NEW INEQUALITIES FOR THE FUNCTION $y = t \ln t$

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Abstract. The main aim of this note, which can be viewed as a certain ad-
dendum to the paper [1], is to propose several new inequalities for the function
$y = t \ln t$. We consider the local behaviour of this function near the point $t = 1$,
as well as the global behaviour of this function on the intervals $[1, \infty)$ and $(0, 1]$.

1. Introduction

The reading of paper [1] by C. Chesneau and Y. J. Bagul has strongly influenced
us to write this note. In Theorem 2, we give new abstract local bounds for the
function $y = t \ln t$ near the point $t = 1$. The obtained inequalities can be used to
improve the main results of paper [1], Proposition 1 and Proposition 2. We also
present an interesting result with regards to these propositions, which claims that
there is no rational real function which intermediates the functions $\ln(1 + x)$ and
$f(x)/\sqrt{x+1}$ for $x \geq 0$ ($x \in (-1, 0]$); here and hereafter,

$$f(x) := \pi + \frac{1}{2}(4 + \pi)x - 2(x + 2)\arctan \sqrt{x+1}, \quad x \geq -1.$$

The following inequalities are well known (see also [3, Problem 3.6.19, p. 274]
and [4]):

$$\ln(1 + x) \leq \frac{x}{\sqrt{x+1}}, \quad x \geq 0, \quad \ln(1 + x) \leq \frac{x(2 + x)}{2(1 + x)}, \quad x \geq 0, \quad (1)$$

$$\ln(1 + x) \leq \frac{x(6 + x)}{2(3 + 2x)}, \quad x \geq 0 \quad \text{and} \quad \ln(1 + x) \leq \frac{(x + 2)[(x + 1)^3 - 1]}{3(1 + x)(x + 1)^2 + 1}, \quad x \geq 0. \quad (2)$$

Taken together, the first inequality in (1) and the second inequality in (2) are known
in the existing literature as Karamata’s inequality [2]. As clarified in [4], all these
inequalities are weaker than the inequality:

$$\ln(1 + x) \leq \frac{f(x)}{\sqrt{x+1}}, \quad x \geq 0. \quad (3)$$

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This inequality has been proved in [1] Proposition 1. In [1] Proposition 2, the authors have proved that
\[ \ln(1 + x) \geq \frac{f(x)}{\sqrt{x + 1}}, \quad x \in (-1, 0], \] (4)
as well.

Our approach leans heavily on the use of substitution \( t = \sqrt{x + 1} \). Then the inequalities (3) and (4) become
\[ 2\ln t \leq \frac{f(t^2 - 1)}{t}, \quad t \geq 1 \quad \text{and} \quad 2\ln t \geq \frac{f(t^2 - 1)}{t}, \quad t \in (0, 1], \]
i.e.,
\[ 2t \ln t \leq f(t^2 - 1), \quad t \geq 1 \quad \text{and} \quad 2t \ln t \geq f(t^2 - 1), \quad t \in (0, 1]. \] (5)
We can prove (5) in the following way. Notice that
\[ \left[ \ln t - \left( \frac{1}{2}(4 + \pi)t - 2t \arctan t - 2 \right) \right]'(t) = -(t^2 - 1)^2 t^{-2} (t^2 + 1)^{-2}, \quad t > 0. \]
Using an elementary argumentation, this estimate implies
\[ \ln t \leq \frac{1}{2}(4 + \pi)t - 2t \arctan t - 2, \quad t > 0. \]
Define \( R(t) := 2t \ln t - f(t^2 - 1), t > 0 \). Since \( R'(t) = 2(1 + \ln t) -(4 + \pi)t + 4t \arctan t + 2, t > 0 \), the previous inequality yields \( R'(t) \leq 0, t > 0 \) and (5). Moreover, by taking the limit of function \( R(\cdot) \) as \( t \to 0^+ \), we get that \( 2t \ln t - f(t^2 - 1) \in (2 - (\pi/2), 0] \) for \( t \in (0, 1] \).

In this paper, we will first generalize the inequalities in (5) by considering the local behaviour of the function \( y = t \ln t \) near the point \( t = 1 \). We will use the following simple lemmata, which is known from the elementary courses of mathematical analysis:

**Lemma 1** Suppose \( t_0 \in \mathbb{R}, a > 0, n \in \mathbb{N} \) and function \( f : (t_0 - a, t_0 + a) \to \mathbb{R} \) is \( 2n \)-times differentiable. If \( f^{(i)}(t_0) = 0 \) for all \( i = 1, \ldots, 2n - 1 \) and \( f^{(2n)}(t_0) > 0 \) \( (f^{(2n)}(t_0) < 0) \), then the function \( y = f(t) \) has a local minimum (maximum) at \( t = t_0 \).

**Lemma 2** We have
\[ (\arctan x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1 + x^2)^{n/2}} \sin(n\pi/2 - n \arctan x), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \]

After that, we will prove the following result with regards to [1] Proposition 1, Proposition 2):

**Theorem 1**

(i) There do not exist real polynomials \( P(\cdot) \) and \( Q(\cdot) \) such that \( Q(x) \neq 0 \) for \( x \geq 0 \) and
\[ \ln(1 + x) \leq \frac{P(x)}{Q(x)} \leq \frac{f(x)}{\sqrt{x + 1}}, \quad x \geq 0. \] (6)

(ii) There do not exist real polynomials \( P(\cdot) \) and \( Q(\cdot) \) such that \( Q(x) \neq 0 \) for \( x \in (-1, 0] \) and
\[ \ln(1 + x) \geq \frac{P(x)}{Q(x)} \geq \frac{f(x)}{\sqrt{x + 1}}, \quad x \in (-1, 0]. \] (7)
2. The main results and their proofs

We start this section by stating the following result:

**Theorem 2** Suppose that $a \in (0, 1)$, $P : (1 - a, 1 + a) \to \mathbb{R}$ is a function and $P(1) = 0$. Then the following holds:

(i) If $P'(1) \geq 2$ and there exists an odd natural number $n$ such that $P(\cdot)$ is $(n + 2)$-times differentiable, $P^{(n+2)}(1) + 2(-1)^{n+1}n! > 0$ and
\[
P^{(j)}(1) + 2(-1)^{j+1}(j - 2)! = 0 \quad \text{for all } j = 2, 3, \ldots, n + 1,
\]
then there exists a real number $\zeta \in (0, a]$ such that
\[
2t \ln t \leq P(t), \quad t \in [1, 1 + \zeta] \quad \text{and} \quad 2t \ln t \geq P(t), \quad t \in [1 - \zeta, 1].
\]

(ii) Assume that there exists an even natural number $n \geq 6$ such that $P(\cdot)$ is $(n + 1)$-times differentiable, $P^{(n+1)}(1) + 2(-1)^{n}(n - 1)! > 0$ and
\[
P^{(j)}(1) + 2(-1)^{j+1}(j - 2)! = 0 \quad \text{for all } j = 1, 2, \ldots, n.
\]
Then there exists a real number $\eta \in (0, a]$ such that
\[
2t \ln t \leq P(t) \leq f(t^2 - 1), \quad t \in [1, 1 + \eta]
\quad \text{and} \quad 2t \ln t \geq P(t) \geq f(t^2 - 1), \quad t \in [1 - \eta, 1].
\]

(iii) Assume that there exists an even natural number $n \geq 6$ such that $P(\cdot)$ is $(n + 1)$-times differentiable,
\[
P^{(j)}(1) + 2(-1)^{j+1}(j - 2)! = 0 \quad \text{for all } j = 1, 2, 3, 4,
\]
and, for every $j = 5, 6, \ldots, n,$
\[
P^{(j)}(1) + 4 \left[ \frac{(-1)^{n}n!}{2^{(n+1)/2}} \sin((n + 1)\pi/4) + \frac{(-1)^{n+1}n!}{2^{n/2}} \sin(n\pi/4) \right] < 0
\]
and $\eta, t \in [1 - \eta, 1].$

Then there exists a real number $\eta \in (0, a]$ such that (9) holds.

(iv) If $P(\cdot)$ is five times differentiable, $P^{(5)}(1) \in (-12, -8),$ then there exists a real number $\eta \in (0, a]$ such that (1) holds.

**Proof.** Define $G(t) := P(t) - 2t \ln t, t > 0.$ Then, for every real number $t > 0,$ we have $G'(t) = P'(t) - 2(1 + \ln t), G''(t) = P''(t) - (2/t) \ln t + 2(-1)^{n+1}(n - 2)! \cdot t^{1-n}, n \geq 3.$ The assumptions made in (i) imply that $G'(1) \geq 0,$ $G'(1) = 0$ for $1 \leq j \leq n$ and $G^{(n+1)}(1) > 0.$ Applying Lemma 1, we get that the function $G'(t)$ has a local minimum at $t = 1.$ Since $G'(1) \geq 0,$ we get that the function $t \mapsto G'(t)$ is non-negative in an open neighborhood of point $t = 1,$ so that the mapping $t \mapsto G(t)$ is increasing in an open neighborhood of point $t = 1.$ This finishes the proof of (i). For the proof of (ii), define $Q(t) := P(t) - f(t^2 - 1),$ $t > 0.$ Then a simple computation yields that, for every real number $t > 0,$ we have $Q'(t) = P'(t) - (4 + \pi)t + 4t \arctan t + 2$ and $Q''(t) = P''(t) - (4 + \pi) + 4 \arctan t + \frac{4t}{t^2 + 1}.
Using Leibniz rule and Lemma 2, for every real number \( t > 0 \) and for every natural number \( n \geq 3 \), we can show that

\[
Q^{(n)}(t) = P^{(n)}(t) + 4 \left[ \arctan \left( \frac{(n-1)!}{1 + t^2} \right) \right]^{(n-1)}(t)
\]

\[
= P^{(n)}(t) + 4 \left[ \frac{(-1)^{n-1}(n-1)!}{1 + t^2} \sin(n\pi/2 - n \arctan t) \right.
\]

\[
+ \frac{(-1)^n(n-1)!}{(1 + t^2)^{(n-1)/2}} \sin((n-1)\pi/2 - (n-1) \arctan t) \right].
\]

Arguing as in the proof of (i), we have that \((Q')^{(j)}(1) = 0\) for \( j = 0, 1, 2, 3 \) and \((Q')^{(4)}(1) < 0\); hence, the function \( t \mapsto Q'(t) \) has a local maximum at \( t = 1 \) and the mapping \( t \mapsto Q(t) \) is decreasing in an open neighborhood of point \( t = 1 \). Similarly, \((G')^{(j)}(1) = 0\) for \( j = 0, 1, 2, \ldots, n-1 \) and \((G')^{(n)}(1) > 0\); hence, the function \( t \mapsto G'(t) \) has a local minimum at \( t = 1 \) and the mapping \( t \mapsto G(t) \) is increasing in an open neighborhood of point \( t = 1 \). This completes the proof of (ii). The proof of (iii) can be deduced similarly, by interchanging the roles of \( G(t) \) and \( Q(t) \). If the assumptions of (iv) holds, then we can apply Lemma 1, with \( n = 2 \), in order to see that the function \( t \mapsto G'(t) \) has a local minimum at \( t = 1 \), as well as the function \( t \mapsto G'(t) \) is non-negative in an open neighborhood of point \( t = 1 \); hence, the mapping \( t \mapsto G(t) \) is increasing in an open neighborhood of point \( t = 1 \). Similarly, we can show that the mapping \( t \mapsto Q(t) \) is decreasing in an open neighborhood of point \( t = 1 \). The proof of the theorem is thereby complete.

**Remark 1** Define \( H(t) := f(t^2 - 1), \ t \in \mathbb{R}. \) Concerning the conditions used in Theorem 2, it is worth noting that the function \( H(\cdot) \) satisfies \( H(1) = 0, \ H'(1) = H''(1) = 2, \ H'''(1) = -2, \ H^{(iv)}(1) = 4 \) and \( H^{(v)}(1) = -8 \). This implies that the values of terms appearing at the right-hand sides of (10) and (11) coincide for \( j = 1, 2, 3, 4 \) and differ for \( j = 5 \) (observe that \( G^{(v)}(1) = P^{(v)}(1) + 12 \)).

**Remark 2** The parts (ii)-(v) of Theorem 2 ensure the existence of a large class of elementary functions for which we can further refine the inequalities in (3) locally around the point \( t = 1 \). Compared with the function \( H(\cdot) \), the most simplest example of function which provides a better estimate describing the local behaviour of function \( y = t \ln t \) around the point \( t = 1 \) is given by the function \( t \mapsto H(t) - \epsilon (t-1)^5, \ t > 0, \) where \( \epsilon \in (0, 1/30) \).

Concerning the global behaviour of function \( y = t \ln t, \ t > 0 \), it is clear that the inequalities in (3) give some very uninteresting estimates with regards to the asymptotic behaviour of function \( y = t \ln t \) when \( t \to +\infty \) or \( t \to 0+ \); on the other hand, the importance of estimate (3) lies in the fact that it gives some bounds for the behaviour of function \( y = t \ln t \) on any compact interval \([a, b]\), where \( 0 < a < 1 < b \). It is clear that there exists a large class of infinitely differentiable functions \( P : (0, \infty) \to \mathbb{R} \) such that

\[
2t \ln t \leq P(t) \leq f(t^2 - 1), \ t \geq 1
\]

and

\[
2t \ln t \geq P(t) \geq f(t^2 - 1), \ t \in (0, 1].
\]

(12)

Finding new elementary functions \( P(\cdot) \) for which the equation (12) holds is without scope of this paper.

We close the paper by giving the proof of Theorem 1:
Theorem 1. Suppose that (10) holds for some real polynomials $P(\cdot)$ and $Q(\cdot)$ such that $Q(x) \neq 0$ for $x \geq 0$. Without loss of generality, we may assume that $Q(x) > 0$, $x \geq 0$. Using the substitution $t = \sqrt{x+1}$, we get that

$$2 \ln t \leq \frac{P(t^2 - 1)}{Q(t^2 - 1)} \leq \frac{f(t^2 - 1)}{t}, \quad t \geq 1.$$ 

If $P(t) = \sum_{j=0}^{n} a_j t^j$ and $Q(t) = \sum_{j=0}^{m} b_j t^j$ for some non-negative integers $m$, $n$ and some real numbers $a_j$, $b_j$ ($a_0 b_m \neq 0$; clearly, we cannot have $P(x) \equiv 0$), we get

$$t \sum_{j=0}^{n} a_j (t^2 - 1)^j \leq f(t^2 - 1) \sum_{j=0}^{m} b_j (t^2 - 1)^j, \quad t \geq 1$$

and

$$\sum_{j=0}^{n} a_j (t^2 - 1)^j \geq 2 \ln t \sum_{j=0}^{m} b_j (t^2 - 1)^j, \quad t \geq 1.$$ 

Since $f(t^2 - 1) \sim 2 - (\pi/2) t^2$, $t \to +\infty$, the estimate (13) implies $n \leq m$. The positivity of polynomial $Q(\cdot)$ on the non-negative real axis implies $b_m > 0$ so that (14) gives $a_n > 0$. Considering the asymptotic behaviour of terms appearing in (14), we get that the inequality $n < m$ cannot be satisfied so that $m = n$. Dividing the both sides of (14) with $t^{2n}$ and letting $t \to +\infty$ in the obtained expression, we get that $a_n / 2b_n \geq +\infty$, which is a contradiction. This completes the proof of (i).

To prove (ii), suppose that the estimates hold for some real polynomials $P_0(\cdot)$ and $Q_0(\cdot)$ such that $Q_0(x) \neq 0$ for $x \in (-1, 0]$. Then (7) holds for some real polynomials $P(\cdot)$ and $Q(\cdot)$ such that $Q(x) > 0$ for $x \in (-1, 0]$. Letting $x \to -1^-$ in (7), we get that $Q(-1) = 0$. If $P(x) = \sum_{j=0}^{n} a_j x^j$ and $Q(x) = \sum_{j=0}^{m} b_j x^j$ for some non-negative integers $m$, $n$ and some real numbers $a_j$, $b_j$ ($a_0 b_m \neq 0$; again, we cannot have $P(x) \equiv 0$), this implies

$$\ln(1 + x) \cdot \sum_{j=0}^{m} b_j x^j \geq \sum_{j=0}^{n} a_j x^j \geq \frac{f(x)}{\sqrt{x+1}} \sum_{j=0}^{m} b_j x^j, \quad x \in (-1, 0].$$

Letting $x \to 0^-$ in this expression, we get that $a_0 = 0$ so that $n \geq 1$ and $x |P(x)$. Define $P_1(x) := P(x)/x$ and $Q_1(x) := Q(x)/(x+1)$. Then $P_1(x)$ and $Q_1(x)$ are real polynomials, $Q_1(x) > 0$ for $x \in (-1, 0]$ and after multiplication with $x + 1 \over 2Q_1(x)$ we have

$$\frac{x+1}{x^2} \ln(1 + x) \leq \frac{P_1(x)}{Q_1(x)} \leq \frac{f(x)}{x+1}, \quad x \in (-1, 0).$$

Letting $x \to -1^-$ in this expression, we get that $\lim_{x \to -1^-} \frac{P_1(x)}{Q_1(x)} = 0$, which implies $P_1(-1) = 0$. Since $P_1(x)$ is a non-zero polynomial, we get that $x + 1 |P_1(x)$. Multiplying the equation (16) with $x + 1 \over x + 1$, we get

$$\ln(1 + x) \geq \frac{P_1(x)}{Q_1(x)} \geq \frac{f(x)}{x+1}, \quad x \in (-1, 0).$$
Letting \( x \to 0^- \), we get
\[
\ln(1 + x) \geq \frac{P_1(x)}{Q_1(x)} \geq \frac{f(x)}{\sqrt{x + 1}}, \quad x \in (-1, 0].
\]
Repeating this procedure, we get that for every natural number \( k \) we have \((x + 1)^k|Q(x)\), which is a contradiction.

References