Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 326-345.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

COMMON FIXED POINTS THEOREM FOR FOUR MAPPINGS ON METRIC SPACE SATISFYING CONTRACTIVE CONDITIONS OF INTEGRAL TYPE

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Abstract. This paper presents some extensions of the results in [19], [21], [28]. We obtain a result on common fixed point theorem in metric space for four maps using integral type contractions. Here we try to investigate some results concerning mappings which share CLR property and subsequentially continuous mappings. Some examples to justify our results are given.

1. Introduction

In the field of non linear analysis fixed point theory is one of the important topic. Due to its wide application in the areas like economics, engineering, etc. it has got the huge attention of use number of researches. One of the fundamental result is due the celebrated Banach contraction Principle [5]. Since then there had been a large number of extension of this result. Brainciari [6] presented the integral type of contraction analogous to Banach contraction to obtain the unique fixed point. Articles related to finding of common fixed point of two , or three or four mappings came into the picture. Along with this many researches defined a class of mappings, like for example, compatible mappings, weakly compatible , subsequentially continuous, EA property, CLR-property etc. and obtained unique common fixed point for these mappings. In this paper also we have obtained unique common fixed point for four mappings taking new contraction conditions which generalised the existing results. In support we have presented some examples.

In 1986 Jungck [13] gave the concept of Compatible mappings.

Definition 1.1. Let $(X,\, d)$ be a metric space. A pair of self-mapping $f, h\colon X \to X$ is compatible if $\lim_{n\to\infty} d(fhx_n, hfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X, such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z,\ for\ \ some\ z \in X.$$

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Integral type contraction condition, Common CLR Property, Weakly Compatible mappings and Fixed point.
Submitted Dec. 4, 2019. Revised Jan. 4, 2020.

In 1996 Jungck [12] generalizes this concept of compatible maps to Weakly Compatible maps to study common fixed point theorems.

Definition 1.2. Let $(X,\, d)$ be a metric space. A pair of self-mapping $f, h\colon X \to X$ is Weakly Compatible if they commute at their coincidence points, that is, if there exists a point $x \in X$, such that $fhx = hfx$, whenever $fx = hx$.

In the study of common fixed points of Weakly Compatible mappings, we often require the assumption of completeness of the space or subspace or continuity of mappings involved besides some contractive condition. Aamri and El Moutawakil [1] introduced the notion of (E.A) property, which requires only the closedness of the subspace.

Definition 1.3. [1] Let (X, d) be a metric space and $f, h\colon X \to X$ be two self-maps. The pair (f, h) is said to satisfy the (E.A) property if there exist sequence $\{x_n\}$ in X and some $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z \in X.$$

Liu et al. [15] extended the (E.A) property to common the (E.A) property as follows.

Definition 1.4. Let (X, d) be a metric space and f, g, h, and $J\colon X \to X$ be four self-maps. The pairs (f, h) and (g, J) satisfy the common (E.A) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = t \in X.$$

Definition 1.5. [7] Let (X, d) be a metric space and $f, h\colon X \to X$ be self-maps. The pair (f, h) is called subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z,\ for\ some\ z \in X.$$

and

$$\lim_{n\to\infty} fhx_n = fz,\ \lim_{n\to\infty} hfx_n = hz.$$

Sintunavarat and Kumam [29] introduced the notion of the (CLR) property,which never requires any condition on closedness of the space or subspace

Definition 1.6. Let (X, d) be a metric space and $f, h\colon X \to X$ be self-maps. The pair (f, h) said to satisfy the common limit in the range of h property if

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = hx,\ for\ \ some\ x \in X.$$

Here we can observe that if the pair of mapping (f, h) satisfy E.A property together with the condition that $h(X)$ is closed then the pair also satisfies the common limit in the range of h property.
Imdad et al. [11] introduced the common (CLR) property which is an extension of the (CLR) property.

Definition 1.7. Let (X, d) be a metric space and f, g, h, and $J\colon X \to X$ be four self-maps. The pairs (f, h) and (g, J) satisfy the common limit range property with respect to mappings h and J, denoted by CLR_{hJ} if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = t \in h(X) \cap J(X).$$

Rhoades [26] proved the following fixed point theorems for the weakly contraction mapping which generalizations of the celebrated Banach fixed point theorem.

Theorem 1.8. *Let (X, d) be a complete metric space and $f\colon X \to X$ such that for each $x, y \in X$*

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y))$$

where, $\psi\colon R^+ \times R^+$ is continuous and nondecreasing such that ψ is positive on $R^+ - 0$, $\psi(0) = 0$ and $\psi(t) > 0, \forall t > 0$. Then f will have a unique fixed point in X.

Further Dutta and Choudhury [9] came with a generalization given in the following theorem

Theorem 1.9. [9] *Let (X, d) be a complete metric space and $f\colon X \to X$ such that for each $x, y \in X$*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

where, $\psi, \phi\colon R^+ \times R^+$ is continuous and monotone nondecreasing functions such that, $\psi(t) = \phi(t) = 0$ iff $t = 0$. Then f will have a unique fixed point in X.

Let us begin by considering some classes of mappings .

$\Phi_1 = \{\varphi\colon R^+ \to R^+$ satisfies that φ is Lebesgue integrable, summable on each compact subset of R^+ and for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0\}$.

$\Phi_2 = \{\varphi\colon R^+ \to R^+$ is nondecreasing continuous function on $R^+ - 0$ and $\varphi(t) = 0 \iff t = 0$ $\}$.

$\Phi_3 = \{\varphi\colon R^+ \to R^+$ is lower semicontinuous function and $\varphi(t) > 0$ for each $t > 0\} = \{\varphi\colon R^+ \to R^+$ is lower semicontinuous function and $\varphi(t) = 0$ iff $t = 0.\}$

$\Phi_4 = \{\varphi\colon R^+ \to R^+$ is upper semicontinuous function on $R^+ - 0$ and $\varphi(0) = 0$ and $\varphi(t) < t$, for each $t > 0$ $\}$.

$\Phi_5 = \{\varphi\colon R^+ \to R^+$ is upper semicontinuous function and $\lim_{n\to\infty} a_n = 0$ for each sequence $\{a_n\}_{n\in N} \subset R^+$ with $a_{n+1} \leq \psi(a_n), \forall n \in N\}$.

$\Phi_6 = \{\varphi\colon R^+ \to R^+$ is continuous function and $\varphi(t) < t$, for each $t > 0\}$.

Let us consider the following notations,

$m_1(x, y) = max\{d(hx, Jy), d(fx, hx), d(gy, Jy), \frac{d(hx,gy)+d(Jy,fx)}{2}, \frac{d(fx,hx)d(gy,Jy)}{1+d(hx,Jy)}, \frac{d(fx,Jy)d(gy,hx)}{1+d(hx,Jy)}, d(fx, hx)\frac{1+d(hx,gy)+d(Jy,fx)}{1+d(fx,hx)+d(Jy,gy)}\}$,

$m_2(x, y) = max\{d(hx, Jy), d(fx, hx), d(gy, Jy), \frac{d(hx,gy)+d(Jy,fx)}{2}, \frac{1+d(fx,hx)}{1+d(hx,Jy)}d(Jy, gy), \frac{1+d(gy,Jy)}{1+d(hx,Jy)}d(hx, fx), d(Jy, gy)\frac{1+d(hx,gy)+d(Jy,fx)}{1+d(fx,hx)+d(Jy,gy)}\}$,

$m_3(x, y) = max\{d(hx, Jy), d(fx, hx), d(gy, Jy), \frac{d(hx,gy)+d(Jy,fx)}{2}, \frac{d(fx,hx)d(gy,Jy)}{1+d(fx,gy)}, \frac{d(fx,Jy)d(gy,hx)}{1+d(gy,fx)}, d(hx, fx)\frac{1+d(hx,gy)+d(Jy,fx)}{1+d(fx,hx)+d(Jy,gy)}\}$.

$m_4(x,y) = max\{d(hx,Jy), d(fx,hx), d(gy,Jy), \frac{d(hx,gy)+d(Jy,fx)}{2}\}$.

$m_5(x,y) = max\{d(hx,Jy), d(fx,hx), d(gy,Jy), \frac{d(hx,gy)+d(Jy,fx)}{2}, \frac{d(fx,hx)d(gy,Jy)}{1+d(hx,Jy)}, \frac{d(fx,Jy)d(gy,hx)}{1+d(hx,Jy)}, \frac{d(fx,Jy)d(gy,hx)}{1+d(fx,gy)}\}$.

In the year 2015, Liu et. al.,[16] came up with the results that provides the unique common fixed point for four mappings in a metric space. They proved the following theorem:

Theorem 1.10. *Let (X,d) be a metric space and $f,g,h,J\colon X \to X$ such that $f(X) \subset h(X)$ and $g(X) \subset J(X)$ share (CLR_{hJ}) and one of $f(X), h(X), g(X)$, and $J(X)$ is complete and for each x, $y \in X$,*

$$d(fx,gy) \leq \psi(m_1(x,y)), \ \textit{where}\ \psi \in \Phi_5.$$

Then if the pairs (f,h) and (g,J) are Weakly Compatible, then f,g,h, and J will have a unique common fixed point in X.

Also the authors in [16] proved the following lemma

Lemma 1.11. *Let (X,d) be a metric space where the mappings $f,g,h,J\colon X \to X$, satisfy*

$$\psi(d(fx,gy)) \leq \psi(m_i(x,y)) - \varphi(m_i(x,y)), \ \forall x,y \in X$$

Where

$$(\psi,\varphi) \in \Phi_2 \times \Phi_3 \ \textit{and}\ i = 1,2.$$

Assume that $I\colon R^+ \to R^+$ is the identity mapping and

$$\psi_1(t) = (\psi + I)^{-1}(\psi + I - \varphi)(t), \forall t \in R^+.$$

Then

$$\psi_1 \in \Phi_5 \ \textit{and}\ d(fx,gy) \leq \psi_1(m_i(x,y)), \ \forall\ x,y \in X.$$

Braiciari [6] generalised the Banach contraction principal by introducing the integral type contraction condition as follows:

Theorem 1.12. *Let (X,d) be a complete metric space and $f\colon X \to X$ such that for each $x,y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c\int_0^{d(x,y)} \varphi(t)dt$$

where $c \in [0,1)$ is a constant and $\varphi \in \Phi_1$. Then f will have a unique fixed point in $a \in X$ such that $\lim_{n\to\infty} f^n x = a$.

Further Rhoades [27] obtained the common fixed point for two mappings by using the integral type contraction condition. The author came with the following theorem:

Theorem 1.13. *Let (X,d) be a complete metric space and $f\colon X \to X$ such that for each $x,y \in X$*

$$\int_0^{d(fx,fy)} \phi(t)dt \leq c\int_0^{m(x,y)} \phi(t)dt$$

where,

$$m(x,y) = max\{d(x,y), d(fx,x), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\},$$

and $c \in [0,1)$ is a constant and $\varphi \in \Phi_1$. Then f will have a unique fixed point in $a \in X$ such that $\lim_{n\to\infty} f^n x = a$.

Liu et. al [19] proved the following theorem for finding unique fixed point in a complete metric space.

Theorem 1.14. *Let (X,d) be a complete metric space and $f\colon X \to X$ satisfying and for each $x, y \in X$*

$$\int_0^{d(fx,fy)} \varphi(t)dt) \leq \int_0^{d(x,y)} \varphi(t)dt) - \int_0^{\psi(d(x,y))} \varphi(t)dt, \text{ where } (\varphi,\psi) \in \Phi_1 \times \Phi_3.$$

Then f will have a unique fixed point in $a \in X$ such that $\lim_{n\to\infty} f^n x = a$.

Theorem 1.15. *Let (X,d) be a complete metric space and $f\colon X \to X$ such that and*

$$for\ each\ x, y \in X, \int_0^{d(fx,fy)} \varphi(t)dt \leq \int_0^{m(x,y)} \varphi(t)dt - \int_0^{\psi(m(x,y))} \varphi(t)dt.$$

where, $(\varphi,\psi) \in \Phi_1 \times \Phi_3$ and $m(x,y) = max\{d(x,y), d(fx,x), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}$. Then f will have a unique fixed point in $a \in X$ such that $\lim_{n\to\infty} f^n x = a$.

In 2015, Sarwar et.al., [28] obtained the unique common fixed point for four mappings in a metric space. That means their result is true for incomplete metric space also. The author made uses of CLR property and weakly compatible maps that enables one to get the result without the given space being complete also one does not require the closedness of the subspace concerned. By considering the concept of commom CLR property for the two pairs of mappings one can understand that the traditional way of finding the fixed point such as constructing a sequence showing it as Cauchy and by the help of completeness of the space choosing its limit which ultimately becomes the fixed point is not required in this case. The authors in [28] proved the following theorem:

Theorem 1.16. *Let (X,d) be a metric space and $f, g, h, J\colon X \to X$ such that (f,h) and (g,J) share (CLR_{hJ}) property and for each $x, y \in X$,*

$$\int_0^{d(fx,gy)} \varphi(t)dt \leq \psi(\int_0^{m_i(x,y)} \varphi(t)dt) \text{ where } (\varphi,\psi) \in \Phi_1 \times \Phi_4.$$

and $i = 1, 2$. Then if the pairs (f,h) and (g,J) are weakly compatible that f, g, h, and J will have a unique common fixed point in X.

Theorem 1.17. [21] *Let (X,d) be a metric space and $f, g, h, J\colon X \to X$ such that (f,h) and (g,J) are compatible and subsequentially continuous(alternately subcompatible and reciprocally continuous), then the pair (f,h) has a coincident point and the pair (g,J) has a coincident point. For all $x, y \in X$*

$$\int_0^{d(fx,gy)} \varphi(t)dt \leq \int_0^{m_i(x,y)} \varphi(t)dt - W(\int_0^{m_i(x,y)} \varphi(t)dt), \text{ where, } (\varphi, W) \in \Phi_1 \times \Phi_6,$$

and $i = 4, 5$. Then f, g, h, and J will have a unique common fixed point in X.

Remark 1.18. From the proof of the above Theorem 1.17 it follows that if (X,d) be a metric space and $f, g, h, J\colon X \to X$ such that (f,h) and (g,J) are compatible and subsequentially continuous(alternately subcompatible and reciprocally continuous) and for all $x, y \in X$,

$$\int_0^{d(fx,gy)} \varphi(t)dt \leq \int_0^{m_i(x,y)} \varphi(t)dt) - W(\int_0^{m_i(x,y)} \varphi(t)dt)$$

where, $(\varphi, W) \in \Phi_1 \times \Phi_6$ *and* $i = 4, 5$. Then the pair (f, h) and (g, J) share common CLR_{hJ} property and the limit serves as the required unique common fixed point of f, g, h, and J. Also the pair (f, h) and the pair (g, J) has a coincident point. Also if (X, d) be a metric space and $f, g, h, J \colon X \to X$ such that (f, h) and (g, J) share common E.A. property and for all $x, y \in X$

$$\int_0^{d(fx,gy)} \varphi(t)dt \le \int_0^{m_2(x,y)} \varphi(t)dt) - W(\int_0^{m_2(x,y)} \varphi(t)dt),$$

where $(\varphi, W) \in \Phi_1 \times \Phi_6$. Then the pair (f, h) and (g, J) share common CLR_{hJ} property and the limit serves as the required unique common fixed point of f, g, h, and J and the pairs (f, h) and (g, J) has a coincident point.

Recently Liu et. al., have obtained the result where unique fixed point have been found for mapping satisfying integral type contractive conditions on a complete metric space. The author in [20] defined
$M_1(x, y) = max\{d(x, y), d(fx, x), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(fx,x)d(y,fy)}{1+d(fx,fy)}, \frac{d(fx,y)d(fy,x)}{1+d(fx,fy)}, \frac{d(x,fx)d(y,fy)}{1+d(y,x)}, \frac{d(x,fy)d(y,fx)}{1+d(x,y)}\}, \forall x, y \in X.$

$M_2(x, y) = max\{d(x, y), d(fx, x), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(fy,x)[1+d(y,fx)]}{2[1+d(x,y)]}, \frac{d(fx,y)[1+d(fy,x)]}{2[1+d(x,y)]}, \frac{d(x,fx)[1+d(y,fy)]}{1+d(y,x)}, \frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}\}, \forall x, y \in X.$

$M_3(x, y) = max\{d(x, y), d(fx, x), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2}\}.$

Using the above definitions the author in [20] proved the following theorem:

Theorem 1.19. *Let* (X, d) *be a complete metric space and* $f \colon X \to X$ *such that for each* $x, y \in X$

$$\phi(\int_0^{d(fx,fy)} \varphi(t)dt) \le \phi(\int_0^{M_i(x,y)} \varphi(t)dt) - \int_0^{\psi(M_i(x,y))} \varphi(t)dt$$

where, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. *Then* f *will have a unique fixed point in* $a \in X$ *such that* $\lim_{n\to\infty} f^n x = a$. *The theorem is true for* $i = 1, 2, 3$.

Definition 1.20. [3] Let $\mathcal{F}$ be the family of lower semi - continuous functions $F(t1, ..., t6) \colon R_+^6 \to R$ satisfying the following conditions:
$(F1) : F(t, 0, t, 0, 0, t) > 0, \ \forall t > 0;$
$(F2) : F(t, 0, 0, t, t, 0) > 0, \ \forall t > 0;$
$(F3) : F(t, t, 0, 0, t, t) > 0, \ \forall t > 0.$

For examples we refer [3] to the readers.

Definition 1.21. Let f and g, be self-maps on set X. If $fx = gx = w$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 1.22. [14] An altering distance is a function $\psi \colon [0, 1) \to [0, 1)$ satisfying:
(1) ψ is continuous and increasing.
(2) $\psi(t) = 0$, if and only if $t = 0$.

Denote $C(f, g) = \{x : fx = gx\}$ is the collection of all coincidence points of selfmaps f and g of a metric space X.

Theorem 1.23. [22] *Let (X, d) be a metric space and $f, g, h,$ and T be self mappings of X satisfying the inequality*

$$F(\psi(d(fx, hy)), \psi(d(gx, Ty)), \psi(d(gx, fx)), \psi(d(Ty, hy)), \psi(d(gx, hy)), \psi(d(Ty, fx))) \leq 0,$$

for all $x, y \in X$, $F \in \mathcal{F}$ and ψ is an altering distance. If $f, g,$ and T satisfy $CLR_{(f,g),T}$ - property, then

(1) $C(f, g) \neq \phi$.

(2) $C(h, T) \neq \phi$.

Moreover, if (f, g) and (h, T) are weakly compatible, then $f, g, h,$ and T have a unique common fixed point.

Lemma 1.24. [2] . *Let f and g be weakly compatible self mappings of a nonempty set X. If f and g have a unique point of coincidence $w = fx = gx$ for some $x \in X$, then w is the unique common fixed point of f and g.*

Lemma 1.25. [18] *Let $\phi \in \Phi_1$ and $\{r_n\}_n \in N$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then we have*

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)dt = \int_0^a \phi(t)dt.$$

2. Main results

Theorem 2.1. *Let (X, d) be a metric space and $f, g, h, J \colon X \to X$ such that (f, h) and (g, J) are compatible and subseqentially continuous and for each $x, y \in X$*

$$\psi(\int_0^{d(fx,gy)} \varphi(t)dt) \leq \psi(\int_0^{m_1(x,y)} \varphi(t)dt) - \int_0^{\phi(m_1(x,y))} \varphi(t)dt$$

where, $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. Then $f, g, h,$ and J will have a unique common fixed point in X.

Proof: Since the pair (f, h) is subseqentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z, \ for\ some\ z \in X$$

and

$$\lim_{n\to\infty} fhx_n = fz, \ \lim_{n\to\infty} hfx_n = hz.$$

Now from compatibility of (f, h), we get

$$\lim_{n\to\infty} d(fhx_n, hfx_n) = 0.$$

$$\Rightarrow d(fz, hz) = 0.$$

Thus $hz = gz$. So $C(f, h) \neq \phi$. Similarly, the pair (g, J) is subseqentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = w.$$

for some $w \in X$ and

$$\lim_{n\to\infty} gJy_n = gw, \ \lim_{n\to\infty} Jgy_n = Jw.$$

Again (g, J) is compatible therefore we have $gw = Jw$.
Let us suppose that $fz \neq gw$. Then on putting $x = z$ and $y = w$ in the theorem , we have

$$\phi(\int_0^{d(fz,gw)} \varphi(t)dt) \leq \phi(\int_0^{m_1(z,w)} \varphi(t)dt) - \int_0^{\psi(m_1(z,w))} \varphi(t)dt.$$

$m_1(z, w) = max\{d(hz, Jw), d(fz, hz), d(gw, Jw), \frac{d(hz,gw)+d(Jw,fz)}{2}, \frac{d(fz,hz)d(gw,Jw)}{1+d(hz,Jw)}, \frac{d(fz,Jw)d(gw,hz)}{1+d(hz,Jw)}, d(fz, hz)\frac{1+d(hz,gw)+d(Jw,fz)}{1+d(fz,hw)+d(Jw,gw)}\}$
$m_1(z, w) = max\{d(fz, gw), 0, 0, d(gw, fz), 0, \frac{d(fz,gw)d(gw,fz)}{1+d(fz,gw)}, 0\}$.
Now, $d(fz, Jw) < d(fz, Jw)+1 \Rightarrow \frac{1}{d(fz,Jw)+1} < \frac{1}{d(fz,Jw)} \Rightarrow \frac{d(fz,Jw)^2}{d(fz,Jw)+1} < \frac{d(fz,Jw)^2}{d(fz,Jw)} = d(fz, Jw)$.

$$m_1(u, v) = d(fz, gw).$$

Since, $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we get,

$$\psi(\int_0^{d(fz,gw)} \varphi(t)dt) \leq \psi(\int_0^{m_1(z,w)} \varphi(t)dt) - \int_0^{\phi(m_1(z,w))} \varphi(t)dt < \psi(\int_0^{d(fz,gw)} \varphi(t)dt).$$

Which is a contradiction. Thus we have $fz = gw$.

$$\Rightarrow hz = fz = gw = Jw = q,$$

for some $q \in X$. If suppose $t \neq z$, and $ft = ht$. Suppose $ft \neq Jw$. Then on putting $x = t$ and $y = w$ in the theorem we get,

$$\psi(\int_0^{d(ft,gw)} \varphi(t)dt) \leq \psi(\int_0^{m_1(t,w)} \varphi(t)dt) - \int_0^{\phi(m_1(t,w))} \varphi(t)dt.$$

$m_1(t, w) = max\{d(ht, Jw), d(ft, ht), d(gw, Jw), \frac{d(ht,gw)+d(Jw,ft)}{2}, \frac{d(ft,ht)d(gw,Jw}{1+d(ht,Jw)}, \frac{d(ft,Jw)d(gw,ht)}{1+d(ht,Jw)}, d(ft, ht)\frac{1+d(ht,gw)+d(Jw,ft)}{1+d(ft,ht)+d(Jw,gw)}\}$.
$m_1(t, w) = max\{d(ft, Jw), 0, 0, d(ft, Jw), 0, \frac{d(ft,Jw)^2}{1+d(ft,Jw)}, 0\}$.
$m_1(t, v) = d(ft, gw)$.

$$\psi(\int_0^{d(ft,gw)} \varphi(t)dt) \leq \psi(\int_0^{d(ft,gw)} \varphi(t)dt) - \int_0^{\phi d(ft,gw)} \varphi(t)dt < \psi(\int_0^{d(ft,gw)} \varphi(t)dt).$$

$\Rightarrow ft = gw$. So, we have $ht = ft = gw = Jw = q$.
Consequently we have $fz = hz = q = ht = ft$.
From this it follows that q is the unique point of coincidence of (f, h). Similarly q is the unique point of coincidence of (g, J). Now that both the pairs (f, h) and (g, J) are compatible implies that they are weakly compatible. On applying Lemma 1.24 q is the unique common fixed point of f, g, h and J.

Theorem 2.2. *Let (X, d) be a metric space and $f, g, h, J : X \to X$ such that (f, h) and (g, J) are compatible and subseqentially continuous and for each $x, y \in X$*

$$\psi(\int_0^{d(fx,gy)} \varphi(t)dt) \leq \psi(\int_0^{m_i(x,y)} \varphi(t)dt) - \int_0^{\phi(m_i(x,y))} \varphi(t)dt$$

where, $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and $i = 2, 3, 4, 5$. Then $f, g, h,$ and J will have a unique common fixed point in X.

Proof: The proof is same as the proof of Theorem 2.1.

Theorem 2.3. *Let (X,d) be a metric space and $f,g,h,J\colon X\to X$ such that (f,h) and (g,J) are compatible and subseqentially continuous and for each $x,y\in X$*

$$\int_0^{d(fx,gy)}\varphi(t)dt \le \int_0^{m_i(x,y)}\varphi(t)dt - \int_0^{m_i(x,y)}\varphi(t)dt$$

where, $\varphi\in\Phi_1$ and $i=2,3,4,5$. Then if the pairs (f,h) and (g,J) are weakly compatible, $f,g,h,$ and J will have a unique common fixed point in X.

Proof: Taking ψ and ϕ as an Identity map in the Theorem 2.2 we get the result.

Theorem 2.4. *Let (X,d) be a metric space and $f,g,h,J\colon X\to X$ such that (f,h) and (g,J) are compatible and subseqentially continuous and for each $x,y\in X$*

$$\phi(\int_0^{d(fx,gy)}\varphi(t)dt) \le \phi(\int_0^{m_i(x,y)}\varphi(t)dt)\ -\psi(\int_0^{m_i(x,y)}\varphi(t))dt.$$

where, $(\varphi,\phi,\psi)\in\Phi_1\times\Phi_2\times\Phi_3$ and $i=1,2,3,4,5$. Then if the pairs (f,h) and (g,J) are weakly compatible, $f,g,h,$ and J will have a unique common fixed point in X.

Proof: The proof is same as the proof of Theorem 2.1.

Theorem 2.5. *Let (X,d) be a metric space and $f,g,h,J\colon X\to X$ such that (f,h) and (g,J) are compatible and subseqentially continuous and for each $x,y\in X$*

$$\int_0^{d(fx,gy)}\varphi(t)dt \le \int_0^{m_i(x,y)}\varphi(t)dt)\ -\psi(\int_0^{(m_i(x,y))}\varphi(t))dt.$$

where, $(\varphi,\psi)\in\Phi_1\times\Phi_3$ and $i=1,2,3,4,5$. Then if the pairs (f,h) and (g,J) are weakly compatible, $f,g,h,$ and J will have a unique common fixed point in X.

Proof: Taking ϕ as an Identity map in the Theorem 2.4 we get the result.

In the above theorem we have considered that $\phi\in\Phi_3$, but we know that lower semicontinuity is weaker than continuity. Hence the result is also true for if $\phi\in\Phi_6$. So, taking $\phi\in\Phi_6$ and

1)$i=5$ in Theorem 2.5 we get the Theorem 3. of [21].

2)$i=4$ in Theorem 2.5 we get the Corollary 1. of [21].

It is clear from [23], that Φ_5 is a subclass of Φ_4. Hence any result concerning the use of control functions, the class of mappings belonging to Φ_4 would serve more general than the class of mappings belonging to Φ_5. We now present here a lemma involving class of function Φ_4 and satisfying integral type condition, whose proof is similar to that of the Lemma 1.11 above:

Lemma 2.6. *Let (X,d) is a metric space where the mappings $f,g,h,J\colon X\to X$, satisfy*

$$\psi(\int_0^{d(fx,gy)}\varphi(t)dt) \le \psi(\int_0^{m_i(x,y)}\varphi(t)dt) - \phi(\int_0^{m_i(x,y)}\varphi(t)dt) \tag{1}$$

for all $x, y \in X$, Where $(\psi, \phi, \varphi) \in \Phi_2 \times \Phi_3 \times \Phi_1$ and $i = 1, 2$. Assume that $I : R^+ \to R^+$ is the identity mapping and

$$\psi_1(t) = (\psi + I)^{-1}(\psi + I - \phi)(t), \forall t \in R^+. \tag{2}$$

Then $\psi_1 \in \Phi_4$ and

$$\int_0^{d(fx,gy)} \varphi(t)dt \leq \psi_1(\int_0^{m_i(x,y)} \varphi(t)dt), \forall x, y \in X. \tag{3}$$

Proof: That ψ_1 is a upper semicontinuous function is true from Lemma 1.11. Now, since $\phi(t) > 0, \forall t > 0$. Therefore,

$$\psi(t) + t + \phi(t) > \psi(t) + t, \forall t > 0.$$
$$\Rightarrow \psi(t) + t > \psi(t) + t - \phi(t), \forall t > 0.$$
$$\Rightarrow (\psi + I)t > (\psi + I - \phi)t, \forall t > 0.$$
$$\Rightarrow t > (\psi + I)^{-1}(\psi + I - \phi)t, \forall t > 0.$$
$$\Rightarrow \psi_1(t) < t, \forall t > 0.$$

Thus $\psi_1 \in \Phi_4$.
Now we will consider two cases
Case 1: If $m_i(x_0, y_0) = 0$, for some $x_0, y_0 \in X$. Then we get,
$d(hx_0, Jy_0) = d(hx_0, fx_0) = d(Jy_0, gy_0)$. Which gives, $fx_0 = hx_0 = gy_0 = Jy_0$. Therefore,

$$\int_0^{d(fx_0,gy_0)} \varphi(t)dt = 0.$$

and since $\psi_1 \in \Phi_4$, we have, $\psi_1(\int_0^{m_i(x_0,y_0)} \varphi(t)dt) = 0 = \int_0^{d(fx_0,gy_0)} \varphi(t)dt$.
Case 2: If $m_i(x, y) > 0$, for all $x, y \in X$. From 2.1 we get,

$$\psi(\int_0^{d(fx,gy)} \varphi(t)dt) \leq \psi(\int_0^{m_i(x,y)} \varphi(t)dt) - \phi(\int_0^{m_i(x,y)} \varphi(t)dt),$$

Since, $\phi \in \Phi_3$, we get,

$$\psi(\int_0^{d(fx,gy)} \varphi(t)dt) < \psi(\int_0^{m_i(x,y)} \varphi(t)dt). \tag{4}$$

Now that ψ is nondecreasing function we get,

$$\Rightarrow \int_0^{d(fx,gy)} \varphi(t)dt < \int_0^{m_i(x,y)} \varphi(t)dt.$$

$$So,\ (\psi + I)\int_0^{d(fx,gy)} \varphi(t)dt = \psi(\int_0^{d(fx,gy)} \varphi(t)dt) + \int_0^{d(fx,gy)} \varphi(t)dt.$$

From equations 1, 2 and 4we get,

$$\begin{aligned}
(\psi + I)\int_0^{d(fx,gy)} \varphi(t)dt &\leq \psi(\int_0^{m_i(x,y)} \varphi(t)dt) - \phi(\int_0^{m_i(x,y)} \varphi(t)dt) + \int_0^{d(fx,gy)} \varphi(t)dt. \\
&< \psi(\int_0^{m_i(x,y)} \varphi(t)dt) - \phi(\int_0^{m_i(x,y)} \varphi(t)dt) + \psi(\int_0^{m_i(x,y)} \varphi(t)dt. \\
&< (\psi - \phi + \psi)\int_0^{m_i(x,y)} \varphi(t)dt.
\end{aligned}$$

$$\Rightarrow \int_0^{d(fx,gy)} \varphi(t)dt < (\psi + I)^{-1}(\psi - \phi + \psi) \int_0^{m_i(x,y)} \varphi(t)dt.$$

$$\Rightarrow \int_0^{d(fx,gy)} \varphi(t)dt < \psi_1(\int_0^{m_i(x,y)} \varphi(t)dt).$$

Theorem 2.7. *Let (X, d) be a metric space and $f, g, h, J \colon X \to X$ such that (f, h) and (g, J) share (CLR_{hJ}) property and for each $x, y \in X$*

$$\psi(\int_0^{d(fx,gy)} \varphi(t)dt) \leq \psi(\int_0^{m_i(x,y)} \varphi(t)dt) - \phi(\int_0^{m_i(x,y)} \varphi(t)dt)$$

where, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_3 \times \Phi_2$ and $i = 1, 2, 3, 4, 5$. Then if the pairs (f, h) and (g, J) are weakly compatible then f, g, h, and J will have a unique common fixed point in X.

Proof: By using the Lemma 2.6 in the Theorem 1.16 we get the desired result.

Theorem 2.8. *Let (X, d) be a metric space and $f, g, h, J \colon X \to X$ such that (f, h) and (g, J) share (CLR_{hJ}) property and for each $x, y \in X$*

$$\int_0^{d(fx,gy)} \varphi(t)dt \leq \int_0^{m_i(x,y)} \varphi(t)dt - \phi(\int_0^{m_i(x,y)} \varphi(t)dt)$$

where, $(\varphi, \phi) \in \Phi_1 \times \Phi_3$ and $i = 1, 2, 3, 4, 5$. Then if the pairs (f, h) and (g, J) are weakly compatible then f, g, h, and J will have a unique common fixed point in X.

Proof: If we take $\psi(t) = t$ in the above Theorem 2.7 then we get the desired result.

Remark 2.9. In the above theorem we have considered that $\phi \in \Phi_3$, but we know that lower semicontinuity is weaker than continuity. Hence the result is also true for if $\phi \in \Phi_6$.

Remark 2.10. Here we make an observation that using Remark 1.18, and Remark 2.9 it follows that Theorem 3. of [21] implies Theorem 2.8 for $i = 5$, in this paper. But the converse is not true as it is clear from the example given below. This proves that Theorem 2.8 for $i = 5$ is in fact an extension of Theorem 3.1 of [21].

Example 2.11. Let $X = [\frac{1}{3}, 1)$ be a metric space with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$ and f, g, h, and $J \colon X \to X$, defined by

$$f(x) = \begin{cases} \frac{1}{3}, & \text{for } x \in [\frac{1}{3}, \frac{2}{3}), \\ \frac{2}{3}, & \text{for } x \in [\frac{2}{3}, 1), \end{cases} \quad g(x) = \begin{cases} \frac{1}{2}, & \text{for } x \in [\frac{1}{3}, \frac{2}{3}), \\ \frac{2}{3}, & \text{for } x \in [\frac{2}{3}, 1) \end{cases}$$

$$h(x) = \begin{cases} \frac{3}{4}, & \text{for } x \in [\frac{1}{3}, \frac{2}{3}), \\ \frac{2}{3}, & \text{for } x \in [\frac{2}{3}, 1) \end{cases} \quad J(x) = \begin{cases} \frac{1}{3}, & \text{for } x \in [\frac{1}{3}, \frac{2}{3}), \\ 1 - \frac{x}{2}, & \text{for } x \in [\frac{2}{3}, 1) \end{cases}$$

Let $\{x_n\} = \{\frac{2}{3} + \frac{1}{n+3}\}$ and $\{y_n\} = \{\frac{2}{3} + \frac{1}{n+4}\}$ be two sequences in X. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f(\{\frac{2}{3} + \frac{1}{n+3}\}) = \frac{2}{3};$$
$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} h(\{\frac{2}{3} + \frac{1}{n+3}\}) = \frac{2}{3};$$
$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} g(\{\frac{2}{3} + \frac{1}{n+4}\}) = \frac{2}{3};$$

$$\lim_{n\to\infty} Jy_n = \lim_{n\to\infty} J(\{\frac{2}{3}+\frac{1}{n+4}\}) = \lim_{n\to\infty}(\{1-\frac{1}{2}(\frac{2}{3}+\frac{1}{n+4})\}) = \lim_{n\to\infty}(\frac{2}{3}-\frac{1}{2(n+4)})\}) = \frac{2}{3}.$$

Thus

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = \frac{2}{3} \in h(X)\cap J(X).$$

That is, the pair (f,h) and (g,J) share (CLR_{hJ}) property.
Also, we have $f(\frac{2}{3}) = h(\frac{2}{3}) = \frac{2}{3}$, and $f(h(\frac{2}{3})) = \frac{2}{3} = h(f(\frac{2}{3}))$.
So (f,h) is weakly compatible. Similarly (g,J) is also weakly compatible mappings.
Now,

$$\lim_{n\to\infty} fhx_n = \lim_{n\to\infty} fh\{\frac{2}{3}+\frac{1}{n+3}\} = \lim_{n\to\infty} f\{\frac{2}{3}\} = \frac{2}{3} = f(\frac{2}{3}).$$

$$\lim_{n\to\infty} hfx_n = \lim_{n\to\infty} hf\{\frac{2}{3}+\frac{1}{n+3}\} = \lim_{n\to\infty} h\{\frac{2}{3}\} = \frac{2}{3} = h(\frac{2}{3}).$$

Therefore, the pair (f,h) is compatible and subsequentially continuous.
Again,

$$\lim_{n\to\infty} gJy_n = \lim_{n\to\infty} gJ\{\frac{2}{3}+\frac{1}{n+4}\} = \lim_{n\to\infty} g\{1-\frac{1}{2}(\frac{2}{3}+\frac{1}{n+4})\} = \lim_{n\to\infty} g\{\frac{2}{3}-\frac{1}{2(n+4)})\} = \frac{1}{2} \neq g(\frac{2}{3}).$$

Hence, the pair (g,J) is not subsequentially continuous.
Let $\{x_n\} = \{\frac{1}{3}+\frac{1}{n+3}\}$ and $\{y_n\} = \{\frac{1}{3}+\frac{1}{n+4}\}$ be two sequences in X. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f(\{\frac{1}{3}+\frac{1}{n+3}\}) = \frac{1}{3};$$

$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} h(\{\frac{1}{3}+\frac{1}{n+3}\}) = \frac{3}{4};$$

Now,

$$\lim_{n\to\infty} fhx_n = \lim_{n\to\infty} fh\{\frac{1}{3}+\frac{1}{n+3}\} = \lim_{n\to\infty} f\{\frac{3}{4}\} = \frac{2}{3} = f(\frac{2}{3}).$$

$$\lim_{n\to\infty} hfx_n = \lim_{n\to\infty} hf\{\frac{1}{3}+\frac{1}{n+3}\} = \lim_{n\to\infty} h\{\frac{1}{3}\} = \frac{3}{4} = h(\frac{1}{3}).$$

Therefore, the pair (f,h) is not subsequentially continuous.
Which implies that Corollary 1. of [21] cannot be applied here. But this example satisfies the conditions of the Theorem 2.8. We will now check the contractive condition as it is taken in Corallary 1 of [21]. and Theorem 2.8 for $i=4$ i.e, for

$$m_4(x,y) = max\{d(hx,Jy), d(fx,hx), d(gy,Jy), \frac{d(hx,gy)+d(Jy,fx)}{2}\}.$$

Let us consider $\varphi(t): R^+ \to R^+$ by $\varphi(t) = 2t$ and $W(t): R^+ \to R^+$ by $W(t) = \frac{t}{2}, \forall t > 0$.
Case 1: When $x,y \in [\frac{1}{3},\frac{2}{3})$. Then $fx = \frac{1}{3}, hx = \frac{3}{4}, gy = \frac{1}{2}, Jy = \frac{1}{3}$ and $d(fx,gy) = \frac{1}{6}$.
Therefore, $\int_0^{d(fx,gy)} \varphi(t)dt = \frac{1}{36}$. Now,

$$\begin{aligned} m_4(x,y) &= max\{d(hx,Jy), d(fx,hx), d(gy,Jy), \frac{d(hx,gy)+d(Jy,fx)}{2}\} \\ &= max\{\frac{5}{12},\frac{5}{12},\frac{1}{6},\frac{1}{8}\} \\ &= \frac{5}{12}. \end{aligned}$$

Also $\int_0^{\frac{3}{4}} \varphi(t)dt - W(\int_0^{\frac{3}{4}} \varphi(t)dt) = \frac{25}{144} - \frac{25}{288} = \frac{25}{288}$.
Thus we have,

$$\int_0^{d(fx,gy)} \varphi(t)dt \leq \int_0^{m_4(x,y)} \varphi(t)dt) - W(\int_0^{m_4(x,y)} \varphi(t)dt).$$

for $x,y \in [\frac{1}{3},\frac{2}{3})$.
Case 2: When $x,y \in [\frac{2}{3},1)$. Then $fx = \frac{2}{3}$, $hx = \frac{2}{3}$, $gy = \frac{2}{3}$, $Jy = 1-\frac{y}{2}$, and $d(fx,gy) = 0$.

Then $\int_0^{d(fx,gy)} \varphi(t)dt = 0$, where $\frac{2}{3} \leq y < 1$.
Now,

$$\begin{aligned} m_4(x,y) &= max\{-\frac{1}{3}+\frac{x}{2}, 0, -\frac{1}{3}+\frac{x}{2}, \frac{1}{2}(\frac{x}{2}-\frac{1}{3})\} \\ &= \frac{x}{2}-\frac{1}{3},\ \frac{2}{3} \leq y < 1. \end{aligned}$$

Thus we have,

$$\int_0^{m_4(x,y)} \varphi(t)dt - W(\int_0^{(m_4(x,y))} \varphi(t)dt) > \int_0^{d(fx,gy)} \varphi(t)dt.$$

Hence, it follows from above cases that $\int_0^{d(fx,gy)} \varphi(t)dt \leq \int_0^{m_4(x,y)} \varphi(t)dt - W(\int_0^{(m_4(x,y))} \varphi(t)dt)$. Thus the above example satisfies all the conditions of Theorem 2.8 and the unique common fixed point of f, g, h, and J is $\frac{2}{3} \in X$.

Theorem 2.12. *Let (f,h) and (g,J) be two pairs of self mappings on a metric space (X,d). Then if f,g,h and J posesses a unique common fixed point in X, then the pair (f,h) and (g,J) satisfies common limit in the range of h and J property.*

Proof: Suppose w is the unique fixed point of f, g, h and J, that is $fw = gw = hw = Jw = w$.
Let $x_n = \{w + \frac{1}{n}\}$, $y_n = \{w - \frac{1}{n}\}$,$\forall n \in N$ be a sequence in X. Then $x_n \to w$ as $n \to \infty$ and since w is a fixed point of f, so f is continuous at w. Therefore, $fx_n \to fw = w$ as $n \to \infty$. Similarly we get, $\lim_{n\to\infty} hx_n = w$, and $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = w$.
Thus we have,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = w \in h(X) \cap J(X).$$

We present an example for the above Theorem 2.12, to show that the existence of unique common fixed point of four mappings in a metric space will guarantee that the four mappings satisfy common limit range property.

Example 2.13. Let $X = [0, \infty)$ be a metric space with the usual metric $d(x,y) = |x-y|$, for all $x, y \in X$ and f, g, h, and $J \colon X \to X$, defined by

$$f(x) = \begin{cases} \frac{x}{6}, & \text{for } x \in [0,2), \\ 2x-2, & \text{for } x \in [2,\infty), \end{cases} \quad h(x) = \begin{cases} \frac{x}{4}, & \text{for } x \in [0,2), \\ 3x-4, & \text{for } x \in [2,\infty) \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{for } x \in [0,2), \\ \frac{x}{2}-1, & \text{for } x \in [2,\infty) \end{cases} \quad J(x) = \begin{cases} x, & \text{for } x \in [0,2), \\ x-2, & \text{for } x \in [2,\infty) \end{cases}$$

Here 0 is the unique common fixed point of f, g, h, and J.
Let $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{\frac{1}{n+1}\}$ be two sequences in X. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} (\frac{1}{6n}) = 0;$$

$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} h(\frac{1}{n}) = \lim_{n\to\infty} \frac{1}{4n} = 0;$$

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} g(\frac{1}{n+1}) = 0;$$

$$\lim_{n\to\infty} Jy_n = \lim_{n\to\infty} J(\frac{1}{n+1}) = \lim_{n\to\infty} (\frac{1}{n+1}) = 0.$$

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = 0 \in h(X) \cap J(X).$$

But the converse of the Theorem 2.12 is not true. That is the condition in Theorem 2.12 is only sufficient, but not necessary. Below we present an example to show that the mappings f, g, h, and J in a metric space satisfy common limit in the range of h and J property but it has no common fixed point in X.

Example 2.14. Let $X = [0, \infty)$ be a metric space with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$ and f, g, h, and $J\colon X \to X$, defined by

$$f(x) = \begin{cases} \frac{x}{3}, & \text{for } x \in [0,2), \\ 2x-2, & \text{for } x \in [2,\infty), \end{cases} \quad h(x) = \begin{cases} \frac{x}{2}, & \text{for } x \in [0,2), \\ 3x-4, & \text{for } x \in [2,\infty) \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{2}, & \text{for } x \in [0,2), \\ \frac{x}{2}-1, & \text{for } x \in [2,\infty) \end{cases} \quad J(x) = \begin{cases} \frac{3}{4}, & \text{for } x \in [0,2), \\ x-2, & \text{for } x \in [2,\infty) \end{cases}$$

Let $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{2 + \frac{1}{n}\}$ be two sequences in X. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} (\frac{1}{3n}) = 0;$$
$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} h(\frac{1}{n}) = \lim_{n\to\infty} \frac{1}{2n} = 0 = h(0);$$
$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} g(2 + \frac{1}{n}) = \lim_{n\to\infty} (\frac{1}{2n}) = 0;$$
$$\lim_{n\to\infty} Jy_n = \lim_{n\to\infty} J(2 + \frac{1}{n}) = \lim_{n\to\infty} (\frac{1}{n}) = 0 = J(2).$$
$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = 0 \in h(X) \cap J(X).$$

But there is no common fixed point of f, g, h, and J in X.

Now we present a necessary condition for the existence of a unique common fixed point of four mappings in a metric space which satisfies common limit range property. First we prove a lemma.

Lemma 2.15. *Let (X, d) be a metric space and $f, g, h, J\colon X \to X$ such that (f, h) and (g, J) share (CLR_{hJ}) property and for each $x, y \in X$*

$$\phi(\int_0^{d(fx,gy)} \varphi(t)dt) \leq \phi(\int_0^{m_1(x,y)} \varphi(t)dt) - \int_0^{\psi(m_1(x,y))} \varphi(t)dt.$$

where, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. Then if the pairs $C(f, h) \neq \phi$ and $C(g, J) \neq \phi$.

Proof: Assume that the pairs (f, h) and (g, J) share (CLR_{hJ}) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = z, \tag{5}$$

for some $z \in h(X) \cap J(X)$.
Since $z \in h(X)$, there exists a point $u \in X$ such that $hu = z$. Thus equation 5 becomes

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = z = hu.$$

Now, we claim that $fu = hu$. If not then $fu \neq hu$.
Then on putting $x = u$ and $y = y_n$ in the condition of the theorem , we have

$$\phi(\int_0^{d(fu,gy_n)} \varphi(t)dt) \quad \leq \quad \phi(\int_0^{m_1(u,y_n)} \varphi(t)dt) - \int_0^{\psi(m_1(u,y_n))} \varphi(t)dt \tag{6}$$

Now, $m_1(u, y_n) = max\{d(hu, Jy_n), d(fu, hu), d(gy_n, Jy_n), \frac{d(hu,gy_n)+d(Jy_n,fu)}{2}, \frac{d(fu,hu)d(gy_n,Jy_n)}{1+d(hu,Jy_n)}, \frac{d(fu,Jy_n)d(gy_n,hu)}{1+d(hu,Jy_n)}, d(fu, hu)\frac{1+d(hu,gy_n)+d(Jy_n,fu)}{1+d(fu,hu)+d(Jy_n,gy_n)}\}$.

Taking the limit $n \to \infty$ we get,
$\lim_{n\to\infty} m_1(u, y_n) = max\{d(z, z), d(fu, z), d(z, z), \frac{d(z,z)+d(z,fu)}{2}, \frac{d(fu,z)d(z,z)}{1+d(z,Jy_n)}, \frac{d(fu,z)d(z,z)}{1+d(z,z)}, d(fu, z)\frac{1+d(z,z)+d(z,fu)}{1+d(fu,z)+d(z,z)}\}$.

$$\lim_{n\to\infty} m_1(u, y_n) = max\{0, d(fu, z), 0, \frac{d(z, fu)}{2}, 0, 0, d(fu, z)\} = d(fu, z). \quad (7)$$

Now, taking the upper limit as $n \to \infty$ in equation 6 and since $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we get,

$$\begin{aligned}
\phi(\int_0^{d(fu,z)} \varphi(t)dt) &= \lim_{n\to\infty} sup\, \phi(\int_0^{d(fu,gy_n)} \varphi(t)dt) \\
&\leq \lim_{n\to\infty} sup\{\phi(\int_0^{m_1(u,y_n)} \varphi(t)dt) - \int_0^{\psi(m_1(u,y_n))} \varphi(t)dt\} \\
&\leq \lim_{n\to\infty} sup\, \phi(\int_0^{m_1(u,y_n)} \varphi(t)dt) - \lim_{n\to\infty} inf \int_0^{\psi(m_1(u,y_n))} \varphi(t)dt. \\
&\leq \lim_{n\to\infty} sup\ \phi(\int_0^{m_1(u,y_n)} \varphi(t)dt) - \lim_{n\to\infty} inf \int_0^{\psi(m_1(u,y_n))} \varphi(t)dt.
\end{aligned}$$

By using Lemma 1.25 we get,

$$\phi(\int_0^{d(fu,z)} \varphi(t)dt) \leq \phi(\int_0^{d(fu,z)} \varphi(t)dt) - \int_0^{\psi d(fu,z)} \varphi(t)dt < \phi(\int_0^{d(fu,z)} \varphi(t)dt).$$

which leads to a contradiction. Hence, $fu = hu$. So

$$fu = hu = z. \quad (8)$$

Similarly if one considers that $z \in J(X)$, then there exists a point $v \in X$ such that $Jv = z$. Thus equation 5 becomes

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = z = Jv.$$

Here, we claim that $d(Jv, gv) = 0$. Suppose if $d(Jv, gv) > 0$.
Then on putting $x = x_n$ and $y = v$ in the theorem one gets,

$$Jv = gv = z. \quad (9)$$

Thus by combining equation 8 and 9 we get, $fu = hu = z = Jv = gv$.
Thus the result follows.

Theorem 2.16. *Let (X, d) be a metric space and $f, g, h, J : X \to X$ such that (f, h) and (g, J) share (CLR_{hJ}) property and for each $x, y \in X$*

$$\phi(\int_0^{d(fx,gy)} \varphi(t)dt) \leq \phi(\int_0^{m_1(x,y)} \varphi(t)dt) - \int_0^{\psi(m_1(x,y))} \varphi(t)dt$$

where, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. Then if the pairs (f, h) and (g, J) are weakly compatible mappings, $f, g, h,$ and J will have a unique common fixed point in X.

Proof: From the proof of the above Lemma 2.15, we get,

$$fu = hu = z = Jv = gv.$$

If suppose that $ft = ht$ for some $t \neq u$. Then on putting $x = t$ and $y = y_n$ in the theorem we get, $ft = z$. So we have $ht = ft = z = hu = fu$, which implies that z is the unique point of coincidence of (f, h), since (f, h) is weakly compatible it follows from Lemma 1.24 that z is the unique common fixed point of f and h. Similarly one can easily prove that z is the unique common fixed point of g and J. Thus z is the unique common fixed point of f, g, h and J.

Example 2.17. Let (X, d) be a metric space with the usual metric $d(x, y) = |x-y|$, for all $x, y \in X$ and $f, g, h, J \colon X \to X$, defined by

$$f(x) = \begin{cases} 1, & \text{for } x \in (0,1], \\ \frac{1}{6}, & \text{for } x \in (1,2), \end{cases} \quad g(x) = \begin{cases} 1, & \text{for } x \in (0,1], \\ \frac{1}{8}, & \text{for } x \in (1,2), \end{cases}$$

$$h(x) = \begin{cases} 1, & \text{for } x \in (0,1], \\ \frac{1}{2}, & \text{for } x \in (1,2), \end{cases} \quad J(x) = \begin{cases} 1, & \text{for } x \in (0,1], \\ \frac{1}{3}, & \text{for } x \in (1,2), \end{cases}$$

Let $\{x_n\} = \frac{1}{n}$ and $\{y_n\} = \frac{1}{n+2}$ be two sequences in X. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f(\frac{1}{n}) = 1;$$

$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} h(\frac{1}{n}) = 1;$$

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} g(\frac{1}{n+1}) = 1;$$

$$\lim_{n\to\infty} Jy_n = \lim_{n\to\infty} J(\frac{1}{n+1}) = 1.$$

Thus

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = 1 \in h(x) \cap J(X).$$

That is, the pair (f, h) and (g, J) share (CLR_{hJ}) property.
Also, we have $fx = hx, \forall x \in (0, 1]$ and $f(h(x)) = 1 = h(f(x))$. So (f, h) is weakly compatible. Similarly (g, J) is also weakly compatilbe mappings.
Let us consider

$$\varphi(t) : R^+ \to R^+ \; by \; \varphi(t) = 2t,$$
$$\phi(t) : R^+ \to R^+ \; by \; \phi(t) = t,$$

and

$$\psi(t) : R^+ \to R^+ \; by \; \psi(t) \; as \; ceiling \; function.$$

Case 1: When $x, y \in (0, 1]$.
Then $fx = hx = gy = Jy = 1$ and $m_1(x, y) = 0$. Therefore, $\phi(\int_0^{d(fx,gy)} \varphi(t)dt) = 0$.
Also $\phi(\int_0^{m_1(x,y)} \varphi(t)dt) - \int_0^{\psi(m_1(x,y))} \varphi(t)dt = 0$
Case 2: When $x, y \in (1, 2])$. Then $fx = \frac{1}{6}, hx = \frac{1}{3}, gy = \frac{1}{8}, Jy = \frac{1}{2}$, and

$$m_1(x, y) = max\{\frac{1}{6}, \frac{1}{6}, \frac{3}{8}, \frac{13}{48}, \frac{3}{50}, \frac{1}{15}, \frac{1}{6}\} = \frac{3}{8}.$$

Thus we have,
$\phi(\int_0^{d(fx,gy)} \varphi(t)dt) = \phi(\frac{1}{576}) = \frac{1}{576}$, and
$\phi(\int_0^{m_1(x,y)} \varphi(t)dt) - \int_0^{\psi(m_1(x,y))} \varphi(t)dt = \frac{9}{64} > \phi(\int_0^{d(fx,gy)} \varphi(t)dt)$.

Hence, it follows from above cases that
$\phi(\int_0^{d(fx,gy)}\varphi(t)dt) \leq \phi(\int_0^{m_1(x,y)}\varphi(t)dt) - \int_0^{\psi(m_1(x,y))}\varphi(t)dt$.
Thus the above example satisfies all the conditions of Theorem 2.16 and The unique common fixed point of f, g, h, and J is $1 \in X$.

We now make one observation here that the Theorem 2.5 and Theorem 2.16 differ only by the contraction conditions of integral type. We present an example to conclude that these two theorems are independent of each other.

Example 2.18. Let us take $\varphi\colon R^+ \to R^+$ by $\varphi(x) = 2x$, $\psi\colon R^+ \to R^+$ by $\psi(x) = x$ and

$$\phi(x) = \begin{cases} \frac{3x}{4}, & \text{for } x \in (0, \frac{3}{4}), \\ \frac{x}{4}, & \text{for } x \in [\frac{3}{4}, \infty), \end{cases}$$

Clearly, $(\phi, \psi, \varphi) \in \Phi_3 \times \Phi_2 \times \Phi_1$.

Suppose $a > 0$, then $\phi(\int_0^a \varphi(t)dt) = \phi(a^2) = \begin{cases} \frac{3a^2}{4}, & \text{for } a \in (0, \frac{3}{4}), \\ \frac{a^2}{4}, & \text{for } a \in [\frac{3}{4}, \infty), \end{cases}$

and $\int_0^{\phi(a)}\varphi(t)dt = (\phi(a))^2 = \begin{cases} (\frac{3a}{4})^2, & \text{for } a \in (0, \frac{3}{4}), \\ (\frac{a}{4})^2, & \text{for } a \in [\frac{3}{4}, \infty), \end{cases}$

Which implies that $\phi(\int_0^a \varphi(t)dt) > \int_0^{\phi(a)}\varphi(t)dt$.

Now we take another example:

Example 2.19. Let us take $\varphi\colon R^+ \to R^+$ by $\varphi(x) = 2x$, $\psi\colon R^+ \to R^+$ by $\psi(x) = x$ and

$$\phi(x) = \begin{cases} 2x, & \text{for } x \in (0, 1), \\ x + \frac{1}{2}, & \text{for } x \in [1, \infty), \end{cases}$$

Clearly, $(\phi, \psi, \varphi) \in \Phi_3 \times \Phi_2 \times \Phi_1$.

Suppose $a > 0$, then $\phi(\int_0^a \varphi(t)dt) = \phi(a^2) = \begin{cases} 2a^2, & \text{for } a \in (0, 1), \\ a^2 + \frac{1}{2}, & \text{for } a \in [1, \infty), \end{cases}$

and $\int_0^{\phi(a)}\varphi(t)dt = (\phi(a))^2 = \begin{cases} (2a)^2, & \text{for } a \in (0, 1), \\ (a + \frac{1}{2})^2, & \text{for } a \in [1, \infty), \end{cases}$

Which implies that $\phi(\int_0^a \varphi(t)dt) < \int_0^{\phi(a)}\varphi(t)dt$.

Corollary 2.20. *Let (X, d) be a metric space and $f, h, J\colon X \to X$ such that (f, h) and (f, J) share (CLR_{hJ}) property and for each $x, y \in X$*

$$\phi(\int_0^{d(fx,fy)}\varphi(t)dt) \leq \phi(\int_0^{p(x,y)}\varphi(t)dt) - \int_0^{\psi(p(x,y))}\varphi(t)dt$$

*where, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and $p(x, y) = max\{d(hx, Jy), d(fx, hx), d(fy, Jy), \frac{d(hx,fy)+d(Jy,fx)}{2}, \frac{d(fx,hx)d(fy,Jy)}{1+d(hx,Jy)}, \frac{d(fx,Jy)d(fy,hx)}{1+d(hx,Jy)}, d(fx, hx)\frac{1+d(hx,fy)+d(Jy,fx)}{1+d(fx,hx)+d(Jy,fy)}\}$.
Then if the pairs (f, h) and (f, J) are weakly compatible that f, h, J will have a unique common fixed point in X.*

Corollary 2.21. *Let (X, d) be a metric space and $f, J\colon X \to X$ such that (f, J) share (CLR_J) property and for each $x, y \in X$*

$$\phi(\int_0^{d(fx,fy)}\varphi(t)dt) \leq \phi(\int_0^{q(x,y)}\varphi(t)dt) - \int_0^{\psi(q(x,y))}\varphi(t)dt)$$

where $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ *and*
$q(x, y) = max\{d(Jx, Jy), d(fx, Jx), d(fy, Jy), \frac{d(Jx,fy)+d(Jy,fx)}{2}, \frac{d(fx,Jx)d(fy,Jy)}{1+d(Jx,Jy)}, \frac{d(fx,Jy)d(fy,Jx)}{1+d(Jx,Jy)},$
$d(fx, Jx)\frac{1+d(Jx,fy)+d(Jy,fx)}{1+d(fx,Jx)+d(Jy,fy)}\}.$
Then if the pair (f, J) *are weakly compatible that* f *and* J *will have a unique common fixed point in* X.

Theorem 2.22. *Let* (X, d) *be a metric space and* $f, g, h, J : X \to X$ *such that* (f, h) *and* (g, J) *share* (CLR_{hJ}) *property and for each* $x, y \in X$

$$\phi(\int_0^{d(fx,gy)} \varphi(t)dt) \le \phi(\int_0^{m_2(x,y)} \varphi(t)dt) - \int_0^{\psi(m_2(x,y))} \varphi(t)dt$$

where, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. *Then if the pairs* (f, h) *and* (g, J) *are weakly compatible that* f, g, h, J *will have a unique common fixed point in* X.

Proof: Proof is same as Theorem 2.16.

Theorem 2.23. *Let* (X, d) *be a metric space and* $f, g, h, J : X \to X$ *such that* (f, h) *and* (g, J) *share* (CLR_{hJ}) *property and for each* $x, y \in X$

$$\phi(\int_0^{d(fx,gy)} \varphi(t)dt) \le \phi(\int_0^{m_3(x,y)} \varphi(t)dt) - \int_0^{\psi(m_3(x,y))} \varphi(t)dt)$$

where, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. *Then if the pairs* (f, h) *and* (g, J) *are weakly compatible then* f, g, h, J *will have a unique common fixed point in* X.

Proof: Proof is same as Theorem 2.16.

Theorem 2.24. *Let* (X, d) *be a metric space and* $f, g, h, J : X \to X$ *such that* (f, h) *and* (g, J) *share* (CLR_{hJ}) *property and for each* $x, y \in X$

$$\phi(\int_0^{d(fx,gy)} \varphi(t)dt) \le \phi(\int_0^{m_4(x,y)} \varphi(t)dt) - \int_0^{\psi(m_4(x,y))} \varphi(t)dt$$

where, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. *Then if the pairs* (f, h) *and* (g, J) *are weakly compatible then* f, g, h, J *will have a unique common fixed point in* X.

Proof: Proof is same as Theorem 2.16.

Theorem 2.25. *Let* (X, d) *be a metric space and* $f, g, h, J : X \to X$ *such that* (f, h) *and* (g, J) *share* (CLR_{hJ}) *property and for each* $x, y \in X$

$$\phi(\int_0^{d(fx,gy)} \varphi(t)dt) \le \phi(\int_0^{M_3(x,y)} \varphi(t)dt) - \int_0^{\psi(M_3(x,y))} \varphi(t)dt$$

where, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. *Then if the pairs* (f, h) *and* (g, J) *are weakly compatible then* f, g, h, J *will have a unique common fixed point in* X.

Proof: Proof is same as Theorem 2.16.

The following theorem is Theorem 2.2 in [19].

Theorem 2.26. *Let* (X, d) *be a metric space and* $f, g, h, J : X \to X$ *such that* (f, h) *and* (g, J) *share* (CLR_{hJ}) *property and for each* $x, y \in X$

$$\int_0^{d(fx,gy)} \varphi(t)dt \le \int_0^{M_3(x,y)} \varphi(t)dt - \int_0^{\psi(M_3(x,y))} \varphi(t)dt$$

where, $(\varphi, \psi) \in \Phi_1 \times \Phi_3$. Then if the pairs (f, h) and (g, J) are weakly compatible then f, g, h, J will have a unique common fixed point in X.

Proof: Consider $\phi(t) = t$ in the Theorem 2.25 we get the result.

If we consider $M_3(x, y) = d(x, y)$ in the above theorem then we get the Theorem 2.1 from [19].

Acknowledgement. The authors of this article are thankful to the referee for the comments made for this article.

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