Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 272-290.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

A NOTE ON FACTORIZATION FOR CENTRO-SYMMETRIC REAL MATRICES THAT PRESERVES CENTRO-SYMMETRY

VINEET BHATT, K.C. PETWAL AND S. KUMAR

Abstract. In this paper, we show that QR factorization of a centro-symmetric matrix is not possible by the normal QR-factorization. We further investigate QR-factorization of a centro-symmetric matrix in which Q and R are centro-symmetric but R is not upper-triangular unlike the previous studies.

1. Introduction

This paper deals with special type of QR factorization of a matrix. To conduct this factorization some special types of products i.e., R_n Inner product, Reverse of a vector, Reverse Identity matrix, Centro-symmetric matrix etc. have been used during the analysis. If the simple QR factorization is done for a matrix $A = (a_{i \times j})_{(m \times n)}$ (See example 1), then it will get converted into an orthogonal matrix Q and a upper triangular matrix R [1], but if we conduct QR factorization of a centro-symmetric matrix, to obtain centro-symmetric Q and R matrix, then we cannot find it by the above stated method i.e., QR factorization which is generally used nowadays. In this paper, we wish to analyse the QR factorization of a centro-symmetric matrix such that Q and R which are obtained as the result should also be centro-symmetric. The above stated objective can also be obtained by use of simple method but in that case, the following problems arise: If we conduct QR factorization of matrix A then can Q matrix be perplectic (See subsection-2.4.) and orthogonal i.e. $Q^t \times R_n \times Q = R_n$ and $Q \times Q^t = Q^t \times Q = I_n$.
Example1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

Then QR factorization of A is

$$Q = \begin{bmatrix} -0.1961 & -0.5472 & -0.8137 \\ -0.7845 & -0.4104 & 0.4650 \\ -0.5883 & 0.7295 & -0.3487 \end{bmatrix}$$

2010 *Mathematics Subject Classification.* 15A06, 15A23, 15A24.
Key words and phrases. Factorization, Centro-Symmetric, QR factorization and Matrix.
Submitted Sept. 8, 2019. Revised Oct. 30, 2019.

and

$$R = \begin{bmatrix} -5.0990 & -5.4913 & -4.3146 \\ 0 & -1.6871 & -2.5534 \\ 0 & 0 & -0.9300 \end{bmatrix}$$

Above example show that QR factorization of centro-symmetric matrix A, Q is orthogonal matrix and R is upper triangular matrix. We observe in sub-section 2.8. of this paper, if Q is perplectic and orthogonal, then it leads centro-symmetric nature of Q. It is clear in subsection 2.5 that if a matrix A is centro-symmetric then its factorization Q and R must also be centro-symmetric but in the above stated example this does not happen, hence it gives rise to the problem. The second problem is that if we obtain or form Q as perplectic orthogonal by any means, but then too R will always remain upper triangular matrix, but if we try to make matrix R centro-symmetirc then it means that R will be diagonal because an upper triangular centro-symmetric matrix is always diagonal. Therefore it can conclude that in the QR factorization of a centro-symmetric matrix, Q will always be orthogonal and R will be diagonal but, this is not a situation in each case. Author [2] has negated the above statement in his paper through the following example:

Example 2. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is a centro-symmetric matrix then QR factorization of A is

$$Q = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

and

$$R = \begin{bmatrix} -1.4142 & -1.4142 \\ 0 & 0 \end{bmatrix}.$$

In this example R is neither centro-symmetric nor diagonal. So this is the main problem.

This paper is basically divided in five parts. First part is introductory and the second part is a base for finding the results of third part. In section 2, every theorem, propositions and properties are discussed with numerical example, therefore it is an important base for the result of this paper and we are surveying reverse identity matrix, R_n inner product, reverse vector, centro-symmetric metric, Moore Penros inverse etc. with numerical example. Section 3 is related to the steps of factorization of a centro-symmetric matrix along with 2 numerical examples and section 4 is the application of such type QR factorization. The last section of this paper is the conclusion. Algorithm, references, notation and their meanings in this context are attached at the last.

2. Notation and Preliminaries

In this review, we tried to remain as consistent as possible with terminology that would be familiar to applied mathematicians. Vectors are denoted by boldface lowercase letters, e.g., $\boldsymbol{a}$. Matrices are denoted by capital letters, e.g., A. Scalars are denoted by lowercase letters, e.g.,$a, b, \alpha, \beta, a_1, a_2$.... The product are denoted by $\times$, e.g., $A \times B$. The $(i, j)^{th}$ entry of a matrix A is denoted by $(a_{i\times j}$ and order of a matrix is denoted by subscript of (a_{ij}), e.g., $(a_{ij})_{(m\times n)}$. I_n denoted identity matrix of order

TABLE 1. Determinant behavior of reverse identity matrix for different order

Order of R_n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Determinant of R_n	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1

$(n \times n)$ and reverse identity matrix of order $(n \times n)$ is denoted by R_n. Real vector space is denoted by R^n. The transpose of scalar, vector and matrix is denoted by superscript t, e.g., A^t is transpose of matrix A. Moore-Penrose generalized inverse of matrix A is denoted by A^+. Perplectic scalar product is denoted by $[a, b]_{R_n}$. Scalars product of two vector is denoted by $< a \times b >$. Conjugate transpose of A is denoted by $(A)^{ct}$. In this section, we will give a brief introduction of the basic concept of centro-symmetric matrix to find its QR factorization, which preserves centro-symmetric properties. Every property of centro-symmetric matrix is explained with numerical example:

2.1. **Reverse Identity matrix:** The reverse identity matrix R_n of an identity matrix I_n is a reverse form of I_n, denoted by

$$R_n = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ .. & .. & .. & .. & .. \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n}$$

or we can say that if we take reverse of each column of an identity matrix then we can find R_n. For example,

$$R_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a reverse identity matrix of order (3×3).

Proposition 2.1. Square of a reverse identity matrix is an identity matrix of same order i.e., $R_n^2 = I_n \forall n \in N$. For instance $R_3^2 = I_3$ i.e., $R_3 \times R_3 =$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 2.2. Transpose of a reverse identity matrix is same matrix i.e., $R_n^t = R_n \forall n \in N$.

Proposition 2.3. Determinant behavior (determinants of reverse identity matrix for different order) of reverse identity matrix is very interesting. We observed it for n = 1 to 19 and found that, if order is n=2,3,6,7,10,11.. then determinant is -1 and if order is n=1,4,5,8,9, then determinant is 1 (See table 1.), R_n is also know as exchange matrix and standard involutory permutation.

2.2. **Reverse of a vector:** In case of any vector $\boldsymbol{a} \in R^n$, reverse of $\boldsymbol{a}$ is a reverse arrangement of elements of vector $\boldsymbol{a}$ i.e., if $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_n \end{pmatrix} \in R^n$ then reverse of $\boldsymbol{a}$ is $\boldsymbol{a}_R = \begin{pmatrix} \alpha_n \\ \alpha_{n-1} \\ . \\ . \\ . \\ \alpha_1 \end{pmatrix}$.

Example 3: If $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in R^3$ then $\boldsymbol{a_R} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in R^3$ or we can find reverse of a vector $\boldsymbol{a}^R = R_n \times \boldsymbol{a}$ i.e. reverse of above vector $\boldsymbol{a}$ is $\boldsymbol{a}_R = R_3 \times \boldsymbol{a} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

2.3. R_n **Inner Product (Perplectic Inner product:** Generally inner product of any two vector $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\boldsymbol{a}, \boldsymbol{b}$ [3], but perplectic inner product is different type of inner product in which reverse identity matrix perform an enormous role.

Theorem 2.1. The perplectic inner product of two vector $\boldsymbol{a}$ and $\boldsymbol{b}$ are$[\boldsymbol{a}, \boldsymbol{b}]_{R_n} = < \boldsymbol{a}, R_n \times \boldsymbol{b} >$. For instance, let two vector $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} \in R^3$ then their perplectic inner product $[\boldsymbol{a}, \boldsymbol{b}]_{R_3} =< \boldsymbol{a}, R_3 \times \boldsymbol{b} >=< \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} >= 24$. Similarly general norm [3] of two vector is always ≥ 0, but in case of perplectic norm not always ≥ 0, so it is not norm of a vector, we can define perplectic norm of any vector $\boldsymbol{a}$ as $N[\boldsymbol{a}] = [\boldsymbol{a}, \boldsymbol{a}]_{R_n} =< \boldsymbol{a}, R_n \times \boldsymbol{a} >$. For this statement, let we take an example, if vector$\begin{pmatrix} -4 \\ 2 \end{pmatrix} \in R^2$ then $N(\boldsymbol{a}) = -16$ i.e., value of$N(\boldsymbol{a})$ for this vector is not positive, so perplectic norm is not a norm but this type norm is very useful in reaching the goal of this paper.

Proposition 2.4. For all vectors $\boldsymbol{a} \in R^n, N(a)^2 \leq \| \boldsymbol{a} \|_2^4$.

Example 4: Let $\boldsymbol{a} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$, for this vector $N(\boldsymbol{a}) = -16$ and value of $\| \boldsymbol{a} \|_2^4 = (\sqrt{< \boldsymbol{a}, \boldsymbol{a} >})^4 = 400$, so $\boldsymbol{a} \in R^n, N(a)^2 \leq \| \boldsymbol{a} \|_2^4$ i.e., $(-16)^2 < 400$.

2.4. R_n**-Orthogonal matrix (Perplectic matrix):** Theorem 2.2. A square matrix A is R_n orthogonal and perplectic if $A^t \times A = A \times A^t = I_n$ and $A^t \times R_n \times A = R_n$.

The set of all square matrices, which preserve the perplectic product is denoted by$P(n)$ i.e., $P(n) = \{A \in R^{n\times n} : [A \times \boldsymbol{a}, A \times \boldsymbol{b}] = [\boldsymbol{a}, \boldsymbol{b}] \forall \boldsymbol{a}, \boldsymbol{b} \in R^n\}$. The set $P(n)$ is a group with respect to matrix multiplication and is called real perplectic group [4], [5].

2.5. **Centro-symmetric matrix:** In matrix theory centro-symmetric matrices are symmetric about its center [6]. In case of square matrix $A = (a_{ij})_{(n\times n)}$ is centro-symmetric when its $(i,j)^{th}$ elements satisfy the conditions $a_{ij} = a_{(n-i+1,n-j+1)}$ for $i \geq 1, n \geq j$ or matrix A is centro-symmetric if and only if $A \times R_n = R_n \times A$. For instance, all matrix of order (2×2) and (3×3) are denoted by $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $\begin{bmatrix} a & d & c \\ f & b & f \\ c & d & a \end{bmatrix}$.

We can simply arrange second and third ordered centro-symmetric matrix but in case of higher order arrangement of elements, the case is different. For arrangement of centro-symmetric matrix of order $(4 \times 4))$, we are using above centro-symmetric condition for arranging the elements of matrix i.e., $a_{ij} = a_{(n-i+1,n-j+1)}$ for $i \geq 1, n \geq j$.

For instance, let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, then $a_{11} = a_{(4-1+1,4-1+1)} = a_{4,4} = \alpha, a_{12} = a_{(4-1+1,4-2+1)} = a_{4,3} = \beta, a_{13} = a_{(4-1+1,4-3+1)} = a_{4,2} = \gamma, a_{14} = a_{(4-1+1,4-4+1)} = a_{41} = \delta, a_{21} = a_{(4-2+1,4-1+1)} = a_{34} = \epsilon, a_{22} = a_{(4-2+1,4-2+1)} = a_{33} = \zeta, a_{23} = a_{(4-2+1,4-3+1)} = a_{32} = s, a_{24} = a_{((4-2+1,4-4+1))} = a_{31} = t$ and we find centro-symmetric matrix $A = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \zeta & \eta & \theta & \kappa \\ \kappa & \theta & \eta & \zeta \\ \zeta & \gamma & \beta & \alpha \end{bmatrix}$.

Theorem 2.3. If A and B are centro-symmetric matrices over a given field F, then $A + B$ and $c \times A(c \in F)$ are centro-symmetric.

Theorem 2.4. For every square centro-symmetric matrix, multiplication of two centro-symmetric matrices is also centro-symmetric.

In case of non-square matrix, multiplication of two centro-symmetric matrices is also a centro-symmetric. Let A be an $(m \times n)$ and B be $(n \times p)$ centro-symmetric matrix. Then matrix $A \times B$ is also centro-symmetric.

Theorem 2.5. $A^{(-1)}, A^{+}$ (If exist for a matrix) andA^t are also centro-symmetric for a centro-symmetric matrix A [9].

Example 5: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 0.375 & -0.50 & 0.875 \\ -1.00 & 1.00 & -1.00 \\ 0.875 & -0.50 & 0.375 \end{bmatrix}$, $A^t = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ and $A^+ = \begin{bmatrix} 0.375 & -0.50 & 0.875 \\ -1.00 & 1.00 & -1.00 \\ 0.875 & -0.50 & 0.375 \end{bmatrix}$. This example shows that inverse, transpose and Moore-Penrose inverse of a centro-symmetric matrix is centro-symmetric[9].

2.5.1. *Centro-symmetric matrix of order* $(m \times n)$ *if* $m \neq n$*:* : In case of non-square matrix ?? $= (a_{ij})_{m\times n}$ is centro-symmetric iff $A = R_m \times A \times R_n$. For instance, let matrix $A = (a_{ij})_{3\times 4} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix}$, where $m = 3$ and $n = 4$, then this matrix should be equal to $R_3 \times A \times R_4$ i.e., $R_3 \times A \times R_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{bmatrix}$.
After comparing corresponding elements of matrix A with resultant matrix $R_3 \times A \times R_4$, we find $\alpha_1 = \gamma_4, \alpha_2 = \gamma_3, \alpha_3 = \gamma_2, \alpha_4 = \gamma_1, \beta_1 = \beta_4, \beta_2 = \beta_3, \beta_3 = \beta_2 and \beta_4 = \beta_1$, so we can arrange a rectangular centro-symmetric matrix by the help of above example. According the above process $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 9 & 9 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ is an example of centro-symmetric matrix of order (3×4). If we want to find a centro-symmetric matrix by rearranging column vector of a matrix then we can use the vector reverse property: If $A = (a_{ij})_{(m\times n)}$ is a matrix then$A$ is centro-symmetric if and only if $a_i = a^R_{(n-i+1)}$ for $i = 1, 2, 3, ., n$. So we can simply arrange a centro-symmetric matrix as $\boldsymbol{a}_i$ (i^th column of a matrix) $= \boldsymbol{a}^R_{n-i+1}$ (reverse of $(n-i+1)^{th}$ column). In above centro-symmetric matrix A, $\boldsymbol{a}_1$ (first column)=$\boldsymbol{a}^R_{(4-1+1)} = \boldsymbol{a}^R_4$=reverse of 4^{th} column, similarly $\boldsymbol{a}_2$ (second column)=$\boldsymbol{a}^R_3$ (reverse of third column).

2.6. **Two-Column sub matrix:** To find the result in section-3, we need two-column sub-matrix of a centro-symmetric matrix $A = (a_{(i,j)})_{(m\times n)}$, which is defined as $\begin{bmatrix} a_{k,k} & a_{k,n-k+1} \\ . & . \\ . & . \\ . & . \\ a_{m-k+1,k} & a_{m-k+1,n-k+1} \end{bmatrix}$, for each$k = 1, 2, ..., min?(m, n)/2$. Let us take an example, $A = (a_{i,j})_{(4\times 5)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}$. The value of k for A of order (4×5) is $k = 1, 2, , min?(4, 5)/2 = 1, 2.$. Consequently we find two two-column sub-matrix, which are $A_1 = \begin{bmatrix} a_{11} & a_{15} \\ a_{21} & a_{25} \\ a_{31} & a_{35} \\ a_{41} & a_{45} \end{bmatrix}$, and $A_2 = \begin{bmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{bmatrix}$.
The value of k is different in case of even and odd value of $min?(m, n)$. If k is odd, then $k = 1, 2, ..., (min?(m, n) + 1)/2$ and if it is even, $k = 1, 2, ..., min?(m, n)/2$. For each $k = 1, 2, ., d$, where $d = \begin{cases} \frac{min(m,n)+1}{2} & for\ min(m, n)\ is\ odd \\ \frac{min(m,n)}{2} & for\ min(m, n)\ is\ even \end{cases}$ it is

easy to verify the condition $A = R_m \times A \times R_n$ i.e., $R_{m-2k+2} \times A_k \times R_2 = \begin{bmatrix} a_{m-k+1,n-k+1} & a_{m-k+1,k} \\ . & . \\ . & . \\ . & . \\ a_{k,n-k+1}. & a_{k,k} \end{bmatrix} = \begin{bmatrix} a_{k,k} & a_{k,n-k+1} \\ . & . \\ . & . \\ . & . \\ a_{m-k+1,k} & a_{m-k+1,n-k+1} \end{bmatrix}$.

2.6.1. *Double-Cone Matrix* [2]*:* Proposition 2.5. Matrix $A = (a_{ij})_{m\times n}$ is a double cone matrix when the two-column sub-matrix $A = \begin{bmatrix} a_{k+1,k} & a_{k+1,n-k+1} \\ . & . \\ . & . \\ . & . \\ a_{m-k,k} & a_{m-k,n-k+1} \end{bmatrix}$ has only zero entries for each $k = 1, 2, ..., \frac{min(m,n)}{2}$.

2.7. **Moore-Penrose generalized inverse of matrix** A [7]**:** Some conditions and proposition related to Moore-Penrose generalized inverse are discussed here: Proposition 2.6. If $A \in (a_{i,j})_{(m\times n)}$, then there exists a unique$A^+ \in (a_{i,j})_{(n\times m)}$ that satisfies four Penrose conditions:$A \times A^+ \times A = A$, $A^+ \times A \times A^+ = A^+$, $A^+ \times A = (A^+ \times A)^{ct}$ and $A \times A^+ = (A \times A^+)^{ct}$. For a non-singular real matrix A, it is clear that $A^+ = A^{-1}$, which trivially satisfies the above four conditions. Pseudo-inverse of a nonsingular matrix is same as the ordinary inverse or we can say that if, $A \in (a_{i,j})_{(n,n)}$, then $(A^t)^+ = (A^+)^t$.
Theorem 2.6. Let A, B and C are $n \times n$ matrices for which $C = A \times B$, if either of the following two conditions satisfy: first condition is A has orthogonal columns i.e., $A^t \times A = I_n$ and the second condition, B has orthogonal rows i.e.,$B \times B^t = I_n$; then $C^+ = B^+ \times A^+$.
Proposition 2.7. A collection of m vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_m$ is orthonormal, if i)The vector have unit norm i.e., $\|\boldsymbol{a}_i\| = 1$. ii)They are mutually orthogonal i.e.,$\boldsymbol{a}_i^t\boldsymbol{a}_j = 0$ if $i \neq j$. We can understand above theorem 2.6. and proposition 2.7. by an example. The example given below has orthonormal rows and columns. Let matrix $A = \begin{bmatrix} 0 & 0 & -1 \\ 0.7071 & 0.7071 & 0 \\ 0.7071 & -0.7071 & 0 \end{bmatrix}$ then $A^t = \begin{bmatrix} 0 & 0.7071 & 0.7071 \\ 0 & 0.7071 & -0.7071 \\ -1 & 0 & 0 \end{bmatrix}$. because matrix A has orthogonal columns, so $A^t \times A = I_3$.

Similarly, let matrix $B = \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ 0 & 0 & -1 \\ 0.7071 & -0.7071 & 0 \end{bmatrix}$ and its transpose $B^t = \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0.7071 & 0 & -0.7071 \\ 0 & -1 & 0 \end{bmatrix}$, as matrix B has orthogonal rows, so $B \times B^t = I_3$. Thus matrix $C = A \times B = \begin{bmatrix} -0.7071 & 0.7071 & 0 \\ 0.5 & 0.5 & -0.7071 \\ 0.5 & 0.5 & 0.7071 \end{bmatrix}$. Moore Penrose Psedo-Inverse of matrix C, A and$B$ areC^+, A^+ and B^+ respectively i.e., $C^+ = \begin{bmatrix} -0.7071 & 0.5 & 0.5 \\ 0.7071 & 0.5 & 0.5 \\ 0.0 & -0.7071 & 0.7071 \end{bmatrix}$, $A^+ = \begin{bmatrix} 0 & 0.7071 & 0.7071 \\ 0 & 0.7071 & -0.7071 \\ -1 & -0.0 & 0.0 \end{bmatrix}$ and $B^+ =$

$\begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0.7071 & 0 & -0.7071 \\ 0 & -1 & 0 \end{bmatrix}$. Now we can verify that $C^+ = B^+ \times A^+$.

Proposition 2.8. For a centro-symmetric matrix $B = (b_{ij})_{(m\times n)}$, the matrix $B^t \times R_m \times B$ is always centro-symmetric. For instance matrix $B = (b_{ij})_{3\times 4} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 9 & 9 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}$,

then $B^t \times R_m \times B = \begin{bmatrix} 33 & 56 & 59 & 42 \\ 56 & 93 & 94 & 59 \\ 59 & 94 & 93 & 56 \\ 42 & 59 & 56 & 33 \end{bmatrix}$ is a centro-symmetric matrix.

2.8. **Perplectic Orthogonal group:** :The set of Perplectic orthogonal matrices of order n is $PO(n) = P(n) \cap O(n)$ for every $n \in N$ and $PO(n)$ is a group with respect to matrix multiplication, such types groups are known as perplectic orthogonal group [5]. Matrices belongs to the groups $PO(n)$are centro-symmetric. More precisely, we have the following proposition.

Proposition 2.9. For a square real matrix $A = (a)_{(n\times n)}$, if A satisfies $A^t \times R_n \times A = R_n, A^t \times A = A \times A^t = I_n$ and $R_n \times A = A \times R_n$, then any two of these properties implies third i.e., $A^t \times R_n \times A = R_n, A^t \times A = A \times A^t = I_n \Rightarrow R_n \times A = A \times R_n$,
$A^t \times R_n \times A = R_n, R_n \times A = A \times R_n \Rightarrow A^t \times A = A \times A^t = I_n$
$A^t \times A = A \times A^t = I_n, R_n \times A = A \times R_n \Rightarrow A^t \times R_n \times A = R_n$.

We have three cases to prove from the proposition 2.9. In first case, if we take matrix $A = (a)_{(n\times n)}$ is orthogonal and centro-symmetric i.e., $A^t \times A = A \times A^t = I_n$ and $R_n \times A = A \times R_n$ then $A^t \times R_n \times A = A^t \times A \times R_n = I_n \times R_n = R_n$ i.e., matrix A is perplectic. Case second, if we take matrix$A = (a)_(n \times n)$ is orthogonal and perplectic i.e., $A^t \times A = A \times A^t = I_n$ and $A^t \times R_n \times A = R_n$ then matrix A must be centro-symmetric i.e., $R_n \times A = I_n \times (R_n \times A) = A \times (A^t \times R_n \times A) = A \times R_n$ i.e., matrix A is centro-symmetric. Finally in third case, if matrix$A = (a)_{(n\times n)}$ is perplectic and centro-symmetric i.e., $A^t \times R_n \times A = R_n$ and $R_n \times A = A \times R_n$ then $A^t \times A = A^t \times I_n \times A = A^t \times R_n^2 \times A = A^t \times R_n \times R_n \times A = (A^t \times R_n \times A) \times R_n = R_n \times R_n = R_n^2 = I_n$. Similarly$A \times A^t = A \times R_n^2 \times A^t = A \times R_n \times R_n \times A^t = (A \times R_n \times A^t) \times R_n = R_n \times R_n = R_n^2 = I_n$ i.e., matrix A is orthogonal. Hence, we can conclude that if any square matrix is orthogonal and perplectic, then matrix is surely centro-symmetric and if orthogonal and centro-symmetric, then perplectic similarly if centro-symmetric and perplectic, then orthogonal.

2.9. **Block perplectic reflectors:** In subsection (2.3.), we have defined perplectic inner product of any two vectors as $[\boldsymbol{a}, \boldsymbol{b}]_{R_n} =< \boldsymbol{a}, R_n \times \boldsymbol{b} >$, similarly we can find perplectic inner product of any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ associated with a square matrix $A = (a_{ij})_{n\times n}$ through $[\boldsymbol{a}, \boldsymbol{b}]_A =< \boldsymbol{a}, A \times \boldsymbol{b} >$.
Proposition 2.10. The product $[\boldsymbol{ab}]_A$ is known as real ortho-symmetric if $A^t = \epsilon \times A$, where $\epsilon \in R, |\epsilon| = 1$.

Example 6: Let vector $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} \in R^3$, then their perplectic inner product associated with $A = \begin{pmatrix} 0 & 2 & 7 \\ -2 & 0 & -8 \\ -7 & 8 & 0 \end{pmatrix}$ is $[\boldsymbol{a}, \boldsymbol{b}]_A =< \boldsymbol{a}, A \times \boldsymbol{b} >= 12$. This type of product is also known as real ortho-symmetric because $A^t = -A = |-1|A = \epsilon \times A$. Inner product associated with any square matrix is used for finding out a matrix $H(V)$, which is necessary for finding QR factorization of a centro-symmetric matrix preserving centro-symmetric properties. Theorem 4.3 from [8] gives sufficient condition for existence of a matrix $H(V)$ linked with the scalar inner product $[a, b]_D$ (where $D = diag(\pm 1, \pm 1, ..., \pm 1)$) of the form $H(V) = I_n - 2V \times (V^t \times D \times V)^+ \times V^t \times D$ having the property $H(V) \times N = M$ where $M = (m_{ij})_{n \times m}, N = (n_{ij})_{n \times m}$ are two real matrices and $V = M - N$. It was found by [8] that the same form can be used for any ortho-symmetric scalar product. Moreover it was proved by [8] that when such $H(V)$ exists, it is D-orthogonal and an involution i.e. $H(V)^2 = I_n$. Author [8] used R_n at the place of D and also proved two necessary condition for existence of such a mapping. We can define $H(V)$ by equation

$$H(V) = I_n - 2V \times (V^t \times R_n \times V)^+ \times V^t \times A \tag{1}$$

Proposition 2.11. The necessary condition for matrix $M = (m_{ij})_{(n \times m)}$ and $N = (n_{ij})_{(n \times m)}$ used in V are:

$$M^t \times R_n \times M = N^t \times R_n \times N \tag{2}$$

$$M^t \times R_n \times N = N^t \times R_n \times M \tag{3}$$

Theorem 2.7. If matrices $M = (m_{ij})_{(n \times m)}, N = (n_{ij})_{(n \times m)}$, satisfying equation (2) and (3) then $H(V) \times N = M$ if and only if $Rank(V^t \times R_n \times V) = Rank(V)$.
Theorem 2.8. If N and M are centro-symmetric matrix satisfying equation (2) and (3) then $H(V) \times N = M$.
Proposition 2.12. If N and M are centro-symmetric matrix then V is also centro-symmetric since $R_n \times V = V \times R_m$.
Proposition 2.13. If $V = M - N = (v_{ij})_{(n \times m)}$ be a centro-symmetric matrix then $H(V)$ is also centro-symmetric. The proof of this property were given in [2].
Proposition 2.14. If $V = M - N = (v_{ij})_{(n \times m)}$ is centro-symmmetric matrix, then $H(V)$ is perplectic orthogonal.
If $H(V)$ is perplectic orthogonal i.e., $H(V)^2 = I_n$, if$H(V)^2 = I_n$ i.e.,$H(V)$ is real symmetric and if $H(V)$ real symmetric i.e., $H(V)^t = H(V)$. So we have to prove that $H(V)^t = H(V)$: $H(V)^t = \{I_n - 2V \times [V^t \times R_n \times V]^+ \times V^t \times R_n\}^t = I_n^t - \{2V \times [V^t \times R_n \times V]^+ \times V^t \times R_n\}^t = I_n^t - 2\{V \times [V^t \times R_n \times V]^+ \times V^t \times R_n\}^t = I_n - 2\{R_n^t \times V^{t^t} \times [V^t \times R_n \times V]^{+^t} \times V^t\} = I_n - 2\{V \times [V^t \times R_n \times V]^+ \times V^t \times R_n = H(V)$. So $H(V)$ is symmetric and we can prove that $H(V)^t \times H(V) = H(V) \times H(V)^t = H(V)^2 = I_n$, so $H(V)$ is orthogonal as well as perplectic.

2.9.1. *Embeddings:* Theorem 2.9. If $m, n \in N, n \geq m$ such that $n - m$ is even then embedding of matrix $A = (a_{ij})_{(m \times m)}$ into I_n is a matrix $E_n = \begin{bmatrix} I_k & & \\ & A & \\ & & I_k \end{bmatrix}$, where $k = (n-m)/2$. Embedding of a matrix is perplectic orthogonal matrix i.e. $E_n^t(A) \times R_n \times E_n(A) = R_n$ and $E_n^t(A) \times E_n(A) = E_n(A) \times E_n^t(A) = I_n$ has proved in [2].

3. Centro-symmetric QR factorization maintaining centro-symmetric property:

In this section, we are going to find QR-factorization of a centro-symmetric matrix where Q and R are also centro-symmetric. Section-2 is a strong and complete base for finding the results. Now we are going to explain few steps of algorithm with numerical examples to find QR factorization.

3.1. Let Matrix $A = \begin{bmatrix} \alpha_1 & \alpha_2 & ... & ... & \alpha_n \\ \alpha_2 & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ \alpha_n & ... & ... & ... & \alpha_1 \end{bmatrix}$ =is a centro-symmetric matrix, where $\alpha_1, \alpha_2, ..., \alpha_n \in R$ and the two-column sub-matrix (See subsection 2.6.) of matrix A are $N = \begin{bmatrix} \alpha_1 & \alpha_n \\ \alpha_2 & \alpha_{n-1} \\ ... & ... \\ \alpha_n & \alpha_1 \end{bmatrix}_{n\times 2}$ and $M = \begin{bmatrix} \delta_1 & \delta_2 \\ 0 & 0 \\ ... & ... \\ ... & ... \\ \delta_2 & \delta_1 \end{bmatrix}_{n\times 2}$, here δ_1 and δ_2 are parameters left to be determined. We can also use $M = \begin{bmatrix} \delta_1 & \delta_2 \\ 0 & 0 \\ ... & ... \\ ... & ... \\ \delta_2 & \delta_1 \end{bmatrix}_{n\times 2} =$ $(\delta_1 \times \boldsymbol{i}_1 + \delta_2 \times \boldsymbol{i}_n \delta_2 \times \boldsymbol{i}_1 + \delta_1 \times \boldsymbol{i}_n)$ where i_1, i_n are column vectors of I_n i.e.,

$$I_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{n\times n}$$

$= \begin{pmatrix} \boldsymbol{i_1} & \boldsymbol{i_2} & ...... & \boldsymbol{i_n} \end{pmatrix}$. To find matrix V (See subsection 2.9.), we are using $V = M - N = \begin{bmatrix} \delta_1 & \delta_2 \\ 0 & 0 \\ ... & ... \\ ... & ... \\ \delta_2 & \delta_1 \end{bmatrix}_{n\times 2} - \begin{bmatrix} \alpha_1 & \alpha_n \\ \alpha_2 & \alpha_{n-1} \\ ... & ... \\ \alpha_n & \alpha_1 \end{bmatrix}_{n\times 2} = \begin{bmatrix} \delta_1 - \alpha_1 & \delta_2 - \alpha_n \\ \alpha_2 & \alpha_{n-1} \\ ... & ... \\ \delta_2 - \alpha_n & \delta_1 - \alpha_1 \end{bmatrix}_{n\times 2} =$ $\begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a^R} \end{pmatrix}$. Now we are going to find values of parameters δ_1 and δ_2, where we need $N^t \times R_n \times N = M^t \times R_n \times M$ and $N^t \times R_n \times M = M^t \times R_n \times N$. Here $N^t = \begin{bmatrix} \alpha_1 & \alpha_n & ... & \alpha_n \\ \alpha_n & \alpha_{n-1} & ... & \alpha_1 \end{bmatrix}$ $and M^t = \begin{bmatrix} \delta_1 & 0 & ... & \delta_2 \\ \delta_2 & 0 & ... & \delta_1 \end{bmatrix}$. Then $N^t \times R_n \times N = M^t \times R_n \times M$

$$\Rightarrow \begin{bmatrix} \alpha_1 \times \alpha_n + ... + \alpha_n \times \alpha_1 & \alpha_n^2 + \alpha_{n-1}^2 + ... + \alpha_1^2 \\ \alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 & \alpha_1 \times \alpha_n + ... + \alpha_n \times \alpha_1 \end{bmatrix} = \begin{bmatrix} 2\delta_1 \times \delta_2 & \delta_1^2 + \delta_2^2 \\ \delta_1^2 + \delta_2^2 & 2\delta_1 \times \delta_2 \end{bmatrix}$$

After comparing elements of both sides, then we have:

$$\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = \delta_1^2 + \delta_2^2 \Rightarrow \|\boldsymbol{\alpha}\|_2^2 = \delta_1^2 + \delta_2^2 \tag{4}$$

$$\alpha_1 \times \alpha_n + \alpha_2 \times \alpha_{n-1} + ... + \alpha_1 \times \alpha_n = 2\delta_1 \times \delta_2. \tag{5}$$

Since

$$N(\boldsymbol{\alpha}) = [\boldsymbol{\alpha}\boldsymbol{\alpha}]_{R_2} = \alpha_1\alpha_n + \alpha_2\alpha_{n-1} + ... + \alpha_1\alpha_n \tag{6}$$

therefore

$$2\delta_1\delta_2 = N(\boldsymbol{\alpha}) \tag{7}$$

Solving equation (4) and (7) for finding the values of parameters. From (7) putting the values of $\delta_2 = N(\boldsymbol{\alpha})/2\delta_1$ in (4), we have $\|\boldsymbol{\alpha}\|_2^2 = \delta_1^2 + (N(\alpha)/2\delta_1)^2 \Rightarrow \delta_1^2 = N(\alpha)^2/4\delta_1^2 = \|\boldsymbol{\alpha}\|_2^2 \Rightarrow 4\delta_1^4 + (N(\boldsymbol{\alpha}))^2 = 4\delta_1^2\|\boldsymbol{\alpha}\|_2^2 \Rightarrow 4\delta_1^4 + (N(\boldsymbol{\alpha}))^2 - 4\delta_1^2\|\boldsymbol{\alpha}\|_2^2 = 0$. Solving by Shridharacharya formula $\delta_1^2 = \frac{\|\boldsymbol{\alpha}\|_2^2 \pm \sqrt{\|\boldsymbol{\alpha}\|_2^4 - (N(\boldsymbol{\alpha}))^2}}{2} \Rightarrow \delta_1 = \sqrt{\frac{\|\boldsymbol{\alpha}\|_2^2 \pm \sqrt{\|\boldsymbol{\alpha}\|_2^4 - (N(\boldsymbol{\alpha}))^2}}{2}}$. Now to find the value of δ_2, we will put the value of δ_1 in (7). In proposition 2.4., the property of $N(\boldsymbol{\alpha})$ has been discussed as $N(\boldsymbol{\alpha})^2 \leq \|\boldsymbol{\alpha}\|_2^4$ for all $\boldsymbol{\alpha} \in R^n$, so the parameters are always real numbers. The parameters δ_1 and δ_2 are useful for finding perplectic block reflector $H(V)$ which has been discussed in section 2.9. Here $H(V)$ is not unique because the solution of equation (4) and (7) is not unique. Thus we can find different solutions for δ_1 and δ_2.

3.2. QR**-factorization of Centro-symmetric matrix:** In the introductory part, we have already discussed QR factorization of a matrix. For preserving centro-symmetric property, QR factorization of a matrix is different.

Theorem 3.1. Let $A = (a_{ij})_{(m \times n)}$ be a Centro-symmetric matrix, then there exist two matrices Q and R with the following properties:
(a) Q is a perplectic orthogonal matrix.
(b) R is double cone centro-symmetric matrix.
(c) $A = Q \times R$.

Author [2], also used above theorem for finding QR factorization of a centro-symmetric matrix with an algorithm. The example 3.3 of the paper [2] shows the QR factorization of a matrix A but in this factorization matrix R is solved to be completely centro-symmetric. Now when we solved the same in example 8, we found that R is approximately centro-symmetric and not completely centro-symmetric. For proper understanding of the reader we have solved two numerical examples step by step using algorithm, where we have explained the examples using approximate values instead using estimated values for matrix R. The above theorem 3.1 represents that we need Q and R matrix for matrix A, which satisfies the above properties. According to section 2.6., to find the two column sub-matrix of any matrix A, firstly, we should determine the value of d i.e., (two column sub-matrix of matrix A is A^k for $k = 1, 2, 3.....d$,where $d = [min?(m, n)/2]$ or $[(min?(m, n) + 1)/2]$ (See subsection 2.6)). We also need a matrix Q_k, which wipes out the entries of k^{th} and $(n - k + 1)^{th}$ column of working matrix $A^{(k-1)}$ but the column $(k - 1)^{th}$ and $(n - k + 2)^{th}$ remains constant. Here we will take working matrix $A^{(k-1)}$ because $A^0 = A$ i.e., when $k = 1$, the working matrix is $A^{(1-1)} = A^0 = A$. In terms of working matrix A^k, we define two column sub-matrices of A such that $N_k = \begin{bmatrix} \alpha^{k-1}_{k,k} & \alpha^{k-1}_{m-k+1,n-k+1} \\ . & . \\ . & . \\ \alpha^{k-1}_{m-k+1,k} & \alpha^{k-1}_{m-k+1,n-k+1} \end{bmatrix}$ and $M_k = \begin{bmatrix} \delta^k_1 & \delta^k_2 \\ 0 & 0 \\ . & . \\ . & . \\ \delta^k_2 & \delta^k_1 \end{bmatrix}$ therefore $V_k = M_k - N_k$.Where δ^k_1 and δ^k_2 are free parameters, we calculate them by previous method based on N_k. V_k is a matrix of order $(m-2k+2)\times 2$. Next step is to calculate a perplectic block reflector $H_k(V_k)$, $H_k(V_k) = I_{m-2k+2} - 2V_k \times [V^t_k \times R_{m-2k+2} \times V_k]^+ \times V^t_k \times R_{m-2k+2}$.

To ensure that each Q_k does not change the columns reduced in the previous step, we use the embedding of $H_k(V_k)$, so we set $Q_k = E_m(H_k(V_k))$. Q_k is proper embedding since $H_k(V_k)$ is a perplectic orthogonal square matrix of order $m-2k+2$. According to the properties of embedding (See subsection 2.9.1.), Q_k is perplectic orthogonal. In relation to above statement, the author [2], have discussed a lemma i.e., if $k < t \leq d$ then $Q_t \times \boldsymbol{\alpha}^k_k$ and $Q_t \times \boldsymbol{\alpha}^k_{n-k+1} = \boldsymbol{\alpha}^k_{n-k+1}$. Our target is to factorize centro-symmetric matrix in to QR centro-symmetric factors. So we need to find the value of A^k for every $k = 1, 2, ..., d$. Where A^k define as $A^0 = A$ and $A^k = Q_k \times A^{k-1})$, if $k = 1$ then $A^1 = Q_1 \times A^0 = Q_1 \times A$ and if $k = 2$ then $A^2 = Q_2 \times A^1 = Q_2 \times A^1 = Q_2 \times Q_1 \times A$, similarly the value of $k = d$ then $A^d = Q_d \times A^{d-1} = (Q_d \times .... \times Q_2 \times Q_1) \times A$. We set $Q = (Q_d \times .... \times Q_2 \times Q_1)^t$ and for $R = A^d$, the property of R is a double cone matrix. We have matrix Q, which is centro-symmetric because $H_k(V_k)$ is centro-symmetric and $A = Q \times R$ i.e.,$Q^t \times A = R$, therefore R must be centro-symmetric being a product of two centro-symmetric matrices and one can also prove that $Q \times R = Q \times Q^t \times A = A$. Now we will understand all the above QR factorization theory of centro-symmetric matrix A with the help of two numerical examples.
Example 7: Let

$$A = \begin{bmatrix} 1 & 4 & 10 & -1 \\ 0.2 & 9 & 7 & 5 \\ 3 & 8 & 8 & 3 \\ 5 & 7 & 9 & 0.2 \\ -1 & 10 & 4 & 1 \end{bmatrix},$$

for centro-symmetric matrix A, number of rows $m = 5$,column $n = 4$ and $d = min?(5,4)/2 = 2$ i.e., $k = 1, 2$. For $k = 1$, two-column sub-matrices of A are $N_1 = \begin{pmatrix} 1 & -1 \\ 0.2 & 5 \\ 3 & 3 \\ 5 & 0.2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\alpha^R} \end{pmatrix}$ and $M_1 = \begin{pmatrix} \delta_1 & \delta_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \delta_2 & \delta_1 \end{pmatrix}$ therefore $V_1 = \begin{pmatrix} \delta_1 - 1 & \delta_2 + 1 \\ -0.2 & -5 \\ -3 & -3 \\ -5 & -0.2 \\ \delta_2 + 1 & \delta_1 - 1 \end{pmatrix}$.
The parameters δ_1 and δ_2 are useful for finding perplectic block reflector $H_1(V_1)$, so we can find parameters d_1 and d_2 by method discussed in subsection 3.1. Therefore the value of $\delta_1 = \sqrt{\frac{\|\boldsymbol{\alpha}\|^2_2 \pm \sqrt{\|\boldsymbol{\alpha}\|^4_2 - (N(\boldsymbol{\alpha}))^2}}{2}} = 5.9862258$ and 0.7517313. For vector

$\boldsymbol{\alpha} = \begin{pmatrix} 1 \\ 0.2 \\ 3 \\ 5 \\ -1 \end{pmatrix}$, Euclidean Vector Norm is $\|\boldsymbol{\alpha}\|_2 = \sqrt{36.4}$ therefore $N(\boldsymbol{\alpha}) = [\boldsymbol{\alpha}\boldsymbol{\alpha}]_{R_2} =$ 9. So $\delta_2 = N(\boldsymbol{\alpha})/2\delta_1 = 0.7572573$ for δ_1=5.9862258 and 5.98618149 for $\delta_1 =$ 0.75173138, according to above calculation the value of $\delta_1 = 5.9862258$ and $\delta_2 =$ 0.7517257367. Putting the value of δ_1 and δ_2 in V_1, $V_1 = \begin{pmatrix} 4.9862258 & 1.75172573 \\ -0.2 & -5 \\ -3 & -3 \\ -5 & -0.2 \\ 1.7517257367 & 4.9862258 \end{pmatrix}$ and $H_1(V_1) = I_5 - 2V_1 \times [V_1^t \times R_5 \times V_1]^+ \times V_1^t \times R_5 = \begin{pmatrix} 0.1857 & -0.076 & 0.4470 & 0.8508 & -0.1897 \\ -0.076 & 0.0133 & -0.3450 & 0.3887 & 0.8508 \\ 0.4470 & -0.3450 & 0.6019 & -0.3450 & 0.4470 \\ 0.8508 & 0.3887 & -0.3450 & 0.0133 & -0.076 \\ -0.1897 & 0.8508 & 0.4470 & -0.0760 & 0.1857 \end{pmatrix}$.

For $k = 2$, two-column sub-matrices of A are $N_2 = \begin{pmatrix} 9 & 7 \\ 8 & 8 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}' & \boldsymbol{\alpha}'^{R} \end{pmatrix}$ and $M_2 = \begin{pmatrix} \delta_1' & \delta_2' \\ 0 & 0 \\ \delta_2' & \delta_1' \end{pmatrix}$ therefore matrix $V_2 = M_2 - N_2 = \begin{pmatrix} \delta_1' - 9 & \delta_2' - 7 \\ 8 & 8 \\ \delta_2' - 7 & \delta_1' - 9 \end{pmatrix}$. The parameters δ_1' and δ_2' are useful for finding out perplectic block reflector $H_2(V_2)$, so we can find out δ_1' and δ_2' by method discussed in subsection 3.1. Hence the value of $\delta_1' = \sqrt{\frac{\|\boldsymbol{\alpha}'\|_2^2 \pm \sqrt{\|\boldsymbol{\alpha}'\|_2^4 - (N(\boldsymbol{\alpha}'))^2}}{2}} = 10.79800395554643$ and 8.797922317002101. For vector $\boldsymbol{\alpha}' = \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$, Euclidean Vector Norm is $\|\alpha'\|_2 = 13.9284$, therefore$N(\boldsymbol{\alpha}') =$ $[\boldsymbol{\alpha}'\boldsymbol{\alpha}']_{R_2} = 190$. So $\delta_2' = N(\boldsymbol{\alpha}')/2\delta_1' = 0.7572573$ forδ_1= 8.797922318893293 for $\delta_1 = 10.79800395554643$. According to above calculation the value of $\delta_1' = 10.79800395554643$ and $\delta_2' = 8.797922318893293$. Putting the value of above parameters in V_2, $V_2 = \begin{pmatrix} 1.79800395554643 & 1.797922318893293 \\ 8 & 8 \\ 1.797922318893293 & 1.79800395554643 \end{pmatrix}$ and $H_2(V_2) = I_3 - 2V_2 \times [V_2^t \times R_3 \times V_2]^+ \times V_2^t \times R_3 = \begin{pmatrix} -0.0918 & 0.4083 & 0.9083 \\ 0.4082 & -0.8165 & 0.4082 \\ 0.9083 & 0.4083 & -0.0918 \end{pmatrix}$. Now we will finding out $Q_k = E_m(H_k(V_k))$, for $k = 1, 2$ matrices Q_1 and Q_2 are:

$$Q_1 = E_5(H_1(V_1)) = H_1(V_1) = \begin{pmatrix} 0.1857 & -0.076 & 0.4470 & 0.8508 & -0.1897 \\ -0.076 & 0.0133 & -0.3450 & 0.3887 & 0.8508 \\ 0.4470 & -0.3450 & 0.6019 & -0.3450 & 0.4470 \\ 0.8508 & 0.3887 & -0.3450 & 0.0133 & -0.076 \\ -0.1897 & 0.8508 & 0.4470 & -0.0760 & 0.1857 \end{pmatrix}$$

and $Q_2 = E_5(H_2(V_2)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -0.0918 & 0.4083 & 0.9083 & 0 \\ 0 & 0.4082 & -0.8165 & 0.4082 & 0 \\ 0 & 0.9083 & 0.4083 & -0.0918 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. The matrices A^k

for$k = 1, 2, , d$ are: $A^0 = A = \begin{pmatrix} 1 & 4 & 10 & -1 \\ 0.2 & 9 & 7 & 5 \\ 3 & 8 & 8 & 3 \\ 5 & 7 & 9 & 0.2 \\ -1 & 10 & 4 & 1 \end{pmatrix}$, $A^1 = Q_1 \times A = \begin{pmatrix} 5.9554 & 7.6936 & 11.800 & 0.7557 \\ -0.0154 & 8.2853 & 3.4751 & 0.0361 \\ 0.0119 & 5.5540 & 5.5540 & 0.0119 \\ 0.0361 & 3.4751 & 8.2853 & -0.0154 \\ 0.7557 & 11.800 & 7.6936 & 5.9554 \end{pmatrix}$

and $A^2 = Q_2 \times A^1 = \begin{pmatrix} 5.9554 & 7.6936 & 11.800 & 0.7557 \\ 0.0391 & 4.6635 & 9.4737 & -0.0124 \\ -0.0013 & 0.2658 & 0.2658 & -0.0013 \\ -0.0124 & 9.4737 & 4.6635 & 0.0391 \\ 0.7557 & 11.800 & 7.6936 & 5.9554 \end{pmatrix}$. We know that $Q = (Q_d \times .... \times Q_2 \times Q_1)^t$ and $R = A^d$ hence $Q = (Q_2 \times Q_1)^t$ and $R = A^2$ i.e., $Q =$

$\begin{pmatrix} 0.1857 & 0.9623 & -0.0487 & 0.0353 & -0.1897 \\ -0.0760 & 0.2110 & 0.4458 & -0.1644 & 0.8508 \\ 0.4470 & -0.0359 & -0.7731 & -0.0360 & 0.4470 \\ 0.8508 & -0.1644 & 0.4458 & 0.2110 & -0.0760 \\ -0.1897 & 0.0353 & -0.0487 & 0.9623 & 0.1857 \end{pmatrix}$ and $R = \begin{pmatrix} 5.9554 & 7.6936 & 11.800 & 0.7557 \\ 0.0391 & 4.6635 & 9.4737 & -0.0124 \\ -0.0013 & 0.2658 & 0.2658 & -0.0013 \\ -0.0124 & 9.4737 & 4.6635 & 0.0391 \\ 0.7557 & 11.800 & 7.6936 & 5.9554 \end{pmatrix} \approx$

$\begin{pmatrix} 5.9554 & 7.6936 & 11.800 & 0.7557 \\ 0 & 4.6635 & 9.4737 & 0 \\ 0 & 0.2658 & 0.2658 & 0 \\ 0 & 9.4737 & 4.6635 & 0 \\ 0.7557 & 11.800 & 7.6936 & 5.9554 \end{pmatrix}$. Therefore, we have Q and R as centro-symmetric factorization of centro-symmetric matrix A. Now, the problem is to find out centro-symmetric QR factorization of a rectangular matrix. This can be understood through the example below,

Example 8: Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0.2 & 4 & 5 \\ 3 & -1 & 3 \\ 5 & 4 & 0.2 \\ -1 & 2 & 1 \end{bmatrix}$$

, is a rectangular centro-symmetric matrix, number of rows $m = 5$,column $n = 3$ and $d = \frac{min?(5,3)+1}{2} = 2$ i.e., $k = 1, 2$. For $k = 1$, two-column sub-matrices of A are

$$N_1 = \begin{pmatrix} 1 & -1 \\ 0.2 & 5 \\ 3 & 3 \\ 5 & 0.2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\alpha^R} \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} \delta_1 & \delta_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \delta_2 & \delta_1 \end{pmatrix} \text{ therefore } V_1 = \begin{pmatrix} \delta_1 - 1 & \delta_2 + 1 \\ -0.2 & -5 \\ -3 & -3 \\ -5 & -0.2 \\ \delta_2 + 1 & \delta_1 - 1 \end{pmatrix}.$$

The parameters $\delta_1 = \sqrt{\frac{\|\boldsymbol{\alpha}\|_2^2 \pm \sqrt{\|\boldsymbol{\alpha}\|_2^4 - (N(\boldsymbol{\alpha}))^2}}{2}} = 5.9862258$ and 0.7517313 and For vector $\boldsymbol{\alpha} = \begin{pmatrix} 1 \\ 0.2 \\ 3 \\ 5 \\ -1 \end{pmatrix}$, Euclidean Vector Norm is $\|\boldsymbol{\alpha}\|_2 = \sqrt{36.4}$ therefore $N(\boldsymbol{\alpha}) = [\boldsymbol{\alpha}\boldsymbol{\alpha}]_{R_2} = 9$. So $\delta_2 = N(\boldsymbol{\alpha})/2\delta_1 = 0.7572573$ for δ_1=5.9862258 and 5.98618149 for $\delta_1 = 0.75173138$, according to above calculation the value of $\delta_1 = 5.9862258$ and $\delta_2 = 0.7517257367$. Putting the value of above parameters in V_1, $V_1 = \begin{pmatrix} 4.9862258 & 1.75172573 \\ -0.2 & -5 \\ -3 & -3 \\ -5 & -0.2 \\ 1.7517257367 & 4.9862258 \end{pmatrix}$ and $H_1(V_1) = I_5 - 2V_1 \times [V_1^t \times R_5 \times V_1]^+ \times V_1^t \times R_5 =$
$\begin{pmatrix} 0.1857 & -0.076 & 0.4470 & 0.8508 & -0.1897 \\ -0.076 & 0.0133 & -0.3450 & 0.3887 & 0.8508 \\ 0.4470 & -0.3450 & 0.6019 & -0.3450 & 0.4470 \\ 0.8508 & 0.3887 & -0.3450 & 0.0133 & -0.076 \\ -0.1897 & 0.8508 & 0.4470 & -0.0760 & 0.1857 \end{pmatrix}$.

For $k = 2$, two-column sub-matrices of A are $N_2 = \begin{pmatrix} 4 & 4 \\ -1 & -1 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}' & \boldsymbol{\alpha}'^R \end{pmatrix}$ and $M_2 = \begin{pmatrix} \delta_1' & \delta_2' \\ 0 & 0 \\ \delta_2' & \delta_1' \end{pmatrix}$ therefore matrix $V_2 = M_2 - N_2 = \begin{pmatrix} \delta_1' - 4 & \delta_2' - 4 \\ 1 & 1 \\ \delta_2' - 4 & \delta_1' - 4 \end{pmatrix}$. The parameters δ_1' and δ_2' are useful for finding out perplectic block reflector $H_2(V_2)$,

so $\delta_1' = \sqrt{\frac{\|\boldsymbol{\alpha}'\|_2^2 \pm \sqrt{\|\boldsymbol{\alpha}'\|_2^4 - (N(\boldsymbol{\alpha}'))^2}}{2}} = 4.0620192023$ and for vecto $\boldsymbol{\alpha}' = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix}$,

Euclidean Vector Norm is $\|\alpha'\|_2 = 5.744562646538$, therefore$N(\boldsymbol{\alpha}') = [\boldsymbol{\alpha}'\boldsymbol{\alpha}']_{R_2} = 33$. So $\delta_2' = N(\boldsymbol{\alpha}')/2\delta_1' = 4.06201920232096$

According to above calculation the value of $\delta_1' = 4.0620192023$ and $\delta_2' = 4.06201920232096$.

Putting the value of above parameters in V_2, $V_2 = \begin{pmatrix} 0.0620192 & 0.0620192 \\ 1 & 1 \\ 0.0620192 & 0.0620192 \end{pmatrix}$

and $H_2(V_2) = I_3 - 2V_2 \times [V_2^t \times R_3 \times V_2]^+ \times V_2^t \times R_3 = \begin{pmatrix} 0.9924 & -0.1231 & -0.0076 \\ -0.1231 & -0.9847 & -0.1231 \\ -0.0076 & -0.1231 & 0.9924 \end{pmatrix}$.

Now we will finding out $Q_k = E_m(H_k(V_k))$, for $k = 1, 2$ matrices Q_1 and Q_2 are:

$$Q_1 = E_5(H_1(V_1)) = H_1(V_1) = \begin{pmatrix} 0.1857 & -0.076 & 0.4470 & 0.8508 & -0.1897 \\ -0.076 & 0.0133 & -0.3450 & 0.3887 & 0.8508 \\ 0.4470 & -0.3450 & 0.6019 & -0.3450 & 0.4470 \\ 0.8508 & 0.3887 & -0.3450 & 0.0133 & -0.076 \\ -0.1897 & 0.8508 & 0.4470 & -0.0760 & 0.1857 \end{pmatrix}$$

and

$$Q_2 = E_5(H_2(V_2)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.9924 & -0.1231 & -0.0076 & 0 \\ 0 & -0.1231 & -0.9847 & -0.1231 & 0 \\ 0 & -0.0076 & -0.1231 & 0.9924 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ The matrices A^k for$k = 1, 2, , d$ are:

$$A^0 = A = \begin{pmatrix} 1 & 2 & -1 \\ 0.2 & 4 & 5 \\ 3 & -1 & 3 \\ 5 & 4 & 0.2 \\ -1 & 2 & 1 \end{pmatrix}, A^1 = Q_1 \times A = \begin{pmatrix} 5.9554 & 2.6443 & 0.7557 \\ -0.0154 & 3.5028 & 0.0361 \\ 0.0119 & -1.5737 & 0.0119 \\ 0.0361 & 3.4751 & -0.0154 \\ 0.7557 & 2.6443 & 5.9554 \end{pmatrix}$$

and $A^2 = Q_2 \times A^1 = \begin{pmatrix} 5.9554 & 2.6443 & 0.7557 \\ -0.0170 & 3.6433 & 0.0345 \\ -0.0143 & 0.6872 & -0.0143 \\ 0.0345 & 3.6433 & -0.0170 \\ 0.7557 & 2.6443 & 5.9554 \end{pmatrix}$.

We know that $Q = (Q_d \times .... \times Q_2 \times Q_1)^t$ and $R = A^d$ hence $Q = (Q_2 \times Q_1)^t$ and $R = A^2$ i.e.,

$$Q = \begin{pmatrix} 0.1857 & -0.1369 & -0.5356 & 0.7899 & -0.1897 \\ -0.0760 & 0.0527 & 0.2902 & 0.4281 & 0.8508 \\ 0.4470 & -0.4138 & -0.5078 & -0.4138 & 0.4470 \\ 0.8508 & 0.4281 & 0.2902 & 0.0527 & -0.0760 \\ -0.1897 & 0.7899 & -0.5356 & -0.1369 & 0.1857 \end{pmatrix} \text{ and } R = \begin{pmatrix} 5.9554 & 2.6443 & 0.7557 \\ -0.0170 & 3.6433 & 0.0345 \\ -0.0143 & 0.6872 & -0.0143 \\ 0.0345 & 3.6433 & -0.0170 \\ 0.7557 & 2.6443 & 5.9554 \end{pmatrix} \approx$$

$$\begin{pmatrix} 5.9554 & 2.6443 & 0.7557 \\ 0 & 3.6433 & 0 \\ 0 & 0.6872 & 0 \\ 0 & 3.6433 & 0 \\ 0.7557 & 2.6443 & 5.9554 \end{pmatrix}$$ .Therefore, we have Q and R as centro-symmetric factorization of rectangular centro-symmetric matrix A. Konrad Burnik [2] factorized a matrix into Q and R but R is not upper-triangular matrix, and also calculated approximate value of matrix R. Our calculation is same as the algorithm used by Konrad Burnik [2]. In example 7 and 8, we calculated that R is not upper-triangle but Q and R both centro-symmetric.

4. Application of QR factorization of a centro-symmetric matrix:

The importance of centro-symmetric QR-factorization in which Q and R are both centro-symmetric is solving a system of linear equation $A \times X = B$, where A is a centro-symmetric square matrix of full rank i.e. $|A| \neq 0$ or we can say that linear system has a unique solution. The method of solution of linear system of equation by above method is same as method of solution of $A \times X = B$ by general QR factorization. The only difference here is that matrix A is centro-symmetric and Q and R both are centro-symmetric. Now for linear system $A \times X = B$, marix $A = Q \times R$, where Q is perplectic orthogonal centro-symmetric matrix and R is double cone centro-symmetric matrix. Let us understand this application with an example:

Example 9: Let us have a system of linear equation $x+2y+3z = 7, 6x+4y+6z = 9$ and $3x + 2y + z = 5$. After arranging above equation in form of $A \times X = B$, where

A =

$$\begin{bmatrix} 1 & 2 & 3 \\ 6 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix}$$

is a centro-symmetric matrix, first we find cento-symmetric QR factorization of A. Using the algorithm (See Appendix-A), the centro-symmetric QR factorization of A are:

$$Q = \begin{bmatrix} -0.2868 & 0.6396 & 0.7132 \\ 0.6396 & -0.4264 & 0.6396 \\ 0.7132 & 0.6396 & -0.2868 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 5.6904 & 3.4112 & 3.6904 \\ 0.00 & 0.8528 & 0.00 \\ 3.6904 & 3.4112 & 5.6904 \end{bmatrix}$$

. We set $y = Q^t \times B = \begin{pmatrix} 7.3148 \\ 3.8376 \\ 9.3148 \end{pmatrix}$ and solve the system $y = R \times X = \begin{pmatrix} 7.3148 \\ 3.8376 \\ 9.3148 \end{pmatrix}$.
Now we have three equation

$$5.6904x + 3.4112y + 3.6904z = 7.3148 \tag{8}$$

$$0.8528y = 3.8376 \tag{9}$$

$$3.6904x + 3.4112y + 5.6904z = 9.3148 \tag{10}$$

After solving equation (8), (9) and (10), we have $x = -1.187, y = 4.5$ and $z = -0.3472$.
In this above example, we investigated that this method is giving correct values for linear system of equation i.e. the values of x, y and z satisfy linear system of equation when R is not upper triangle.

5. Conclusion:

A stepwise numerical method of QR factorization is applied in a centro-symmetric matrix. This method of QR factorization preserved the centro-symmetry of Q and R. For finding such a factorization, perplectic block-reflectors were used to reduce pair of columns of a given centro-symmetric matrix. This was used to achieve a double-cone matrix. In this paper, we calculated the exact value of QR factorization of a centro-symmetric matrix in which Q and R are centro symmetric but R is not upper triangular unlike reported values in literature. We further investigated that this method is giving correct values for linear system of equation.

Acknowledgment: The author expresses hearty thanks and gratefulness to all those scientists whose masterpieces have been consulted during the preparation of the present research article. We are also thankful to C.A. Bijaya Laxmi Thapliyal, and Dr. Anil Singh for major review with grammatical correction of this paper.

References

[1] Roger A. Horn and Charles R. Johnson, Matrix Analysis, Cambridge University Press, 2013.
[2] Konrad Burnik, A structure-preserving QR factorization for centro-symmetric real matrices, Linear Algebra and its Applications, 484, 356378, 2015.
[3] Vineet Bhatt and S. Kumar, A CAS aided survey of CP decomposition and rank-1 approximation of a 3rd-order tensor, Palestine Journal of Mathematics, Vol. 3(2), 299319, 2014.
[4] D. Steven Mackey, Niloufer Mackey, Franoise Tisseur, Structured tools for structured matrices, Electron. J. Linear Algebra 10, 106145, 2003.
[5] D. Steven Mackey, Niloufer Mackey, Daniel M. Dunlavy, Structure preserving algorithms for per-plectic Eigen problems, Electron. J. Linear Algebra 13, 1039, 2005.
[6] Tao, D.; Yasuda, M., A spectral characterization of generalized real symmetric centro-symmetric and generalized real symmetric skew-centro-symmetric matrices, SIAM J. Matrix Anal. Appl. 23 (3),885895, 2002.
[7] Ben-Israel, Adi; Greville, Thomas N.E., Generalized inverses: Theory and applications, New York, NY: Springer, ISBN 0-387-00293-6, 2003.
[8] Sanja Singer, Saa Singer, Ortho-symmetric block reflectors, Linear Algebra Appl. 429(56), 13541385, 2008.
[9] James R. Weaver, Centrosymmetric (Cross-Symmetric) Matrices, Their Basic Properties, Eigenvalues, and Eigenvectors, The American Mathematical Monthly, 92(10), 711-717, 1985.

Appendix-A: Step of Algorithm for Centro-symmetric QR-factorization of matrix A preserving centro-symmetric properties.
Step 1: Take a centro-symmetric matrix $A = [a_{(i,j)}]_{(m\times n)}$.
Step 2: Find $d = min?(m,n)/2$ for min?(m,n) even or d=(min?(m,n)+1)/2 for min?(m,n) odd.
Step 3: Find value of $k = 1, 2, d$.
Step 4: Find $t = m - 2k + 2$.

Step 5: Find a column vector $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{k,k} \\ \alpha_{k+1,k} \\ . \\ . \\ \alpha_{m-k+1,k} \end{pmatrix} = \begin{pmatrix} \alpha_{k,k} & \alpha_{k+1,k} & .... & \alpha_{m-k+1,k} \end{pmatrix}^t$.

Step 6: Find a two column sub matrix $N_k = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\alpha}^R \end{pmatrix} = \begin{pmatrix} \alpha_{k,k} & \alpha_{m-k+1,k} \\ \alpha_{k+1,k} & . \\ . & . \\ . & \alpha_{k+1,k} \\ \alpha_{m-k+1,k} & \alpha_{k,k} \end{pmatrix}$.

Step 7: Find a new two-column sub-matrix with unknown parameters $M_k = \begin{pmatrix} \delta_1 \times \boldsymbol{i_1} + \delta_2 \times \boldsymbol{i_t} & \delta_2 \times \boldsymbol{i_t} + \delta_1 \times \boldsymbol{i_1} \end{pmatrix} = \left(\delta_1 \times \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} + \delta_2 \times \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} \quad \delta_2 \times \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} + \delta_1 \times \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \right)$

Step 8: Find a matrix $V_k = M_k - N_k$.
Step 9: Find N(a) for finding the parameters δ_1 and δ_2.
if $N(\alpha) = 0$ then
Step 11: Find $\delta_1 = \|\boldsymbol{\alpha}\|_2$ and $\delta_2 = 0$.
Step 12: Find $H_k(V_k)$ by $H_k(V_k) = I_t - 2V_k \times [V_k^t \times R_t \times V_k]^+ \times V_k^t \times R_t$.
Step 13: Find $Q_k = E_m(H_k(V_k))$ for every k.
Step 14: Find $Q = (Q_d \times .... \times Q_2 \times Q_1)^t$.
Step 15: Find $A^k = Q_k \times A^{(k-1)}$ where $A^0 = A$ and $A^1 = Q_1 \times A^0 = Q_1 \times A$.

Step 16: Find $R = A^d = Q_d \times A^{(d-1)} = Q_d \times ... \times Q_2 \times Q_1 \times A$.
Now we have two matrix Q and R of QR factorization of matrix A.

Vineet Bhatt
Department of Mathematics, Graphic Era Hill University, Campus Dehradun, Uttarakhand, India
E-mail address: `vineet.bhatt58@gmail.com`

K.C. Petwal
Department of Mathematics, H.N.B. Garhwal University, Campus Tehri, India
E-mail address: `kcpetwal@gmail.com`

S. Kumar
Department of Mathematics, Govt. P.G. College, New Tehri, Uttarakhand, India
E-mail address: `drskbahuguna@gmail.com`