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COEFFICIENT BOUNDS FOR GENERAL CLASS OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH $q-$SĂLĂGEĂN OPERATOR AND CHEBYSHEV POLYNOMIALS

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Abstract. In the present paper, we introduce a general subclass of bi-univalent functions of complex order associated with $q-$Sălăgeăn operator and using Chebyshev polynomials. We obtain coefficient bounds for functions in this class.

1. Introduction

Denote by $\mathcal{A}$ the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\} \tag{1}$$

and by $\mathcal{S}$ the subclass of $\mathcal{A}$ which are univalent in $\mathbb{U}$. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, $f(z)$ is subordinate to $g(z)$ $(f(z) \prec g(z))$, if there exists a function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $f(z) = g(\omega(z))$ and if $g(z)$ is univalent in $\mathbb{U}$, then (see for details [16]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Some of the important and well - investigated subclasses of $\mathcal{S}$ are the classes $S^*(\alpha)$ and $C(\alpha)$ which are, respectively, starlike and convex functions of order α in $\mathbb{U}$ defined by Robertson ([28]) as follows:

$$S^*(\alpha) = \left\{ f : f \in S \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \le \alpha < 1 \right\}, \tag{2}$$

and

$$C(\alpha) = \left\{ f : f \in S \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, 0 \le \alpha < 1 \right\}. \tag{3}$$

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These classes are related to each other by

$$f(z) \in C(\alpha) \Leftrightarrow zf^{'}(z) \in S^{*}(\alpha).$$

It is well known (see Duren [19]) that every function $f \in \mathcal{S}$ has an inverse map f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U}),$$

and

$$f\ (f^{-1}(\omega)) = \omega \qquad (|\omega| < r_0(f); r_0(f) \geq \frac{1}{4}).$$

In fact the inverse function $g = f^{-1}$ is given by

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + ... \tag{4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent function in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Denote by $\triangle$ the class of bi- univalent functions in $\mathbb{U}$. For more study (see [17], [21]).

The Chebyshev polynomials of the first and seconed kinds are well known and defined by (see [1], [10], [18], [20], [26]):

$$T_k(t) = \cos k\theta \qquad \text{and} \qquad U_k(t) = \frac{\sin(k+1)\theta}{\sin\theta} \qquad (-1 < t < 1),$$

where the degree of the polynomial is k and $t = \cos\theta$. Consider the function

$$H(z,t) = \frac{1}{1 - 2tz + z^2}.$$

Note that if $t = \cos\alpha, \alpha \in (\frac{-\pi}{3}, \frac{\pi}{3})$, then for all $z \in \mathbb{U}$

$$\begin{aligned} H(z,t) &= 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha}\ z^k \\ &= 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + ...\ . \end{aligned} \tag{5}$$

Thus, we have (see [34])

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + ...\ (z \in \mathbb{U}, t \in (-1,1)), \tag{6}$$

where $U_{k-1}(t) = \frac{\sin(k \arccos t)}{\sqrt{1-t^2}}$, for $k \in \mathbb{N} = \{1, 2, ...\ \}$, are the second kind of the Chebyshev polynomials. which has the recurrence relation:

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t), \tag{7}$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, ...\ . \tag{8}$$

The first kind of the Chebyshev polynomials $T_k(t), t \in (-1,1)$, have the generating function

$$\sum_{k=0}^{\infty} T_k(t)z^k = \frac{1 - tz}{1 - 2tz + z^2} \quad (z \in \mathbb{U}). \tag{9}$$

The first and second kind of Chebyshev polynomials $T_k(t)$ and $U_k(t)$ are connected by :

$$\frac{dT_k(t)}{dt} = kU_{k-1}(t), \quad T_k(t) = U_k(t) - tU_{k-1}(t), \quad 2T_k(t) = U_k(t) - U_{k-2}(t).... \tag{10}$$

For a function $f(z) \in \mathcal{A}$, given by (1) and $0 < q < 1$, the Jackson's $q-$derivative of a function f is defined by [25] (also see [2], [8], [12], [15], [22], [30], [31], [32], [35]):

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)\, z}, (z \in \mathbb{U}, 0 < q < 1)\,, \tag{11}$$

$D_q f(0) = f^{'}(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (11) we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1}, \tag{12}$$

where

$$[k]_q = \frac{1 - q^k}{1 - q} \quad (0 < q < 1). \tag{13}$$

As $q \to 1^-, [k]_q \to k$. For $f \in \mathcal{A}$, Govindaraj and Sivasubramanian ([23]) defined and discussed the Sălăgean $q-$ difference operator as given below:

$$\begin{aligned} D_q^0 f(z) &= f(z) \\ D_q^1 f(z) &= z D_q f(z) \\ D_q^n f(z) &= z D_q (D_q^{n-1} f(z)) \\ D_q^n f(z) &= z + \sum_{k=2}^{\infty} [k]_q^n \, a_k z^k \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}, 0 < q < 1, z \in \mathbb{U}). \end{aligned} \tag{14}$$

We note that

$$\lim_{q \to 1^-} D_q^n f(z) = D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0, z \in \mathbb{U}) \tag{15}$$

where $D^n f(z)$ is the Sălăgean operator (see [3], [4], [5], [6], [7], [9], [11], [13], [14], [24], [29]).

By using the Sălăgean $q-$difference operator for g of the form (4), Vijaya et al. ([33]) (also see [27]) defined $D_q^n g(\omega)$ by:

$$D_q^n g(\omega) = \omega - a_2 [2]_q^n \omega^2 + (2a_2^2 - a_3)[3]_q^n \omega^3 + \ldots \ . \tag{16}$$

Definition 1. For $0 \le \lambda \le 1, b \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ and $t \in (\frac{1}{2}, 1)$, a function $f \in \Delta$ is said to be in the class $T_{\Delta}^n(b, \lambda, q, t)$ if

$$1 + \frac{1}{b} \left[\frac{(1-\lambda) D_q^{n+1} f(z) + \lambda D_q^{n+2} f(z)}{(1-\lambda) D_q^n f(z) + \lambda D_q^{n+1} f(z)} - 1 \right] \prec H(z, t) = \frac{1}{1 - 2tz + z^2}, \tag{17}$$

and

$$1 + \frac{1}{b} \left[\frac{(1-\lambda) D_q^{n+1} g(\omega) + \lambda D_q^{n+2} g(\omega)}{(1-\lambda) D_q^n g(\omega) + \lambda D_q^{n+1} g(\omega)} - 1 \right] \prec H(\omega, t) = \frac{1}{1 - 2t\omega + \omega^2}. \tag{18}$$

where $z, \omega \in \mathbb{U}$ and g is given by (4).

Specializing the parameters λ, b, q and n, we obtain the following subclasses:

(i) $\lim_{q \to 1^-} T_{\Delta}^n(b, \lambda, q, t) = T_{\Delta}^n(b, \lambda, t)$ where

$$1 + \frac{1}{b} \left[\frac{z(D^n f(z))^{'} + \lambda z^2 (D^n f(z))^{''}}{(1-\lambda) D^n f(z) + \lambda z (D^n f(z))^{'}} - 1 \right] \prec H(z, t) = \frac{1}{1 - 2tz + z^2},$$

and

$$1 + \frac{1}{b} \left[\frac{\omega(D^n g(\omega))^{'} + \lambda \omega^2 (D^n g(\omega))^{''}}{(1-\lambda) D^n g(\omega) + \lambda \omega (D^n g(\omega))^{'}} - 1 \right] \prec H(\omega, t) = \frac{1}{1 - 2t\omega + \omega^2},$$

(ii) $T_{\triangle}^{0}(b,\lambda,q,t)=T_{\triangle}(b,\lambda,q,t)$, where

$$1+\frac{1}{b}\left[\frac{(1-\lambda)zD_qf(z)+\lambda zD_q^2f(z)}{(1-\lambda)f(z)+\lambda zD_qf(z)}-1\right]\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$1+\frac{1}{b}\left[\frac{(1-\lambda)\omega D_qg(\omega)+\lambda\omega D_q^2g(\omega)}{(1-\lambda)g(\omega)+\lambda\omega D_qg(\omega)}-1\right]\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(iii)$T_{\triangle}^{n}(b,0,q,t)=T_{\triangle}^{n}(b,q,t)$,where

$$1+\frac{1}{b}\left[\frac{D_q^{n+1}f(z)}{D_q^nf(z)}-1\right]\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$1+\frac{1}{b}\left[\frac{D_q^{n+1}g(\omega)}{D_q^ng(\omega)}-1\right]\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(iiii) $T_{\triangle}^{n}(1,0,q,t)=T_{\triangle}^{n}(q,t)$, where

$$\frac{D_q^{n+1}f(z)}{D_q^nf(z)}\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$\frac{D_q^{n+1}g(\omega)}{D_q^ng(\omega)}\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(v) $\lim_{q\to1^-}T_{\triangle}^{n}(b,0,q,t)=T_{\triangle}^{n}(b,t)$,where

$$1+\frac{1}{b}\left[\frac{z(D^nf(z))'}{D^nf(z)}-1\right]\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$1+\frac{1}{b}\left[\frac{\omega(D^ng(\omega))'}{D^ng(\omega)}-1\right]\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(vi) $\lim_{q\to1^-}T_{\triangle}^{n}(1,0,q,t)=T_{\triangle}^{n}(t)$, where

$$\frac{z(D^nf(z))'}{D^nf(z)}\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$\frac{\omega(D^ng(\omega))'}{D^ng(\omega)}\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(vii) $T_{\triangle}^{n}(b,1,q,t)=\mathbb{T}_{\triangle}^{n}(b,q,t)$, where

$$1+\frac{1}{b}\left[\frac{D_q^{n+2}f(z)}{D_q^{n+1}f(z)}-1\right]\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$1+\frac{1}{b}\left[\frac{D_q^{n+2}g(\omega)}{D_q^{n+1}g(\omega)}-1\right]\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(viii) $T_{\triangle}^{n}(1,1,q,t)=\mathcal{W}_{\triangle}^{n}(q,t)$, where

$$\frac{D_q^{n+2}f(z)}{D_q^{n+1}f(z)}\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$\frac{D_q^{n+2}g(\omega)}{D_q^{n+1}g(\omega)}\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(viiii) $\lim_{q\to1^-}T_{\triangle}^{0}(b,\lambda,q,t)=T_{\triangle}(b,\lambda,t)$, where

$$1+\frac{1}{b}\left[\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right]\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$1+\frac{1}{b}\left[\frac{\omega g'(\omega)+\lambda\omega^2g''(\omega)}{(1-\lambda)g(\omega)+\lambda\omega g'(\omega)}-1\right]\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(x) $\lim_{q\to 1^-} T^n_\triangle(b,1,q,t)=\mathcal{K}^n_\triangle(b,t)$, where

$$1+\frac{1}{b}\frac{z(D^nf(z))''}{(D^nf(z))'}\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$1+\frac{1}{b}\frac{\omega(D^ng(\omega))''}{(D^ng(\omega))'}\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(xi) $\lim_{q\to 1^-} T^n_\triangle(1,1,q,t)=\mathcal{X}^n_\triangle(t)$, where

$$1+\frac{z(D^nf(z))''}{(D^nf(z))'}\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$1+\frac{\omega(D^ng(\omega))''}{(D^ng(\omega))'}\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(xii) $T^n_\triangle((1-\alpha)e^{i\beta}\cos\beta,\lambda,q,t)=T^n_\triangle(\alpha,\lambda,\beta,q,t)(0\le\alpha<1,|\beta|<\frac{\pi}{2})$, where

$$e^{i\beta}\left[\frac{(1-\lambda)D_q^{n+1}f(z)+\lambda D_q^{n+2}f(z)}{(1-\lambda)D_q^nf(z)+\lambda D_q^{n+1}f(z)}\right]\prec H(z,t)(1-\alpha)\cos\beta+\alpha\cos\beta+i\sin\beta,$$

and

$$e^{i\beta}\left[\frac{(1-\lambda)D_q^{n+1}g(\omega)+\lambda D_q^{n+2}g(\omega)}{(1-\lambda)D_q^ng(\omega)+\lambda D_q^{n+1}g(\omega)}\right]\prec H(\omega,t)(1-\alpha)\cos\beta+\alpha\cos\beta+i\sin\beta,$$

(xiii) $\lim_{q\to 1^-} T^0_\triangle(1,1,q,t)=\mathcal{K}_\triangle(t)$, where

$$1+\frac{zf''(z)}{f'(z)}\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$1+\frac{\omega g''(\omega)}{g'(\omega)}\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

(xiiii) $\lim_{q\to 1^-} T^0_\triangle(1,0,q,t)=T_\triangle(t)$, where

$$\frac{zf'(z)}{f(z)}\prec H(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$\frac{\omega g'(\omega)}{g(\omega)}\prec H(\omega,t)=\frac{1}{1-2t\omega+\omega^2},$$

In this paper, we obtain the initial coefficients bounds and Fekete -Szego problem for functions in the class $T^n_\triangle(b,\lambda,q,t)$.

2. Main Results

Unless otherwise mentioned, we assume in the reminder of this paper that $0\le\lambda\le 1, b\in\mathbb{C}^*, 0<q<1$ and $t\in(\frac{1}{2},1)$.

Theorem 1. Let $f(z)\in T^n_\triangle(b,\lambda,q,t)$. Then

$$|a_2|\le\frac{2t|b|\sqrt{2t}}{\sqrt{|\{q(q+1)[1+\lambda(q+q^2)][3]_q^nb-q(1+\lambda q)^2[2]_q^{2n}\}4bt^2-q^2(1+\lambda q)^2(4t^2-1)[2]_q^{2n}|}},\tag{19}$$

and

$$|a_3|\le\frac{2|b|t}{q}\left(\frac{2|b|t}{q(1+\lambda q)^2[2]_q^{2n}}+\frac{1}{(q+1)[1+\lambda(q+q^2)][3]_q^n}\right).\tag{20}$$

Proof. Let $f(z) \in T_{\vartriangle}^{n}(b, \lambda, q, t)$ and $g = f^{-1}$. From (17) and (18), we have

$$1 + \frac{1}{b}\left(\frac{(1-\lambda)D_q^{n+1}f(z)+\lambda D_q^{n+2}f(z)}{(1-\lambda)D_q^{n}f(z)+\lambda D_q^{n+1}f(z)} - 1\right) = 1 + U_1(t)p(z) + U_2(t)p^2(z) + ... \tag{21}$$

and

$$1 + \frac{1}{b}\left(\frac{(1-\lambda)D_q^{n+1}g(\omega)+\lambda D_q^{n+2}g(\omega)}{(1-\lambda)D_q^{n}g(\omega)+\lambda D_q^{n+1}g(\omega)} - 1\right) = 1 + U_1(t)q(\omega) + U_2(t)q^2(\omega) + ... \tag{22}$$

for some analytic functions

$$p(z) = c_1 z + c_2 z^2 + ... \ (z \in \mathbb{U}), \tag{23}$$

$$q(z) = d_1 \omega + d_2 \omega^2 + ... \ (\omega \in \mathbb{U}), \tag{24}$$

such that $p(0) = q(0) = 0, |p(z)| < 1$ and $|q(\omega)| < 1$. It is well known that if $|p(z)| < 1$ and $|q(\omega)| < 1$ then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } \ j \in \mathbb{N}. \tag{25}$$

From (21) and (22), we have

$$\frac{1}{b}\left(\frac{(1-\lambda)D_q^{n+1}f(z)+\lambda D_q^{n+2}f(z)}{(1-\lambda)D_q^{n}f(z)+\lambda D_q^{n+1}f(z)} - 1\right) = U_1(t)c_1 z + (U_1(t)c_2 + U_2(t)c_1^2)z^2 + ... \tag{26}$$

and

$$\frac{1}{b}\left(\frac{(1-\lambda)D_q^{n+1}g(\omega)+\lambda D_q^{n+2}g(\omega)}{(1-\lambda)D_q^{n}g(\omega)+\lambda D_q^{n+1}g(\omega)} - 1\right) = U_1(t)d_1 \omega + (U_1(t)d_2 + U_2(t)d_1^2)\omega^2 + ... \ . \tag{27}$$

Equating the coefficients in (26) and (27) we get

$$\frac{1}{b}q(1+\lambda q)[2]_q^n a_2 = U_1(t)c_1, \tag{28}$$

$$\begin{aligned}\frac{1}{b}\{q(q+1)\left[1+\lambda(q+q^2)\right][3]_q^n a_3 - q(1+\lambda q)^2[2]_q^{2n}a_2^2\} \\ = (U_1(t)c_2 + U_2(t)c_1^2),\end{aligned} \tag{29}$$

and

$$-\frac{1}{b}q(1+\lambda q)[2]_q^n a_2 = U_1(t)d_1, \tag{30}$$

$$\begin{aligned}\frac{1}{b}\{q(q+1)\left[1+\lambda(q+q^2)\right][3]_q^n (2a_2^2 - a_3) - q(1+\lambda q)^2[2]_q^{2n}a_2^2\} \\ = (U_1(t)d_2 + U_2(t)d_1^2).\end{aligned} \tag{31}$$

From (28) and (30) it follow that

$$c_1 = -d_1 \tag{32}$$

and

$$2q^2(1+\lambda q)^2[2]_q^{2n}a_2^2 = b^2U_1^2(t)(d_1^2 + c_1^2). \tag{33}$$

Also, (29) and (31) yield

$$\begin{aligned}2q(q+1)\left[1+\lambda(q+q^2)\right][3]_q^n a_2^2 - 2q(1+\lambda q)^2[2]_q^{2n}a_2^2 \\ = b\left\{U_1(c_2+d_2) + U_2(c_1^2+d_1^2)\right\},\end{aligned} \tag{34}$$

which by (33), leads to

$$\begin{aligned}\{2q(q+1)\left[1+\lambda(q+q^2)\right][3]_q^n - 2q(1+\lambda q)^2[2]_q^{2n} - \tfrac{2U_2q^2(1+\lambda q)^2[2]_q^{2n}}{bU_1^2}\}a_2^2 \\ = bU_1(c_2+d_2),\end{aligned}$$

that is

$$a_2^2 = \frac{bU_1(c_2+d_2)}{2q(q+1)\left[1+\lambda(q+q^2)\right][3]_q^n - 2q(1+\lambda q)^2[2]_q^{2n} - \frac{2U_2q^2(1+\lambda q)^2[2]_q^{2n}}{bU_1^2}}. \quad (35)$$

From (8) , (25) and (35) , we have (19).
Next , by subtracting (31) from (29), we have

$$\frac{2}{b}\{q(q+1)\left[1+\lambda(q+q^2)\right][3]_q^n(a_3-a_2^2)\} = U_1(c_2-d_2).$$

then

$$a_3 - a_2^2 = \frac{bU_1(c_2-d_2)}{2q(q+1)[1+\lambda(q+q^2)][3]_q^n}$$

$$a_3 = a_2^2 + \frac{bU_1(c_2-d_2)}{2q(q+1)[1+\lambda(q+q^2)][3]_q^n}. \quad (36)$$

Hence using (33) and applaying (8), we get (20). This completes the proof of Theorem 1. □

Taking $\lambda = 0$ in Theorem 1, we have the following corollary:
Corollary 1. Let $f \in T_{\triangle}^n(b,q,t)$. Then

$$|a_2| \leq \frac{2t|b|\sqrt{2t}}{\sqrt{|\{q(q+1)[3]_q^n b - q[2]_q^{2n}\}4bt^2 - q^2[2]_q^{2n}(4t^2-1)|}},$$

$$|a_3| \leq \frac{2|b|t}{q}\left(\frac{2|b|t}{q[2]_q^{2n}} + \frac{1}{(q+1)[3]_q^n}\right).$$

Taking $\lambda = 1$ in Theorem 1, we have the following corollary:
Corollary 2. Let $f \in \mathbb{T}_{\triangle}^n(b,q,t)$. Then

$$|a_2| \leq \frac{2t|b|\sqrt{2t}}{\sqrt{|\{q(q+1)[1+q+q^2][3]_q^n b - q(1+q)^2[2]_q^{2n}\}4bt^2 - q^2(1+q)^2(4t^2-1)[2]_q^{2n}|}},$$

$$|a_3| \leq \frac{2|b|t}{q}\left(\frac{2|b|t}{q(1+q)^2[2]_q^{2n}} + \frac{1}{(q+1)[1+q+q^2][3]_q^n}\right).$$

3. Fekete-Szegö inequalities for the function class $T_{\triangle}^n(b,\lambda,q,t)$

Theorem 2. If $f \in T_{\triangle}^n(b,\lambda,q,t)$, Then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2t|b|}{q(q+1)[1+\lambda(q+q^2)][3]_q^n}, \qquad |\mu - 1| \leq \left|\begin{array}{c} 1 - \frac{(q+1)^{-1}(1+\lambda q)^2[2]_q^{2n}}{[1+\lambda(q+q^2)][3]_q^n} \\ -\frac{q(q+1)^{-1}(1+\lambda q)^2(4t^2-1)[2]_q^{2n}}{[1+\lambda(q+q^2)][3]_q^n 4t^2 b} \end{array}\right| \quad (37)$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{8t^3|b^2||\mu-1|}{\{q(q+1)[1+\lambda(q+q^2)][3]_q^n b - q(1+\lambda q)^2[2]_q^{2n}\}4bt^2 - q^2(1+\lambda q)^2(4t^2-1)[2]_q^{2n}}, \quad (38)$$

when

$$|\mu - 1| \geq \left|1 - \frac{(q+1)^{-1}(1+\lambda q)^2[2]_q^{2n}}{[1+\lambda(q+q^2)][3]_q^n} - \frac{q(q+1)^{-1}(1+\lambda q)^2(4t^2-1)[2]_q^{2n}}{[1+\lambda(q+q^2)][3]_q^n 4t^2 b}\right|.$$

Proof. From (35) and (36)

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\left[\frac{b^2U_1^3(c_2+d_2)}{2\left\{(q(q+1)[1+\lambda(q+q^2)][3]_q^n - q(1+\lambda q)^2[2]_q^{2n})bU_1^2 - q^2(1+\lambda q)^2[2]_q^{2n}U_2\right\}} + \frac{bU_1(c_2-d_2)}{2(q+1)[1+\lambda(q+q^2)][3]_q^n}\right] \\ &= bU_1\left[\left(h(\mu) + \frac{1}{2(q+1)[1+\lambda(q+q^2)][3]_q^n}\right)c_2 + \left(h(\mu) - \frac{1}{2(q+1)[1+\lambda(q+q^2)][3]_q^n}\right)d_2\right], \end{aligned} \quad (39)$$

where

$$h(\mu)=\frac{bU_1^2(1-\mu)}{2\left\{(q(q+1)[1+\lambda(q+q^2)][3]_q^n b-q(1+\lambda q)^2[2]_q^{2n})bU_1^2-q^2(1+\lambda q)^2[2]_q^{2n}U_2\right\}}.$$

Then, by taking the modelus of (39) and considering (8), we have

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{2|b|t}{q(q+1)[1+\lambda(q+q^2)][3]_q^n}, & 0\leq|h(\mu)|\leq\frac{1}{2(q+1)[1+\lambda(q+q^2)][3]_q^n}\\ 4t|b||h(\mu)|, & |h(\mu)|\geq\frac{1}{2(q+1)[1+\lambda(q+q^2)][3]_q^n}\end{cases}.$$

This completes the proof of Theorem 2. □

Taking $\lambda=0$ in Theorem 2, we have the following corollary:
Corollary 3. Let $f\in T_{\vartriangle}^n(b,q,t)$. Then

$$\left|a_3-\mu a_2^2\right|\leq\frac{2t|b|}{q[3]_q^n[2]_q},$$

$$\left|a_3-\mu a_2^2\right|\leq\frac{8t^3|b^2||\mu-1|}{\{q(q+1)[3]_q^n b-q[2]_q^{2n}\}4bt^2-q^2[2]_q^{2n}(4t^2-1)}.$$

Taking $\lambda=1$ in Theorem 2, we have the following corollary:
Corollary 4. Let $f\in\mathbb{T}_{\vartriangle}^n(b,q,t)$. Then

$$\left|a_3-\mu a_2^2\right|\leq\frac{2t|b|}{q(q+1)[1+q+q^2][3]_q^n},$$

$$\left|a_3-\mu a_2^2\right|\leq\frac{8t^3|b^2||\mu-1|}{\{q(q+1)[1+q+q^2][3]_q^n b-q(1+q)^2[2]_q^{2n}\}4bt^2-q^2(1+q)^2(4t^2-1)[2]_q^{2n}}.$$

Taking $\mu=1$ in Theorem 2, we have the following corollary:
Corollary 5. Let $f\in T_{\vartriangle}^n(b,\lambda,q,t)$. Then

$$\left|a_3-a_2^2\right|\leq\frac{2t|b|}{q(q+1)[1+\lambda(q+q^2)][3]_q^n}.$$

Remark:
For different values of q,b,λ, t in Theorems 1 and 2, we obtain results corresponding to the classes mentioned in the introduction.

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