Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 220-230.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

SOLUTION OF COUPLED SYSTEM OF CAUCHY PROBLEM OF NONLOCAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we apply the Adomian decomposition method (ADM) for solving a coupled system of nonlocal Cauchy problem of differential equations. The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are discussed. Two examples are given in order to illustrate our results.

1. Introduction

The Cauchy problems with multi-point or non-local conditions have been extensively studied by several authors in the last two decades. The interested reader can see [1]-[13].

In paper [14], The nonlocal Cauchy problem of the differential equation

$$\frac{dx\,(t)}{dt} = f\,(t, x\,(t))\,, \quad t \in (0, T]$$

with the nonlocal condition

$$x(0) + \sum_{k=0}^{n} a_k x\,(t_k) = x_0, \qquad t_k \in (0, T)\,.$$

is studied. In this paper, we are concerned with the coupled system of nonlocal Cauchy problem of differential equations

$$\frac{dx\,(t)}{dt} = f_1\,(t, y\,(t))\,, \qquad \frac{dy\,(t)}{dt} = f_2\,(t, x\,(t))\,, \qquad t \in (0, T] \tag{1}$$

with the nonlocal conditions

$$x(0) + \sum_{k=0}^{n} a_k x\,(t_k) = x_0, \qquad y(0) + \sum_{j=0}^{m} b_j y\,(t_j) = y_0, \tag{2}$$

where $a_k,\ b_j \in R$ and $t_k,\ t_j \in (0, T)$. The existence and uniqueness of the solution $X = (x, y)^T \in C(J) \times C(J)$, where $C = C(J)$ be the space of all real valued

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2000 *Mathematics Subject Classification.* 34A12, 34A30, 34D20.
Key words and phrases. Nonlocal Cauchy problem; existence and uniqueness of solution; Adomian decomposition method; error analysis.
Submitted Oct. 22, 2019. Revised Jan. 15, 2020.

functions which are continuous on J and $J = [0,T],\ T < \infty$ of the coupled system (1)-(2) will be proved, the integral representation of this solution will be proved and the solution algorithm using ADM will be discussed. Two examples are given in order to illustrate our results.

2. Problem solving

2.1. **Integral representation.** For the integral representation of the solution of the nonlocal problem (1)-(2), we have the following lemma,

Lemma 1. *If* $\left(1+\sum_{k=0}^{n} a_k\right) \neq 0$, *and* $\left(1+\sum_{j=0}^{m} b_j\right) \neq 0$, *then the unique solution of the nonlocal problem (1)-(2) can be expressed by the system of the integral equations*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \left(1+\sum_{k=0}^{n} a_k\right)^{-1}\left(x_0-\sum_{k=0}^{n} a_k \int_0^{t_k} f_1\left(s,y\left(s\right)\right)ds\right)+\int_0^t f_1(s,y(s))ds \\ \left(1+\sum_{j=0}^{m} b_j\right)^{-1}\left(y_0-\sum_{j=0}^{m} b_j \int_0^{t_j} f_2\left(s,x\left(s\right)\right)ds\right)+\int_0^t f_2(s,x(s))ds \end{pmatrix}.$$

Proof. Operating with $I = \int_0^t (.)\,ds$ to both sides of equation (1), we get

$$x\left(t\right) = x(0)+\int_0^t f_1\left(s,y\left(s\right)\right)ds, \tag{3}$$

$$y\left(t\right) = y(0)+\int_0^t f_2\left(s,x\left(s\right)\right)ds. \tag{4}$$

Let $t = t_k$ in equation (3), then we get

$$x\left(t_k\right) = x(0)+\int_0^{t_k} f_1\left(s,y\left(s\right)\right)ds, \tag{5}$$

and

$$\sum_{k=0}^{n} a_k x\left(t_k\right) = x(0)\sum_{k=0}^{n} a_k+\sum_{k=0}^{n} a_k\int_0^{t_k} f_1\left(s,y\left(s\right)\right)ds. \tag{6}$$

Substitute from equation (6) into equations (2) we get,

$$x(0)+x(0)\sum_{k=0}^{n} a_k = x_0-\sum_{k=0}^{n} a_k\int_0^{t_k} f_1\left(s,y\left(s\right)\right)ds,$$

and

$$x(0) = \left(1+\sum_{k=0}^{n} a_k\right)^{-1}\left(x_0-\sum_{k=0}^{n} a_k\int_0^{t_k} f_1\left(s,y\left(s\right)\right)ds\right), \tag{7}$$

Substitute from equation (7) into equation (3) we obtain

$$x(t)=\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\left(x_0-\sum_{k=0}^{n}a_k\int_0^{t_k}f_1\left(s,y\left(s\right)\right)ds\right)+\int_0^{t}f_1(s,y(s))ds. \quad (8)$$

Similarly, we can obtain

$$y(t)=\left(1+\sum_{j=0}^{m}b_j\right)^{-1}\left(y_0-\sum_{j=0}^{m}b_j\int_0^{t_j}f_2\left(s,x\left(s\right)\right)ds\right)+\int_0^{t}f_2(s,x(s))ds. \quad (9)$$

To complete the proof, differentiating (8), (9) we obtain (1). Also, let $t=0$ in (8), (9), then by direct calculations we can get (2). □

2.2. **The solution algorithm.** The solution algorithm of equations (8)-(9) using ADM is

$$\begin{aligned}
x_0\left(t\right) &= x_0\left(1+\sum_{k=0}^{n}a_k\right)^{-1}, \\
x_i\left(t\right) &= -\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\left(\sum_{k=0}^{n}a_k\int_0^{t_k}A_{i-1}\left(s\right)ds\right)+\int_0^{t}A_{i-1}\left(s\right)ds, \quad (10)\\
y_0\left(t\right) &= y_0\left(1+\sum_{j=0}^{m}b_j\right)^{-1}, \\
y_i\left(t\right) &= -\left(1+\sum_{j=0}^{m}b_j\right)^{-1}\left(\sum_{j=0}^{m}b_j\int_0^{t_j}B_{i-1}\left(s\right)ds\right)+\int_0^{t}B_{i-1}\left(s\right)ds, \quad (11)
\end{aligned}$$

where A_i, B_i are Adomian polynomials of the nonlinear terms $f_1\left(t,y\left(t\right)\right)$, $f_2\left(t,x\left(t\right)\right)$ which take the forms,

$$\begin{aligned}
A_i &= \frac{1}{i!}\,\frac{d^i}{d\lambda^i}f_1\left(t,\sum_{\tau=0}^{\infty}\lambda^{\tau}y_{\tau}\right)\Bigg|_{\lambda=0}, \\
B_i &= \frac{1}{i!}\,\frac{d^i}{d\lambda^i}f_2\left(t,\sum_{\tau=0}^{\infty}\lambda^{\tau}x_{\tau}\right)\Bigg|_{\lambda=0}.
\end{aligned}$$

Finally, the solution of problem (1)-(2) will be

$$x\left(t\right)=\sum_{i=0}^{\infty}x_i(t), \qquad y\left(t\right)=\sum_{i=0}^{\infty}y_i(t). \quad (12)$$

3. Convergence analysis

3.1. **Existence of solution.** Let X be the Banach space of all ordered pairs $(x,y)^T,\ x,y\in C(J)$ with the norm $\left\|(x,y)^T\right\|_X=\|x\|+\|y\|$ where $\|x\|=\max_{t\in J}|x(t)|$. It is clear that $(X,\ \|.\|_X)$ is Banach space.

Now assume that the functions $f_i : [0,T] \times R \to R,\ i = 1,2$ are continuous and satisfy the Lipschitz conditions

$$|f_1(t,y) - f_1(t,v)| \leq k_1 |y - v| \qquad (13)$$
$$|f_2(t,x) - f_2(t,u)| \leq k_2 |x - u| \qquad (14)$$

Theorem 2. *Assume that $f_1, f_2 : [0,T] \times R \to R$ satisfies the Lipschitz conditions (13)-(14). In addition, assume that*

$$K = \max\left\{ k_1\ T\ \left(\left|1 + \sum_{k=0}^{n} a_k\right|^{-1} \sum_{k=0}^{n} |a_k| + 1 \right),\ k_2\ T\ \left(\left|1 + \sum_{j=0}^{m} b_j\right|^{-1} \sum_{j=0}^{m} |b_j| + 1 \right) \right\} < 1$$

and $\left(1 + \sum_{k=0}^{n} a_k\right)$ *and* $\left(1 + \sum_{j=0}^{m} b_j\right) \neq 0$. *Then the integral equations (8)-(9), which equivalent to problem (1)-(2) has a unique solution* $X = \begin{pmatrix} x \\ y \end{pmatrix} \in C(J) \times C(J)$.

Proof. Define the operator F associated with the coupled system (8)-(9) as

$$F(x,\, y)(t) = \begin{pmatrix} F_1 y(t) \\ F_2 x(t) \end{pmatrix}$$

where

$$F_1 y(t) = \left(1 + \sum_{k=0}^{n} a_k\right)^{-1} x_0 - \left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \sum_{k=0}^{n} a_k \int_0^{t_k} f_1(s, y(s))\, ds + \int_0^{t} f_1(s, y(s))\, ds,$$

and

$$F_2 x(t) = \left(1 + \sum_{j=0}^{m} b_j\right)^{-1} y_0 - \left(1 + \sum_{j=0}^{m} b_j\right)^{-1} \sum_{j=0}^{m} b_j \int_0^{t_j} f_2(s, x(s)) ds + \int_0^{t} f_2(s, x(s)) ds.$$

We have for $(u,v)^T \in C(J) \times C(J)$

$$F_1 v(t) = \left(1 + \sum_{k=0}^{n} a_k\right)^{-1} x_0 - \left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \sum_{k=0}^{n} a_k \int_0^{t_k} f_1(s, v(s))\, ds + \int_0^{t} f_1(s, v(s))\, ds,$$

and

$$F_2 u(t) = \left(1 + \sum_{j=0}^{m} b_j\right)^{-1} y_0 - \left(1 + \sum_{j=0}^{m} b_j\right)^{-1} \sum_{j=0}^{m} b_j \int_0^{t_j} f_2(s, u(s)) ds + \int_0^{t} f_2(s, u(s)) ds.$$

Now for $(x,y)^T,\ (u,v)^T \in C(J) \times C(J)$, and for any $t \in [0,T]$, we get

$$|F_1 y(t) - F_1 v(t)| = \Bigg| \left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \sum_{k=0}^{n} a_k \int_0^{t_k} [f_1(s, y(s)) - f_1(s, v(s))]\, ds + \int_0^{t} [f_1(s, y(s)) - f_1(s, v(s))]\, ds \Bigg|$$

$$
\begin{aligned}
&\leq \left|1+\sum_{k=0}^{n} a_k\right|^{-1} \sum_{k=0}^{n} |a_k| \int_0^{t_k} |f_1(s,y(s)) - f_1(s,v(s))|\, ds \\
&\quad + \int_0^{t} |f_1(s,y(s)) - f_1(s,v(s))|\, ds \\
&\leq k_1 \left|1+\sum_{k=0}^{n} a_k\right|^{-1} \sum_{k=0}^{n} |a_k| \int_0^{t_k} |y(s)-v(s)|\, ds + k_1 \int_0^{t} |y(s)-v(s)|\, ds \\
&\leq k_1\ T\ ||y-v|| \left(\left|1+\sum_{k=0}^{n} a_k\right|^{-1} \sum_{k=0}^{n} |a_k| + 1 \right).
\end{aligned}
$$

Hence

$$
\|F_1 y - F_1 v\| \ \leq\ k_1\ T\ ||y-v|| \left(\left|1+\sum_{k=0}^{n} a_k\right|^{-1} \sum_{k=0}^{n} |a_k| + 1 \right). \tag{15}
$$

As done above we can obtain

$$
\|F_2 x - F_2 u\| \ \leq\ k_2\ T\ ||x-y|| \left(\left|1+\sum_{j=0}^{m} b_j\right|^{-1} \sum_{j=0}^{m} |b_j| + 1 \right). \tag{16}
$$

It follows from (15) and (16) that

$$
\begin{aligned}
\left\| F(x,\, y)^T - F(u,\, v)^T \right\|_X &= \left\| (F_1 y,\, F_2 x)^T - (F_1 v,\, F_2 u)^T \right\|_X = \left\| (F_1 y - F_1 v,\, F_2 x - F_2 u)^T \right\|_X, \\
&= \|F_1 y - F_1 v\| + \|F_2 x - F_2 u\|, \\
&\leq K\left(||y-v|| + ||x-y||\right).
\end{aligned}
$$

Since $K < 1$, therefore, F is a contraction operator. So, by Banach fixed point theorem, the operator F has a unique fixed point, which is the unique solution of the coupled system of nonlocal Cauchy problem (1)-(2) given by (8)-(9), where

$$
\begin{aligned}
x(0) &= \lim_{t\to 0} x(t) = \left(1+\sum_{k=0}^{n} a_k\right)^{-1} \left(x_0 - \sum_{k=0}^{n} a_k \int_0^{t_k} f_1(s, y(s))\, ds \right) + \int_0^{t} f_1(s, y(s) ds, \\
y(0) &= \lim_{t\to 0} y(t) = \left(1+\sum_{j=0}^{m} b_j\right)^{-1} \left(y_0 - \sum_{j=0}^{m} b_j \int_0^{t_k} f_2(s, x(s))\, ds \right) + \int_0^{t} f_2(s, x(s) ds.
\end{aligned}
$$

and

$$
\begin{aligned}
x(T) &= \lim_{t\to T} x(t) = \left(1+\sum_{k=0}^{n} a_k\right)^{-1} \left(x_0 - \sum_{k=0}^{n} a_k \int_0^{t_k} f_1(s, y(s))\, ds \right) + \int_0^{T} f_1(s, y(s) ds, \\
y(T) &= \lim_{t\to T} y(t) = \left(1+\sum_{j=0}^{m} b_j\right)^{-1} \left(y_0 - \sum_{j=0}^{m} b_j \int_0^{t_k} f_2(s, x(s))\, ds \right) + \int_0^{T} f_2(s, x(s) ds.
\end{aligned}
$$

This completes the proof. □

3.2. **Proof of convergence.**

Theorem 3. *Suppose that hypotheses of Theorem 2 hold and assume that the unique solution* $X = (x, y)^T$ *of (1) with nonlocal conditions given by (2) can be represented in series form* $(x, y)^T = \left(\sum_{i=0}^{\infty} x_i(t), \sum_{i=0}^{\infty} y_i(t)\right)^T$. *Then, the following Adomian decomposition scheme with* $X_i(t) = (x_i(t), y_i(t))^T$, $n \geq 0$ *converges to the unique solution of (1) with nonlocal conditions given by (2), if* $\|X_1\|_X < c$, *where c is a positive constant.*

Proof. Let S_p be arbitrary partial sum and $S_p = \sum_{i=0}^{p} X_i(t)$, define the two sequences $\{S_{1p}\}$ and $\{S_{2p}\}$ such that, $S_{1p} = \sum_{i=0}^{p} x_i(t)$ and $S_{2p} = \sum_{i=0}^{p} y_i(t)$ are the sequences of partial sums from the series solutions $\sum_{i=0}^{\infty} x_i(t)$ and $\sum_{i=0}^{\infty} y_i(t)$. Since

$$f_1(t, y(t)) = \sum_{i=1}^{\infty} A_i, \qquad f_2(t, x(t)) = \sum_{i=1}^{\infty} B_i,$$

then we have

$$f_1(t, S_{2p}) = \sum_{i=1}^{p} A_i, \qquad f_2(t, S_{1p}) = \sum_{i=1}^{p} B_i, \qquad \text{respectively.}$$

We are going to prove that $\{S_p\}$ is a Cauchy sequence in the Banach space X. Let S_p and S_q be arbitrary partial sums with $p > q$. Indeed,

$$\begin{aligned} \|S_p - S_q\|_X &= \|S_{1p} - S_{1q}\| + \|S_{2p} - S_{2q}\|, \\ &= \max_{t \in J} |S_{1p} - S_{1q}| + \max_{t \in J} |S_{2p} - S_{2q}|. \end{aligned} \tag{17}$$

Now, from (10), we have

$$\begin{aligned} |S_{1p} - S_{1q}| &= \left| \sum_{i=0}^{p} x_i(t) - \sum_{i=0}^{q} x_i(t) \right|, \\ &= \left| -\left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} \sum_{i=q+1}^{p} A_{i-1}(s)\, ds \right) + \int_0^t \sum_{i=q+1}^{p} A_{i-1}(s)\, ds \right|, \\ &= \left| -\left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} \sum_{i=q}^{p-1} A_i(s)\, ds \right) + \int_0^t \sum_{i=q}^{p-1} A_i(s)\, ds \right|, \\ &= \left| -\left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} \left[f_1(t, S_{2(p-1)}) - f_1(t, S_{2(q-1)}) \right] ds \right) \right. \end{aligned}$$

$$+\int_0^t \left[f_1(t,S_{2(p-1)})-f_1(t,S_{2(q-1)})\right]ds\Bigg|,$$

$$\leq \left|1+\sum_{k=0}^n a_k\right|^{-1}\sum_{k=0}^n |a_k|\int_0^{t_k}\left|f_1(t,S_{2(p-1)})-f_1(t,S_{2(q-1)})\right|ds$$

$$+\int_0^t \left|f_1(t,S_{2(p-1)})-f_1(t,S_{2(q-1)})\right|ds,$$

$$\leq k_1\ T\left(\left|1+\sum_{k=0}^n a_k\right|^{-1}\sum_{k=0}^n |a_k|+1\right)\left|S_{2(p-1)}-S_{2(q-1)}\right|\leq K\left\|S_{2(p-1)}-S_{2(q-1)}\right\|.$$

Then

$$\|S_{1p}-S_{1q}\|\leq K\left\|S_{2(p-1)}-S_{2(q-1)}\right\|.$$

Let $p=q+1$, then

$$\left\|S_{1(q+1)}-S_{1q}\right\|\leq K\left\|S_{2q}-S_{2(q-1)}\right\|\leq K^2\left\|S_{2(q-1)}-S_{2(q-2)}\right\|\leq\cdots\leq K^q\|S_{21}-S_{20}\|$$

From the triangle inequality we have,

$$\begin{aligned}\|S_{1p}-S_{1q}\| &\leq \left\|S_{1(q+1)}-S_{1q}\right\|+\left\|S_{1(q+2)}-S_{1(q+1)}\right\|+\cdots+\left\|S_{1p}-S_{1(p-1)}\right\|\\ &\leq \left[K^q+K^{q+1}+\cdots+K^{p-1}\right]\|S_{21}-S_{20}\|\\ &= K^q\left[1+K+\cdots+K^{p-q-1}\right]\|S_{21}-S_{20}\|\\ &\leq K^q\left[\frac{1-K^{p-q}}{1-K}\right]\|y_1\|\end{aligned}$$

Now $0<K<1$, and $p>q$ implies that $(1-K^{p-q})\leq 1$. Consequently,

$$\|S_{1p}-S_{1q}\|\leq\frac{K^q}{1-K}\|y_1\|. \tag{18}$$

In the same way, we can obtain that

$$\|S_{2p}-S_{2q}\|\leq\frac{K^q}{1-K}\|x_1\|. \tag{19}$$

Substituting from (18) and (19) into (17), we get

$$\|S_p-S_q\|_X\leq\frac{K^q}{1-K}\|y_1\|+\frac{K^q}{1-K}\|x_1\|=\frac{K^q}{1-K}(\|x_1\|+\|y_1\|)=\frac{K^q}{1-K}\|X_1\|_X.$$

Now, if $\|X_1\|_X<c$ then $\|S_p-S_q\|_X\to 0$ as $q\to\infty$ and hence, $\{S_p\}$ is a Cauchy sequence in this Banach space X. So, the series $\sum_{i=0}^{\infty}x_i(t)$ and $\sum_{i=0}^{\infty}y_i(t)$ converge and this completes the proof. □

3.3. **Error analysis.**

Theorem 4. *The maximum absolute truncation error of the solution (12) to the problem (1)-(2) is estimated to be,*

$$\left\|X-\sum_{i=0}^{q}X_i\right\|_X\leq\frac{K^q}{1-K}\|X_1\|_X,$$

where $X = (x,y)^T$.

Proof. From Theorem 3 we have,

$$\|S_p - S_q\|_X \leq \frac{K^q}{1-K} \|X_1\|_X ,$$

but, $S_p = \left(\sum_{i=0}^{p} x_i(t), \sum_{i=0}^{p} y_i(t) \right)^T$, and as $p \to \infty$ then, $S_p \to X(t)$ so,

$$\|X - S_q\|_X \leq \frac{K^q}{1-K} \|X_1\|_X .$$

so, the maximum absolute truncation error in the interval J is,

$$\left\| X - \sum_{i=0}^{q} X_i \right\|_X \leq \frac{K^q}{1-K} \|X_1\|_X .$$

and this completes the proof. □

4. Numerical examples

Example 1 Consider the following example,

$$\frac{dx}{dt} = \frac{1}{10} ty, \quad \frac{dy}{dt} = \frac{1}{20} \sin x + t^2, \quad t \in [0,1], \tag{20}$$

$$x(0) + 0.1x\left(\frac{1}{4}\right) = 1, \quad y(0) + 0.2y\left(\frac{1}{2}\right) = 2. \tag{21}$$

Here, $k_1 \leq \frac{1}{10}, T = 1, k_2 \leq \frac{1}{20}, k_1\ T \left(\left| 1 + \sum_{k=0}^{n} a_k \right|^{-1} \sum_{k=0}^{n} |a_k| + 1 \right) = \frac{1}{10}\left(1 + \frac{0.1}{1.1}\right) =$ $0.109, k_2\ T \left(\left| 1 + \sum_{j=0}^{m} b_j \right|^{-1} \sum_{j=0}^{m} |b_j| + 1 \right) = \frac{1}{20}\left(1 + \frac{0.2}{1.2}\right) = 0.058,$

$K = \max \left\{ k_1\ T \left(\left| 1 + \sum_{k=0}^{n} a_k \right|^{-1} \sum_{k=0}^{n} |a_k| + 1 \right),\ k_2\ T \left(\left| 1 + \sum_{j=0}^{m} b_j \right|^{-1} \sum_{j=0}^{m} |b_j| + 1 \right) \right\} =$ $0.109 < 1$,then all assumptions of Theorem (1) are satisfied.

Using equations (10)-(11), problem (20)-(21) will be

$$x(t) = \frac{1}{1.1} - \frac{0.1}{1.1} \int_0^{1/4} \left[\frac{1}{10} sy(s) \right] ds + \int_0^t \left[\frac{1}{10} sy(s) \right] ds, \tag{22}$$

$$y(t) = \frac{2}{1.2} - \frac{0.2}{1.2} \int_0^{1/2} \left[\frac{1}{20} \sin x(s) + s^2 \right] ds + \int_0^t \left[\frac{1}{20} \sin x(s) + s^2 \right] ds, \tag{23}$$

Applying ADM to equations (22)-(23), we have

$$x_0(t) = \frac{1}{1.1},$$

$$x_i(t) = -\frac{0.1}{1.1} \int_0^{1/4} \left[\frac{1}{10} sy_{i-1}(s) \right] ds + \int_0^t \left[\frac{1}{10} sy_{i-1}(s) \right] ds, \ i \geq 1. \tag{24}$$

$$\begin{aligned} y_0(t) &= \frac{2}{1.2}, \\ y_i(t) &= -\frac{0.2}{1.2}\int\limits_0^{1/2}\left[\frac{1}{20}A_{i-1}(s)+s^2\right]ds+\int\limits_0^{t}\left[\frac{1}{20}A_{i-1}(s)+s^2\right]ds,\ i\geq 1. \end{aligned} \tag{25}$$

From equations (24) and (25), the solution of the problem (20)-(21) is,

$$x(t)=\lim_{m\to\infty}\sum_{i=0}^{m}x_i(t),\ y(t)=\lim_{m\to\infty}\sum_{i=0}^{m}y_i(t) \tag{26}$$

Figures 1.a and 1.b show ADM solution ($m=3$).

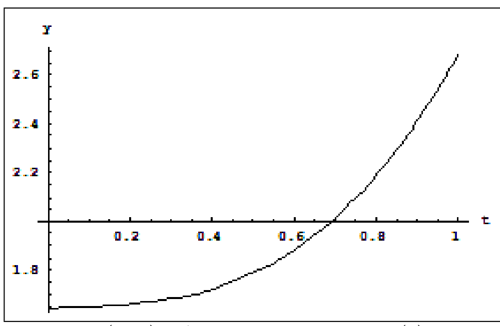

Fig (1-a): ADM solution of $y(t)$.

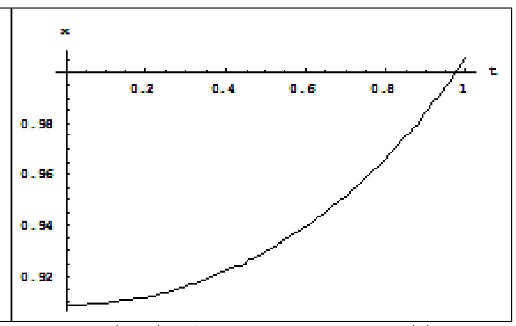

Fig (1-b): ADM solution of $x(t)$.

Example 2 Consider the following example,

$$\frac{dx}{dt}=\frac{1}{5}y+\frac{1}{10}\cos y,\quad \frac{dy}{dt}=\frac{1}{4}x,\quad t\in[0,1], \tag{27}$$

$$x(0)+2x(0.1)+4x(0.3)=4,\quad y(0)+3y(0.4)+5y(0.6)=5 \tag{28}$$

Here, $k_1\leq\frac{1}{10}, T=1, k_2\leq\frac{1}{20}, k_1\ T\ \left(\left|1+\sum\limits_{k=0}^{n}a_k\right|^{-1}\sum\limits_{k=0}^{n}|a_k|+1\right)=\frac{1}{10}\left(1+\frac{6}{7}\right)=$ $0.186, k_2\ T\ \left(\left|1+\sum\limits_{j=0}^{m}b_j\right|^{-1}\sum\limits_{j=0}^{m}|b_j|+1\right)=\frac{1}{4}\left(1+\frac{8}{9}\right)=0.472,$

$$K=\max\left\{k_1\ T\ \left(\left|1+\sum_{k=0}^{n}a_k\right|^{-1}\sum_{k=0}^{n}|a_k|+1\right),\ k_2\ T\ \left(\left|1+\sum_{j=0}^{m}b_j\right|^{-1}\sum_{j=0}^{m}|b_j|+1\right)\right\}=$$

$0.472<1$,then all assumptions of Theorem (1) are satisfied.

Using equations (10)-(11), problem (27)-(28) will be

$$\begin{aligned} x(t) &= \frac{4}{7}-\frac{1}{7}\left[2\int_0^{0.1}\left[\frac{1}{5}y(s)+\frac{1}{10}\cos y(s)\right]ds+4\int_0^{0.3}\left[\frac{1}{5}y(s)+\frac{1}{10}\cos y(s)\right]ds\right] \\ &\quad+\int_0^t\left[\frac{1}{5}y(s)+\frac{1}{10}\cos y(s)\right]ds, \qquad (29) \\ y(t) &= \frac{5}{9}-\frac{1}{9}\left[3\int_0^{0.4}\left[\frac{1}{4}x(s)\right]ds+5\int_0^{0.6}\left[\frac{1}{4}x(s)\right]ds\right] \\ &\quad+\int_0^t\left[\frac{1}{4}x(s)\right]ds. \qquad (30) \end{aligned}$$

Applying ADM to equations (29)-(30), we have

$$\begin{aligned} x_0(t) &= \frac{4}{7}, \\ x_i(t) &= -\frac{1}{7}\left[2\int_0^{0.1}\left[\frac{1}{5}y_{i-1}(s)+\frac{1}{10}A_{i-1}(s)\right]ds+4\int_0^{0.3}\left[\frac{1}{5}y_{i-1}(s)+\frac{1}{10}A_{i-1}(s)\right]ds\right] \\ &\quad+\int_0^t\left[\frac{1}{5}y_{i-1}(s)+\frac{1}{10}A_{i-1}(s)\right]ds,\ i\geq 1. \qquad (31) \end{aligned}$$

$$\begin{aligned} y_0(t) &= \frac{5}{9}, \\ y_i(t) &= -\frac{1}{9}\left[3\int_0^{0.4}\left[\frac{1}{4}x_{i-1}(s)\right]ds+5\int_0^{0.6}\left[\frac{1}{4}x_{i-1}(s)\right]ds\right] \\ &\quad+\int_0^t\left[\frac{1}{4}x_{i-1}(s)\right]ds,\ i\geq 1. \qquad (32) \end{aligned}$$

From equations (31) and (32), the solution of the problem (27)-(28) is,

$$x(t)=\lim_{m\to\infty}\sum_{i=0}^{m}x_i(t),\ y(t)=\lim_{m\to\infty}\sum_{i=0}^{m}y_i(t) \qquad (33)$$

Figures 2.a and 2.b show ADM solution ($m=5$).

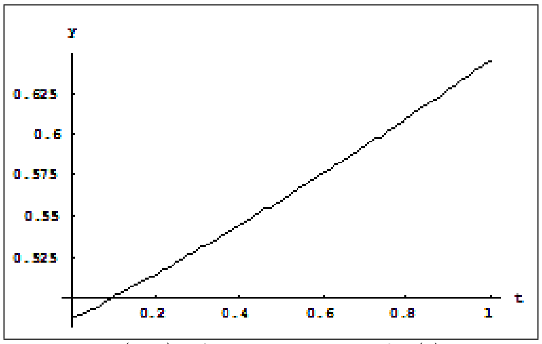

Fig (2-a): ADM solution of $y(t)$.

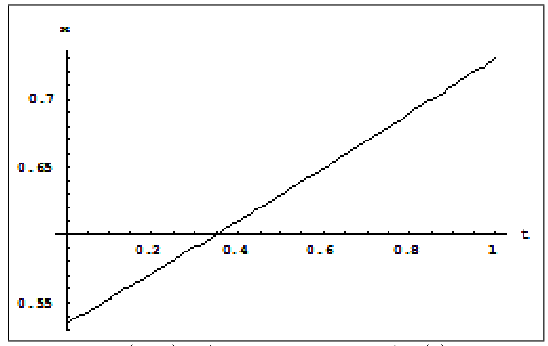

Fig (2-b): ADM solution of $x(t)$.

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