained the result by replacing the constants by suitable non-negative real valued functions.

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Also a generalization of Banach fixed point theorem was given by Jaggi [4] which involved a continuous map satisfying certain inequality involving rational expression. Kannan's study of fixed point theory involving uniformly convex Banach space and strictly Banach space was improved and sharpened by Jaggi [5] in the same year. Of course, this study of Kannan was based on some results obtained by Krasnoselskii. Fisher [2] developed the approach of Kannan and proved analogous results involving two mappings on a complete metric space. Also Ganguly and Bandyopadhyay [3] investigated the properties of fixed points of a family of mappings on complete metric spaces. Khare [8] extended some of the Kannan's result for continuous self map defined on a complete metric space, satisfying an inequality dominated by rational expressions. Koparde and Waghmode [9] extended this analysis for the existence and uniqueness of a fixed point for a sequence of mappings on a Hilbert space satisfying Kannan's type conditions. Pandhare and Waghmode [11] developed this approach of Koparde and Waghmode [9] and proved the fixed point theorem for a self mapping on a closed subset of a Hilbert space satisfying certain condition. There exists an extensive literature on fixed point theorems for various contractive conditions whose comprehensive survey can be found in Rhoades [15, 16] and Smart [21].

Common fixed points were also investigated by Srivastava and Gupta. Motivated by the result of Dass and Gupta [1], Koparde and Waghmode [9] obtained a unique common fixed point for a pair of mappings satisfying certain rational inequality. This analysis was further extended to a sequence of mappings satisfying certain rational inequality. Also Manihar Singh [17] obtained a unique fixed point for a self mapping satisfying certain contractive type condition. Pandhare [12], Veerapandi and Anil Kumar [22] investigated the fixed points for sequences of mappings on a Hilbert space. Sarkhel dealt with the following implication, namely, Banach's fixed point theorem implies Kannan's fixed point theorem. Recently a new approach has been taken by Rashwan [13] in his paper for attaining a common and coincidences fixed points for asymptotically regular mappings over a Hilbert space with various contractive conditions.

In this paper, we proved that a self mapping T satisfying certain rational contraction condition has a unique fixed point on a closed subset X of Hilbert space and again the same result is then extended to a pair of mappings T_1, T_2, some positive integers powers p, q of a pair mappings T_1^p, T_2^q and then further generalized to a sequence of mappings in the space. In the last three cases we have obtained a common fixed point in X. Our results are generalizations of the main result of Pandhare and Waghmode [11].

2. Main Results

Theorem 1. *Let T be a closed subset of a Hilbert space and $T : X \to X$ be a self mapping satisfying the following inequality*

$$\begin{aligned}||Tx-Ty|| \le\ & a_1\frac{||x-Tx||\,[1+||y-Ty||]}{1+||x-y||}+a_2\frac{||y-Ty||\,[1+||y-Tx||]}{1+||x-y||}\\ &+a_3\frac{||x-Ty||\,[1+||y-Tx||]}{1+||x-y||}+a_4\frac{||x-y||\,[1+||Tx-Ty||]}{1+||x-y||}\\ &+a_5\frac{||x-y||\,[1+||x-Tx||]}{1+||y-Ty||}+a_6\frac{||x-Tx||\,[1+||x-Ty||]}{1+||y-Ty||}\\ &+a_7\frac{||x-Ty||\,[1+||Tx-Ty||]}{1+||x-y||}+a_8\frac{||x-y||\,[1+||x-Ty||]}{1+||x-y||}\\ &+a_9\frac{||x-Tx||+||y-Ty||+||x-y||}{1+||x-Tx||\ ||x-Ty||\ ||y-Ty||\ ||x-y||}\\ &+a_{10}\frac{||x-Ty||^2+||y-Tx||^2}{||x-Ty||+||y-Tx||}+a_{11}||x-y||\end{aligned}$$

for all $x, y \in X$ and $x \neq y$, where $a_i (i = 1, 2, 3, ..., 11)$ are non-negative real's with $0 \le a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + 3a_{10} + 2a_{11} < 1$. Then T has a unique fixed point in X.

Proof. For any $x_0 \in X$, we define the sequence $\{x_n\}$ in X by

$$x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, ..., x_{n+1} = Tx_n, \text{for } n = 0, 1, 2, 3, ...$$

Now, we claim that the sequence $\{x_n\}$ is a Cauchy sequence in X. For this we have

$$||x_{n+1} - x_n|| = ||Tx_n - Tx_{n-1}||$$

Then by the hypothesis, we have

$$\begin{aligned}||x_{n+1}-x_n|| \le\ & a_1\frac{||x_n-Tx_n||\,[1+||x_{n-1}-Tx_{n-1}||]}{1+||x_n-x_{n-1}||}+a_2\frac{||x_{n-1}-Tx_{n-1}||\,[1+||x_{n-1}-Tx_n||]}{1+||x_n-x_{n-1}||}\\ &+a_3\frac{||x_n-Tx_{n-1}||\,[1+||x_{n-1}-Tx_n||]}{1+||x_n-x_{n-1}||}+a_4\frac{||x_n-x_{n-1}||\,[1+||Tx_n-Tx_{n-1}||]}{1+||x_n-x_{n-1}||}\\ &+a_5\frac{||x_n-x_{n-1}||\,[1+||x_n-Tx_n||]}{1+||x_{n-1}-Tx_{n-1}||}+a_6\frac{||x_n-Tx_n||\,[1+||x_n-Tx_{n-1}||]}{1+||x_{n-1}-Tx_{n-1}||}\\ &+a_7\frac{||x_n-Tx_{n-1}||\,[1+||Tx_n-Tx_{n-1}||]}{1+||x_n-x_{n-1}||}+a_8\frac{||x_n-x_{n-1}||\,[1+||x_n-Tx_{n-1}||]}{1+||x_n-x_{n-1}||}\\ &+a_9\frac{||x_n-Tx_n||+||x_{n-1}-Tx_{n-1}||+||x_n-x_{n-1}||}{1+||x_n-Tx_n||\ ||x_n-Tx_{n-1}||\ ||x_{n-1}-Tx_{n-1}||\ ||x_n-x_{n-1}||}\\ &+a_{10}\frac{||x_n-Tx_{n-1}||^2+||x_{n-1}-Tx_n||^2}{||x_n-Tx_{n-1}||+||x_{n-1}-Tx_n||}+a_{11}||x_n-x_{n-1}||\end{aligned}$$

$$\begin{aligned}\Rightarrow &(1-a_1-a_6-a_{10}-a_{11})||x_{n+1}-x_n||\\ &+(1-a_1-a_2-a_4-a_5-a_{10}-a_{11})||x_{n+1}-x_n||\ ||x_n-x_{n-1}||\\ &\le \{(a_2+a_4+a_5+a_8+a_9+2a_{10}+a_{11})+\\ &(a_2+a_9+2a_{10}+a_{11})||x_n-x_{n-1}||\}||x_n-x_{n-1}||\end{aligned}$$

and so

$$||x_{n+1} - x_n|| \leq s(n)||x_n - x_{n-1}||$$

wherein

$$s(n) = \frac{(a_2 + a_4 + a_5 + a_8 + a_9 + 2a_{10} + a_{11}) + (a_2 + a_9 + 2a_{10} + a_{11})||x_n - x_{n-1}||}{(1 - a_1 - a_6 - a_{10} - a_{11}) + (1 - a_1 - a_2 - a_4 - a_5 - a_{10} - a_{11})||x_n - x_{n-1}||},$$

for $n = 1, 2, 3, 4, ...$ It is clear that $s(n) < 1$, for all n as

$$0 \leq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + 3a_{10} + 2a_{11} < 1.$$

Using the inequality from the hypothesis successively, we find some $S < 1$, such that

$$||x_{n+1} - x_n|| \leq S^n ||x_1 - x_0||$$

Taking the limit to this inequality, we find that $||x_{n+1} - x_n|| \to 0$, which shows that the sequence $\{x_n\}$ is a Cauchy sequence in X. Using the completeness of X, we can find a $\mu \in X$ such that $x_n \to \mu$ as $n \to \infty$.

Consequently, $\{x_{n+1}\} = \{Tx_n\}$ is a subsequence of $\{x_n\}$ and hence has the same limit μ. Since T is continuous, we obtain

$$\begin{aligned} T(\mu) =& T\left(\lim_{n\to\infty} x_n\right) \\ =& \lim_{n\to\infty} Tx_n \\ =& \lim_{n\to\infty} x_{n+1} \\ =& \mu \end{aligned}$$

Hence μ is a fixed point of T in X. Now, it remains to show that μ is a unique fixed point of T. For this let $\upsilon(\mu \neq \upsilon)$ be another fixed point of T. Then

$$\begin{aligned} ||\mu - \upsilon|| =& ||T\mu - T\upsilon|| \\ \leq& a_1 \frac{||\mu - T\mu|| \left[1 + ||\upsilon - T\upsilon||\right]}{1 + ||\mu - \upsilon||} + a_2 \frac{||\upsilon - T\upsilon|| \left[1 + ||\upsilon - T\mu||\right]}{1 + ||\mu - \upsilon||} \\ &+ a_3 \frac{||\mu - T\upsilon|| \left[1 + ||\upsilon - T\mu||\right]}{1 + ||\mu - \upsilon||} + a_4 \frac{||\mu - \upsilon|| \left[1 + ||T\mu - T\upsilon||\right]}{1 + ||\mu - \upsilon||} \\ &+ a_5 \frac{||\mu - \upsilon|| \left[1 + ||\mu - T\mu||\right]}{1 + ||\upsilon - T\upsilon||} + a_6 \frac{||\mu - T\mu|| \left[1 + ||\mu - T\upsilon||\right]}{1 + ||\upsilon - T\upsilon||} \\ &+ a_7 \frac{||\mu - T\upsilon|| \left[1 + ||T\mu - T\upsilon||\right]}{1 + ||\mu - \upsilon||} + a_8 \frac{||\mu - \upsilon|| \left[1 + ||\mu - T\upsilon||\right]}{1 + ||\mu - \upsilon||} \\ &+ a_9 \frac{||\mu - T\mu|| + ||\upsilon - T\upsilon|| + ||\mu - \upsilon||}{1 + ||\mu - T\mu|| \ ||\mu - T\upsilon|| \ ||\upsilon - T\upsilon|| \ ||\mu - \upsilon||} \\ &+ a_{10} \frac{||\mu - T\upsilon||^2 + ||\upsilon - T\mu||^2}{||\mu - T\upsilon|| + ||\upsilon - T\mu||} + a_{11}||\mu - \upsilon|| \end{aligned}$$

$$\Rightarrow ||\mu - \upsilon|| \leq (a_3 + a_4 + a_5 + a_7 + a_8 + a_9 + a_{10} + a_{11})||\mu - \upsilon||$$

which is a contradiction; for $a_3 + a_4 + a_5 + a_7 + a_8 + a_9 + a_{10} + a_{11} < 1$. Therefore $\mu = \upsilon$, and hence μ is a unique fixed point of T in X. □

Remark 1. *The above presented theorem reduces to the important well known theorems taking variations in the variables/making vanishing values to some real constants as well as and also by reducing the space to another.*

(1) Taking variations in variables and putting $a_4 = a_5 = a_6 = a_7 = a_8 = 0, a_{11} = \beta$, can obtain the result of [7] and Dass and Gupta [1] result can be found by in restricting the Hilbert space to the metric space.
(2) Putting $a_1 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = 0$ can obtain the result of [19].
(3) A comparison reveals that this theorem reduces to [20] on taking $a_2 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = 0$.

Now the following theorem is the refinement of the above theorem for taking two mapping in the inequality.

Theorem 2. *Let X be a closed subset of a Hilbert space and T_1, T_2 be a two self mappings on X satisfying the following condition, then T_1 and T_2 have a common fixed point in X.*

$$\begin{aligned}
||T_1x - T_2y|| \leq\ & a_1\frac{||x-T_1x||\,[1+||y-T_2y||]}{1+||x-y||} + a_2\frac{||y-T_2y||\,[1+||y-T_1x||]}{1+||x-y||} \\
&+ a_3\frac{||x-T_2y||\,[1+||y-T_1x||]}{1+||x-y||} + a_4\frac{||x-y||\,[1+||T_1x-T_2y||]}{1+||x-y||} \\
&+ a_5\frac{||x-y||\,[1+||x-T_1x||]}{1+||y-T_2y||} + a_6\frac{||x-T_1x||\,[1+||x-T_2y||]}{1+||y-T_2y||} \\
&+ a_7\frac{||x-T_2y||\,[1+||T_1x-T_2y||]}{1+||x-y||} + a_8\frac{||x-y||\,[1+||x-T_2y||]}{1+||x-y||} \\
&+ a_9\frac{||x-T_1x||+||y-T_2y||+||x-y||}{1+||x-T_1x||\ ||x-T_2y||\ ||y-T_2y||\ ||x-y||} \\
&+ a_{10}\frac{||x-T_2y||^2+||y-T_1x||^2}{||x-T_2y||+||y-T_1x||} + a_{11}||x-y||
\end{aligned}$$

for all $x, y \in X$ and $x \neq y$, where $a_i (i = 1, 2, 3, ..., 11)$ are non-negative real's with $0 \leq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + 3a_{10} + 2a_{11} < 1$.

Proof. A sequence $\{x_n\}$ for an arbitrary point $x_0 \in X$ defined as follows

$$x_{2n+1} = T_1x_{2n},\ x_{2n+2} = T_2x_{2n+1}, \text{for } n = 0, 1, 2, ...$$

Now consider the following to check the Cauchy sequence nature in X,

$$\begin{aligned}
&||x_{2n+1} - x_{2n}|| = ||T_1x_{2n} - T_2x_{2n-1}|| \\
&\leq a_1\frac{||x_{2n}-T_1x_{2n}||\,[1+||x_{2n-1}-T_2x_{2n-1}||]}{1+||x_{2n}-x_{2n-1}||} + a_2\frac{||x_{2n-1}-T_2x_{2n-1}||\,[1+||x_{2n-1}-T_1x_{2n}||]}{1+||x_{2n}-x_{2n-1}||} \\
&+ a_3\frac{||x_{2n}-T_2x_{2n-1}||\,[1+||x_{2n-1}-T_1x_{2n}||]}{1+||x_{2n}-x_{2n-1}||} + a_4\frac{||x_{2n}-x_{2n-1}||\,[1+||T_1x_{2n}-T_2x_{2n-1}||]}{1+||x_{2n}-x_{2n-1}||} \\
&+ a_5\frac{||x_{2n}-x_{2n-1}||\,[1+||x_{2n}-T_1x_{2n}||]}{1+||x_{2n-1}-T_2x_{2n-1}||} + a_6\frac{||x_{2n}-T_1x_{2n}||\,[1+||x_{2n}-T_2x_{2n-1}||]}{1+||x_{2n-1}-T_2x_{2n-1}||}
\end{aligned}$$

$$+ a_7 \frac{||x_{2n} - T_2x_{2n-1}|| \, [1 + ||T_1x_{2n} - T_2x_{2n-1}||]}{1 + ||x_{2n} - x_{2n-1}||} + a_8 \frac{||x_{2n} - x_{2n-1}|| \, [1 + ||x_{2n} - T_2x_{2n-1}||]}{1 + ||x_{2n} - x_{2n-1}||}$$
$$+ a_9 \frac{||x_{2n} - T_1x_{2n}|| + ||x_{2n-1} - T_2x_{2n-1}|| + ||x_{2n} - x_{2n-1}||}{1 + ||x_{2n} - T_1x_{2n}|| \; ||x_{2n} - T_2x_{2n-1}|| \; ||x_{2n-1} - T_2x_{2n-1}|| \; ||x_{2n} - x_{2n-1}||}$$
$$+ a_{10} \frac{||x_{2n} - T_2x_{2n-1}||^2 + ||x_{2n-1} - T_1x_{2n}||^2}{||x_{2n} - T_2x_{2n-1}|| + ||x_{2n-1} - T_1x_{2n}||} + a_{11}||x_{2n} - x_{2n-1}||$$

which implies that

$$||x_{2n+1} - x_{2n}|| = p(n)||x_{2n} - x_{2n-1}||$$

where

$$p(n) = \frac{(a_2 + a_4 + a_5 + a_8 + a_9 + 2a_{10} + a_{11}) + (a_2 + a_9 + 2a_{10} + a_{11})||x_{2n} - x_{2n-1}||}{(1 - a_1 - a_6 - a_{10} - a_{11}) + (1 - a_1 - a_2 - a_4 - a_5 - a_{10} - a_{11})||x_{2n} - x_{2n-1}||}.$$

Clearly $\lambda = p(n) < 1, \forall n = 1, 2, 3, ...$ Now in general, we get

$$||x_{n+1} - x_n|| = \lambda||x_n - x_{n-1}||$$

Continuing the above process, we get

$$||x_{n+1} - x_n|| = \lambda^n||x_1 - x_0||, n \geq 1,$$

taking $n \to \infty$, we obtain $||x_{n+1} - x_n|| \to 0$. Hence it follows that $\{x_n\}$ is a Cauchy sequence in X and so it has a limit μ in X. Since the sequences $\{x_{2n+1}\} = \{T_1x_{2n}\}$ and $\{x_{2n+2}\} = \{T_2x_{2n+1}\}$ are subsequences of $\{x_n\}$, the sub sequences have the same limit μ in X.

Next, we show that μ a is a common fixed point of T_1 and T_2 . For this by using the inequality, we arrive at

$$\begin{aligned}
||\mu - T_1\mu|| &= ||(\mu - x_{2n+2}) + (x_{2n+2} - T_1\mu)|| \\
&\leq ||\mu - x_{2n+2}|| + ||T_1\mu - T_2x_{2n+1}|| \\
&\leq a_1 \frac{||\mu - T_1\mu|| \, [1 + ||x_{2n+1} - T_2x_{2n+1}||]}{1 + ||\mu - x_{2n+1}||} + a_2 \frac{||x_{2n+1} - T_2x_{2n+1}|| \, [1 + ||x_{2n+1} - T_1\mu||]}{1 + ||\mu - x_{2n+1}||} \\
&+ a_3 \frac{||\mu - T_2x_{2n+1}|| \, [1 + ||x_{2n+1} - T_1\mu||]}{1 + ||\mu - x_{2n+1}||} + a_4 \frac{||\mu - x_{2n+1}|| \, [1 + ||T_1\mu - T_2x_{2n+1}||]}{1 + ||\mu - x_{2n+1}||} \\
&+ a_5 \frac{||\mu - x_{2n+1}|| \, [1 + ||\mu - T_1\mu||]}{1 + ||x_{2n+1} - T_2x_{2n+1}||} + a_6 \frac{||\mu - T_1\mu|| \, [1 + ||\mu - T_2x_{2n+1}||]}{1 + ||x_{2n+1} - T_2x_{2n+1}||} \\
&+ a_7 \frac{||\mu - T_2x_{2n+1}|| \, [1 + ||T_1\mu - T_2x_{2n+1}||]}{1 + ||\mu - x_{2n+1}||} + a_8 \frac{||\mu - x_{2n+1}|| \, [1 + ||\mu - T_2x_{2n+1}||]}{1 + ||\mu - x_{2n+1}||} \\
&+ a_9 \frac{||\mu - T_1\mu|| + ||x_{2n+1} - T_2x_{2n+1}|| + ||\mu - x_{2n+1}||}{1 + ||\mu - T_1\mu|| \; ||\mu - T_2x_{2n+1}|| \; ||x_{2n+1} - T_2x_{2n+1}|| \; ||\mu - x_{2n+1}||} \\
&+ a_{10} \frac{||\mu - T_2x_{2n+1}||^2 + ||x_{2n+1} - T\mu||^2}{||\mu - T_2x_{2n+1}|| + ||x_{2n+1} - T_1\mu||} + a_{11}||\mu - x_{2n+1}| + ||\mu - x_{2n+2}|
\end{aligned}$$

Letting $n \to \infty$, we obtained $||\mu - T_1\mu|| \leq (a_1 + a_6 + a_{10} + a_{11})||\mu - T_1\mu||$, because $a_1 + a_6 + a_{10} + a_{11} < 1$, it follows immediately that $T_1\mu = \mu$.
Similarly from hypothesis, we get $T_2\mu = \mu$ by considering the following

$$||\mu - T_2\mu|| = ||(\mu - x_{2n+1}) + (x_{2n+1} - T_2\mu)||$$

Finally, we want to show that μ is a unique fixed point of T_1,T_2. Let us suppose that $\upsilon(\mu \neq \upsilon)$ is also a common fixed point of T_1 and T_2. Then, in view of hypothesis, we have

$$\begin{aligned}
||\mu-\upsilon|| &= ||T_1\mu - T_2\upsilon|| \\
&\leq a_1 \frac{||\mu - T_1\mu||\,[1+||\upsilon - T_2\upsilon||]}{1+||\mu-\upsilon||} + a_2 \frac{||\upsilon - T_2\upsilon||\,[1+||\upsilon - T_1\mu||]}{1+||\mu-\upsilon||} \\
&+ a_3 \frac{||\mu - T_2\upsilon||\,[1+||\upsilon - T_1\mu||]}{1+||\mu-\upsilon||} + a_4 \frac{||\mu - \upsilon||\,[1+||T_1\mu - T_2\upsilon||]}{1+||\mu-\upsilon||} \\
&+ a_5 \frac{||\mu - \upsilon||\,[1+||\mu - T_1\mu||]}{1+||\upsilon-T_2\upsilon||} + a_6 \frac{||\mu - T_1\mu||\,[1+||\mu - T_2\upsilon||]}{1+||\upsilon-T_2\upsilon||} \\
&+ a_7 \frac{||\mu - T_2\upsilon||\,[1+||T_1\mu - T_2\upsilon||]}{1+||\mu-\upsilon||} + a_8 \frac{||\mu - \upsilon||\,[1+||\mu - T_2\upsilon||]}{1+||\mu-\upsilon||} \\
&+ a_9 \frac{||\mu - T_1\mu|| + ||\upsilon - T_2\upsilon|| + ||\mu-\upsilon||}{1+||\mu - T_1\mu||\;||\mu - T_2\upsilon||\;||\upsilon - T_2\upsilon||\;||\mu-\upsilon||} \\
&+ a_{10} \frac{||\mu - T_2\upsilon||^2 + ||\upsilon - T_1\mu||^2}{||\mu - T_2\upsilon|| + ||\upsilon - T_1\mu||} + a_{11}||\mu-\upsilon||
\end{aligned}$$

$$\Rightarrow ||\mu-\upsilon|| \leq (a_3+a_4+a_5+a_7+a_8+a_9+a_{10}+a_{11})||\mu-\upsilon||$$

This is a contradiction and hence it follows that $\mu = \upsilon$ and so the common fixed point is unique. □

Theorem 3. *Let X be a closed subset of a Hilbert space and T_1, T_2 be two self mappings on X satisfying*

$$\begin{aligned}
||T_1^p x - T_2^q y|| &\leq a_1 \frac{||x - T_1^p x||\,[1+||y - T_2^q y||]}{1+||x-y||} + a_2 \frac{||y - T_2^q y||\,[1+||y - T_1^p x||]}{1+||x-y||} \\
&+ a_3 \frac{||x - T_2^q y||\,[1+||y - T_1^p x||]}{1+||x-y||} + a_4 \frac{||x-y||\,[1+||T_1^p x - T_2^q y||]}{1+||x-y||} \\
&+ a_5 \frac{||x-y||\,[1+||x - T_1^p x||]}{1+||y - T_2^q y||} + a_6 \frac{||x - T_1^p x||\,[1+||x - T_2^q y||]}{1+||y - T_2^q y||} \\
&+ a_7 \frac{||x - T_2^q y||\,[1+||T_1^p x - T_2^q y||]}{1+||x-y||} + a_8 \frac{||x-y||\,[1+||x - T_2^q y||]}{1+||x-y||} \\
&+ a_9 \frac{||x - T_1^p x|| + ||y - T_2^q y|| + ||x-y||}{1+||x - T_1^p x||\;||x - T_2^q y||\;||y - T_2^q y||\;||x-y||} \\
&+ a_{10} \frac{||x - T_2^q y||^2 + ||y - T_1^p x||^2}{||x - T_2^q y|| + ||y - T_1^p x||} + a_{11}||x-y||
\end{aligned}$$

for all $x, y \in X$ and $x \neq y$, where $a_i (i = 1, 2, 3, ..., 11)$ are non-negative real's with $0 \leq a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9+3a_{10}+2a_{11} < 1$. Then T_1 and T_2 have a unique common fixed point in X.

Proof. From Theorem-2, T_1^p and T_2^q have a unique common fixed point $\mu \in X$, so that $T_1^p\mu = \mu$ and $T_2^q\mu = \mu$.
Now, $T_1^p(T_1\mu) = T_1(T_1^p\mu) = T_1\mu$
$\Rightarrow T_1\mu$ is a fixed point of T_1^p.
But μ is a unique fixed point of T_1^p.

$$\therefore\; T_1\mu = \mu$$

Similarly, we get $T_2\mu = \mu$.
$\therefore$ μ is a common fixed point of T_1 and T_2.
For uniqueness, let υ be another fixed point of T_1 and T_2, so that $T_1\upsilon = T_2\upsilon = \upsilon$. Then

$$\begin{aligned}
||\mu-\upsilon|| &= ||T_1^p\mu - T_2^q\upsilon|| \\
&\leq a_1\frac{||\mu - T_1^p\mu||\,[1+||\upsilon - T_2^q\upsilon||]}{1+||\mu-\upsilon||} + a_2\frac{||\upsilon - T_2^q\upsilon||\,[1+||\upsilon - T_1^p\mu||]}{1+||\mu-\upsilon||} \\
&+ a_3\frac{||\mu - T_2^q\upsilon||\,[1+||\upsilon - T_1^p\mu||]}{1+||\mu-\upsilon||} + a_4\frac{||\mu-\upsilon||\,[1+||T_1^p\mu - T_2^q\upsilon||]}{1+||\mu-\upsilon||} \\
&+ a_5\frac{||\mu-\upsilon||\,[1+||\mu - T_1^p\mu||]}{1+||\upsilon - T_2^q\upsilon||} + a_6\frac{||\mu - T_1^p\mu||\,[1+||\mu - T_2^q\upsilon||]}{1+||\upsilon - T_2^q\upsilon||} \\
&+ a_7\frac{||\mu - T_2^q\upsilon||\,[1+||T_1^p\mu - T_2^q\upsilon||]}{1+||\mu-\upsilon||} + a_8\frac{||\mu-\upsilon||\,[1+||\mu - T_2^q\upsilon||]}{1+||\mu-\upsilon||} \\
&+ a_9\frac{||\mu - T_1^p\mu|| + ||\upsilon - T_2^q\upsilon|| + ||\mu-\upsilon||}{1+||\mu - T_1^p\mu||\;||\mu - T_2^q\upsilon||\;||\upsilon - T_2^q\upsilon||\;||\mu-\upsilon||} \\
&+ a_{10}\frac{||\mu - T_2^q\upsilon||^2 + ||\upsilon - T_1^p\mu||^2}{||\mu - T_2^q\upsilon|| + ||\upsilon - T_1^p\mu||} + a_{11}||\mu-\upsilon||
\end{aligned}$$

$$\Rightarrow ||\mu-\upsilon|| \leq (a_3+a_4+a_5+a_7+a_8+a_9+a_{10}+a_{11})||\mu-\upsilon||$$

$$\Rightarrow \mu = \upsilon,\ \text{since } a_3+a_4+a_5+a_7+a_8+a_9+a_{10}+a_{11} < 1.$$

Hence μ is a unique common fixed point of T_1 and T_2 in X.
This completes the proof of the theorem. □

In the upcoming theorem we have taken a sequence of mappings on a closed subset of a Hilbert space converges point wise to a limit mapping and show that if this limit mapping has a fixed point then this fixed point is also the limit of fixed points of the mappings of the sequence.

Theorem 4. *Let X be a closed subset of a Hilbert space and let $\{T_i\}$ be a sequence of self mappings on X converging point wise to T and let*

$$\begin{aligned}
||T_ix - T_iy|| &\leq a_1\frac{||x-T_ix||\,[1+||y-T_iy||]}{1+||x-y||} + a_2\frac{||y-T_iy||\,[1+||y-T_ix||]}{1+||x-y||} \\
&+ a_3\frac{||x-T_iy||\,[1+||y-T_ix||]}{1+||x-y||} + a_4\frac{||x-y||\,[1+||T_ix-T_iy||]}{1+||x-y||} \\
&+ a_5\frac{||x-y||\,[1+||x-T_ix||]}{1+||y-T_iy||} + a_6\frac{||x-T_ix||\,[1+||x-T_iy||]}{1+||y-T_iy||} \\
&+ a_7\frac{||x-T_iy||\,[1+||T_ix-T_iy||]}{1+||x-y||} + a_8\frac{||x-y||\,[1+||x-T_iy||]}{1+||x-y||} \\
&+ a_9\frac{||x-T_ix|| + ||y-T_iy|| + ||x-y||}{1+||x-T_ix||\;||x-T_iy||\;||y-T_iy||\;||x-y||} \\
&+ a_{10}\frac{||x-T_iy||^2 + ||y-T_ix||^2}{||x-T_iy|| + ||y-T_ix||} + a_{11}||x-y||
\end{aligned}$$

for all $x, y \in X$ and $x \neq y$, where $a_i(i = 1, 2, 3, ..., 11)$ are non-negative real's with $0 \leq a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9+3a_{10}+2a_{11} < 1$, if each T_i has a fixed point μ_i and T has a fixed point μ, then the sequence $\{\mu_i\}$ converges to μ.

Proof. Since μ_i is a fixed point of T_i then we have

$$
\begin{aligned}
||\mu - \mu_n|| &= ||T\mu - T_n\mu_n|| \\
&= ||(T\mu - T_n\mu) + (T_n\mu - T_n\mu_n)|| \leq ||T\mu - T_n\mu|| + ||T_n\mu - T_n\mu_n|| \\
&\leq a_1 \frac{||\mu - T_n\mu|| \, [1 + ||\mu_n - T_n\mu_n||]}{1 + ||\mu - \mu_n||} + a_2 \frac{||\mu_n - T_n\mu_n|| \, [1 + ||\mu_n - T_n\mu||]}{1 + ||\mu - \mu_n||} \\
&+ a_3 \frac{||\mu - T_n\mu_n|| \, [1 + ||\mu_n - T_n\mu||]}{1 + ||\mu - \mu_n||} + a_4 \frac{||\mu - \mu_n|| \, [1 + ||T_n\mu - T_n\mu_n||]}{1 + ||\mu - \mu_n||} \\
&+ a_5 \frac{||\mu - \mu_n|| \, [1 + ||\mu - T_n\mu||]}{1 + ||\mu_n - T_n\mu_n||} + a_6 \frac{||\mu - T_n\mu|| \, [1 + ||\mu - T_n\mu_n||]}{1 + ||\mu_n - T_n\mu_n||} \\
&+ a_7 \frac{||\mu - T_n\mu_n|| \, [1 + ||T_n\mu - T_n\mu_n||]}{1 + ||\mu - \mu_n||} + a_8 \frac{||\mu - \mu_n|| \, [1 + ||\mu - T_n\mu_n||]}{1 + ||\mu - \mu_n||} \\
&+ a_9 \frac{||\mu - T_n\mu|| + ||\mu_n - T_n\mu_n|| + ||\mu - \mu_n||}{1 + ||\mu - T_n\mu|| \; ||\mu - T_n\mu_n|| \; ||\mu_n - T_n\mu_n|| \; ||\mu - \mu_n||} \\
&+ a_{10} \frac{||\mu - T_n\mu_n||^2 + ||\mu_n - T_n\mu||^2}{||\mu - T_n\mu_n|| + ||\mu_n - T_n\mu||} + a_{11} ||\mu - \mu_n|| + ||T\mu - T_n\mu||
\end{aligned}
$$

Letting $n \to \infty$, so that $T_n\mu \to T\mu, T_n\mu_n \to \mu_n$ and $T\mu = \mu$, we get

$$
\begin{aligned}
&\lim_{n\to\infty} ||\mu - \mu_n|| \leq (a_4 + a_5 + a_7 + a_8 + a_9 + a_{10} + a_{11}) \lim_{n\to\infty} ||\mu - \mu_n|| \\
&\Rightarrow \lim_{n\to\infty} ||\mu - \mu_n|| = 0, \text{since } a_4 + a_5 + a_7 + a_8 + a_9 + a_{10} + a_{11} < 1. \\
&\Rightarrow \mu_n \to \mu \text{ as } n \to \infty
\end{aligned}
$$

This completes the proof. □

Example 1. *For an Example of Theorem 1, let* $T : [0, 1] \to [0, 1]$ *defined by* $Tx = \frac{x^3}{6}$, *for all* $x \in [0, 1]$. *Obviously 0 is the only fixed point of* T *with usual norm* $||x - y|| =| x - y |$, *for all* $x \in [0, 1]$.

Example 2. *The following is an example of Theorem 2*
Let $T_1, T_2 : [0, 1] \to [0, 1]$ *defined as* $T_1x = \frac{x}{3}$ *and* $T_2x = \frac{x}{4}$, *for all* $x \in [0, 1]$. *Then with usual norm* $||x - y|| =| x - y |$ *one can see that 0 is the only common fixed point of* T_1 *and* T_2.

Conclusions

The Banach's contraction principle has been refined and extended on a closed subset of a Hilbert space to a self mapping involving more number of rational terms in the contractive condition. The same result is extended to a pair of self mappings, positive integers powers of a pair mapping and then again developed the same to a sequence of mappings. In all three different cases we observed the existence and uniqueness of the fixed point.

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N. Seshagiri Rao

Department of Applied Mathematics, School of Applied Natural Science, Adama Science and Technology University, Post Box No. 1888, Adama, Ethiopia.

Email address: seshu.namana@gmail.com

K. Kalyani

Department of Science and Humanities, Vignan's Foundation for Science, Technology & Research, Vadlamudi - 522213, Andhra Pradesh, India.

Email address: kalyani.namana@gmail.com