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COMMON FIXED POINT THEOREMS SATISFYING A CONTRACTIVE CONDITION OF INTEGRAL TYPE

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Abstract. In this paper, using the (CLR) property, common fixed point results for two pairs of weakly compatible mappings satisfying contractive condition of integral type on metric spaces are established.

1. Introduction

Fixed point theory is one of the most fruitful and applicable topics of nonlinear analysis, which is widely used not only in other mathematical theories, but also in many practical problems of natural sciences and engineering. The Banach contraction mapping principle [1] is indeed the most popular result of metric fixed point theory. This principle has many application in several domains, such as differential equations, functional equations, integral equations, economics, wild life, and several others.

Branciari [2] gave an integral version of the Banach contraction principles and proved fixed point theorem for a single-valued contractive mapping of integral type in metric space. Afterwards many researchers [3]-[7] extended the result of Branciari and obtained fixed point and common fixed point theorems for various contractive conditions of integral type on different spaces.

Now, we recollect some known definitions and results from the literature which are helpful in the proof of our main results.

Definition 1.1. A coincidence point of a pair of self-mapping $A, B : X \to X$ is a point $x \in X$ for which $Ax = Bx$.

A common fixed point of a pair of self-mapping $A, B : X \to X$ is a point $x \in X$ for which $Ax = Bx = x$. Jungck [8] initiated the concept of weakly compatible maps to study common fixed point theorems.

Definition 1.2. [8] A pair of self-mapping $A, B : X \to X$ is weakly compatible if they commute at their coincidence points, that is, if there exists a point $x \in X$ such that $ABx = BAx$ whenever $Ax = Bx$.

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In the study of common fixed points of weakly compatible mappings, we often require the assumption of completeness of the space or subspace or continuity of mappings involved besides some contractive condition. Aamri and El Moutawakil [9] introduced the notion of (E.A) property, which requires only the closedness of the subspace and Liu et al. [10] extended the (E.A) property to common the (E.A) property as follows.

Definition 1.3. Let (X, d) be a metric space and $A, B, P, Q : X \to X$ be four self-maps. The pairs (A, Q) and (B, P) satisfy the common (E.A) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Py_n = s \in X.$$

Sintunavarat and Kumam [11] introduced the notion of the (CLR) property, which never requires any condition on closedness of the space or subspace and Imdad et al. [12] introduced the common (CLR) property which is an extension of the (CLR) property.

Definition 1.4. Let (X, d) be a metric space and $A, B, P, Q : X \to X$ be four self-maps. The pairs (A, Q) and (B, P) satisfy the common limit range property with respect to mappings Q and P, denoted by (CLR_{PQ}) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Py_n = s \in QX \cap PX.$$

Definition 1.5. Let Φ be the family of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the following:

(1) ϕ is lower semi continuous.
(2) $\phi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$.
(3) ϕ is discontinuous at $t = 0$.

2. Common Fixed Point Theorems

In this section, we study common fixed point theorems for weakly compatible mappings using (CLR) property and E.A. property.

Theorem 2.1. Let (X, d) be a metric space and A, B, P, Q be four self maps on X satisfying the following:
(1) The pairs (A, P) and (B, Q) share (CLR_{PQ}) property;

$$(2)\ \int_0^{d(Ax,By)} \varphi(t)dt \le \alpha \int_0^{\frac{d(By,Qy)[1+d(Ax,Px)]}{[1+d(Px,Qy)]}} \varphi(t)dt + \beta \int_0^{d(Px,Qy)} \varphi(t)dt, \quad (1)$$

where $\alpha, \beta > 0$ with $\alpha + \beta < 1$ and $\varphi : [0, +\infty[\to [0, +\infty[$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty[$, nonnegative and such that

$$(3)\ \int_0^{\varepsilon} \varphi(t)dt > 0, \qquad \forall \varepsilon > 0.$$

If the pairs (A, P) and (B, Q) are weakly compatible, then A, B, P and Q have a unique common fixed point in X.

Proof. Suppose that the pairs (A, P) and (B, Q) share the (CLR_{PQ}) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z \text{ for some } z \in PX \cap QX. \quad (2)$$

Since $z \in PX$, there exists a point $s \in X$ such that $Ps = z$ From (2.2), we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z = Ps \quad (3)$$

Now we claim that $As = z$. If not, then $d(As, z) > 0$. Putting $x = s$ and $y = y_n$ in (2.1), we get

$$\int_0^{d(As,By_n)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(By_n,Qy_n)[1+d(As,Ps)]}{[1+d(Ps,Qy_n)]}} \varphi(t)dt + \beta \int_0^{d(Ps,Qy_n)} \varphi(t)dt, \quad (4)$$

Taking limit as $n \to \infty$, we get

$$\int_0^{d(As,z)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(z,z)[1+d(As,Ps)]}{[1+d(z,z)]}} \varphi(t)dt + \beta \int_0^{d(z,z)} \varphi(t)dt$$

$$\int_0^{d(As,z)} \varphi(t)dt = 0 \quad (5)$$

Which from (3) implies that $d(As, z) = 0$
Which contradicts the fact that $d(As, z) > 0$, therefore

$$Ps = As = z. \quad (6)$$

Similarly, since $z \in QX$, so there exists a point $v \in X$ such that $Qv = z$. Thus (2.2) becomes

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z = Qv \quad (7)$$

Now we claim that $Bv = z$. If not then $d(Bv, z) > 0$. Then on putting $x = x_n$ and $y = v$ in (2.1), one can get

$$Bv = Qv = z. \quad (8)$$

Therefore, from (2.6) and (2.8), one can write

$$As = Ps = Bv = Qv = z. \quad (9)$$

Next, we show that z is a common fixed point of A, B, P and Q. To this aim, since the pairs (A, P) and (B, Q) are weakly compatible, then using (2.9) we have

$$As = Ps \Rightarrow PAs = APs \Rightarrow Az = Pz, \quad (10)$$

and

$$Bv = Qv \Rightarrow QBv = BQv \Rightarrow Bz = Qz. \quad (11)$$

We will show next that $Az = z$. If not then $d(Az, z) > 0$. Putting $x = z$ and $y = v$ in (2.1), we get

$$\int_0^{d(Az,Bv)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(Bv,Qv)[1+d(Az,Pz)]}{[1+d(Pz,Qv)]}} \varphi(t)dt + \beta \int_0^{d(Pz,Qv)} \varphi(t)dt$$

$$\int_0^{d(Az,z)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(z,z)[1+d(Az,Pz)]}{[1+d(Pz,z)]}} \varphi(t)dt + \beta \int_0^{d(Az,z)} \varphi(t)dt$$

$$(1-\beta)\int_0^{d(Az,z)}\varphi(t)dt \leq 0 \tag{12}$$

Which from (3) implies that $d(Az, z) \leq 0$, a contradiction.
Hence $Az = z$. From (2.10), we can write

$$Az = Pz = z \tag{13}$$

Similarly, setting $x = u, y = z$ in 2.1 and using (2.9), (2.11), we can have

$$Bz = Qz = z. \tag{14}$$

Therefore from (2.13) and (2.14), it follows that

$$Az = Bz = Qz = Pz = z, \tag{15}$$

that is, z is a common fixed point of A, B, Q and P.

Finally, we prove the uniqueness of the common fixed point of A, B, Q and P. Assume that z_1 and z_2 are two distinct common fixed points of A, B, Q and P. Then replacing x by z_1 and y by z_2 in (2.1), we have

$$\int_0^{d(Az_1,Bz_2)}\varphi(t)dt \leq \alpha\int_0^{\frac{d(Bz_2,Qz_2)[1+d(Az_1,Pz_1)]}{[1+d(Pz_1,Qz_2)]}}\varphi(t)dt + \beta\int_0^{d(Pz_1,Qz_2)}\varphi(t)dt$$

$$(1-\beta)\int_0^{d(Az_1,Bz_2)}\varphi(t)dt \leq 0 \tag{16}$$

Which from (3) implies that $d(Az_1, Bz_1) \leq 0$, a contradiction.

Hence $z_1 = z_2$. Therefore A, B, P and Q have a unique common fixed point in X. From theorem 2.1, we can easily deduce the following corollaries.

Corollary 2.1. Let (X, d) be a metric space and A, P, Q be three self maps on X satisfying the following:
(1) The pairs (A, P) and (A, Q) share (CLR_{PQ}) property;

$$(2)\ \int_0^{d(Ax,Ay)}\varphi(t)dt \leq \alpha\int_0^{\frac{d(Ay,Qy)[1+d(Ax,Px)]}{[1+d(Px,Qy)]}}\varphi(t)dt + \beta\int_0^{d(Px,Qy)}\varphi(t)dt,$$

where $\alpha, \beta > 0$ with $\alpha + \beta < 1$ and $\varphi : [0, +\infty[\to [0, +\infty[$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty[$, nonnegative and such that

$$(3)\ \int_0^{\varepsilon}\varphi(t)dt > 0, \qquad \forall\varepsilon > 0.$$

If the pairs (A, P) and (A, Q) are weakly compatible, then A, P and Q have a unique common fixed point in X.

Corollary 2.2. Let (X, d) be a metric space and A, Q be two self maps on X satisfying the following:
(1) The pairs (A, Q) share (CLR_Q) property;

$$(2)\ \int_0^{d(Ax,Ay)}\varphi(t)dt \leq \alpha\int_0^{\frac{d(Ay,Qy)[1+d(Ax,Ax)]}{[1+d(Ax,Qy)]}}\varphi(t)dt + \beta\int_0^{d(Ax,Qy)}\varphi(t)dt,$$

where $\alpha, \beta > 0$ with $\alpha + \beta < 1$ and $\varphi : [0, +\infty[\to [0, +\infty[$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty[$, nonnegative and such that

$$(3) \int_0^{\varepsilon} \varphi(t)dt > 0, \qquad \forall \varepsilon > 0.$$

If the pairs (A, Q) is weakly compatible, then A and Q have a unique common fixed point in X.

Obviously, the (CLR_{MN}) property implies the common property (E.A) but the converse is not true in general. So replacing the (CLR_{MN}) property by the common property (E.A) in Theorem 2.1, we get the following results, the proofs of which can easily be done by following the lines of the proof of Theorem 2.1, because the (E.A) property together with the closedness property of a suitable subspace gives rise to the closed range property.

Corollary 2.3. Let (X, d) be a metric space and A, B, P and Q be four self maps on X satisfying the following:

(1) The pairs (A, P) and (B, Q) share common (E.A.) property such that QX (or NX) is closed subspace of X;

$$(2) \int_0^{d(Ax,By)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(By,Qy)[1+d(Ax,Px)]}{[1+d(Px,Qy)]}} \varphi(t)dt + \beta \int_0^{d(Px,Qy)} \varphi(t)dt,$$

where $\alpha, \beta > 0$ with $\alpha + \beta < 1$ and $\varphi : [0, +\infty[\to [0, +\infty[$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty[$, nonnegative and such that

$$(3) \int_0^{\varepsilon} \varphi(t)dt > 0, \qquad \forall \varepsilon > 0.$$

If the pairs (A, P) and (B, Q) are weakly compatible, then A, B, P and Q have a unique common fixed point in X.

To illustrate Theorem 2.1, we construct the following example.

Example 2.4. Let $X = (0, 7)$ be a metric space with metric $d(x, y) = |x - y|$, where $x, y \in X$ and A, B, P and Q be self-maps of X, defined by

$$A(x) = \begin{cases} 5 & \text{if } x \in (0, 3] \\ 1 & \text{if } x \in (3, 7) \end{cases} \quad ; \quad B(x) = \begin{cases} 5 & \text{if } x \in (0, 3] \\ \frac{1}{2} & \text{if } x \in (3, 7) \end{cases}$$

$$P(x) = \begin{cases} 5 & \text{if } x \in (0, 3] \\ 2 & \text{if } x \in (3, 7) \end{cases} \quad ; \quad Q(x) = \begin{cases} 5 & \text{if } x \in (0, 3] \\ 4 & \text{if } x \in (3, 7) \end{cases}$$

First we verify condition (1) of Theorem 2.1. To this aim, Let $\{x_n\} = \left\{\frac{3n}{n+1}\right\}_{n\geq 1}$ and $\{y_n\} = \left\{\frac{3}{n+1}\right\}_{n\geq 1}$ be two sequences in X. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} A\Big(\frac{3n}{n+1}\Big) = A(3) = 5$$
$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} P\Big(\frac{3n}{n+1}\Big) = P(3) = 5$$
$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} B\Big(\frac{3n}{n+1}\Big) = B(0) = 5$$
$$\lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Q\Big(\frac{3n}{n+1}\Big) = Q(0) = 5$$

Thus $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = 5 \in PX \cap QX$.

That is, (A, P) and (B, Q) satisfies the common (CLR_{PQ}) property.

Next, to verify condition (2) of theorem 2.1 let us define $\varphi(t) = 2t$.

Let $x, y \in (0, 3]$. Then $Ax = Px = By = Qy = 5$ and from equation (2.1)

$$\begin{aligned} L.H.S. &=? \int_0^{d(Ax,By)} \varphi(t)dt \\ &= \int_0^{d(5,5)} 2tdt \\ &= 0 \end{aligned}$$

$$\begin{aligned} R.H.S. &= \alpha \int_0^{\frac{d(By,Qy)[1+d(Ax,Px)]}{[1+d(Px,Qy)]}} \varphi(t)dt + \beta \int_0^{d(Px,Qy)} \varphi(t)dt \\ &= \alpha \int_0^{\frac{d(5,5)[1+d(5,5)]}{[1+d(5,5)]}} \varphi(t)dt + \beta \int_0^{d(5,5)} \varphi(t)dt \\ &= 0 \end{aligned}$$

Therefore L.H.S. = R.H.S.

Now let $x, y \in (3, 7)$

$$\begin{aligned} L.H.S. &=? \int_0^{d(Ax,By)} \varphi(t)dt \\ &= \int_0^{d\left(1,\frac{1}{2}\right)} 2tdt \\ &= 0.25 \end{aligned}$$

$$\begin{aligned} R.H.S. &= \alpha \int_0^{\frac{d(By,Qy)[1+d(Ax,Px)]}{[1+d(Px,Qy)]}} \varphi(t)dt + \beta \int_0^{d(Px,Qy)} \varphi(t)dt \\ &= \alpha \int_0^{\frac{d\left(\frac{1}{2},4\right)[1+d(1,2)]}{[1+d(2,4)]}} 2tdt + \beta \int_0^{d(2,4)} 2tdt \\ &= \alpha \int_0^{\frac{7}{3}} 2tdt + \beta \int_0^{2} 2tdt \\ &= \frac{49}{9}\alpha + 4\beta \end{aligned}$$

Since $\alpha, \beta > 0$ with $\alpha + \beta < 1$, therefore $L.H.S. < R.H.S.$

Therefore from theorem 2.1, the mappings A, B, P and Q have a unique common fixed point, which is $x = 3$.

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