UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING VALUES WITH THEIR $n$-TH DERIVATIVES

DILIP CHANDRA PRAMANIK AND JAYANTA ROY

Abstract. In this paper, we prove some results on the uniqueness of meromorphic functions which share some values with their $n$-th derivatives. Our results improve and generalize the results due to Gopalakrishna and Bhosmurath; Yang; Chen, Chen and Tsai; Lahiri and Pal; R. S. Dyavanal.

1. Introduction and main results

In the paper, by meromorphic functions we always mean meromorphic functions in the open complex plane $\mathbb{C}$. Let $f$ be a non-constant meromorphic function. By $S(r, f)$ we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ is said to be a small function of $f$ if either $a \equiv \infty$ or $T(r, a) = S(r, f)$. We denote by $S(f)$ the collection of all small functions of $f$. Clearly $\mathbb{C} \cup \{\infty\} \in S(f)$ and $S(f)$ is a field over the set of complex numbers.

For a positive integer $p$ and $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_p(a; f)$ the set of those zeros of $f - a$ whose multiplicities do not exceed $p$, each zero is counted according to its multiplicities and $E_p(a; f)$ the set of those distinct zeros of $f - a$ whose multiplicities do not exceed $p$, where we mean by a zero of $f - \infty$ a pole of $f$. Also by $E_{\infty}(a; f)(E_{\infty}(a; f))$ we denote the set of all zeros of $f - a$ counted with multiplicities (ignoring multiplicities). If $E_{\infty}(a; f) = E_{\infty}(a; g)$ $(E_{\infty}(a; f) = E_{\infty}(a; g))$, we say that $f$ and $g$ share a CM(IM). Also we say that a meromorphic function $f(z)$ partially shares $a$ with a meromorphic function $g(z)$ if $E_{\infty}(a; f) \subseteq E_{\infty}(a; g)$.

For $A \subset \mathbb{C}$ we denote by $N_A(r, a; f)$ the reduced counting function of those zeros of $f - a$ which belong to the set $A$, where $a \in \mathbb{C} \cup \{\infty\}$. Clearly if $A = \mathbb{C}$, then $N_A(r, a; f) = N(r, a; f)$.

For a positive integer $p$ and $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N_p(r, a; f)$ the counting function (reduced counting function) of those zeros of $f - a$ whose multiplicities do not exceed $p$. Similarly we define $N_p(r, a; f)$ the counting
function (reduced counting function) of those zeros of \( f - a \) whose multiplicities greater than equal to \( p \). Also we write \( N_\infty (r, a; f) \).

For standard definitions and notations of Nevanlinna theory we refer the reader to [4, 6]. The modern theory of uniqueness of entire and meromorphic functions was initiated by R. Nevanlinna with his two famous theorems: The Five Value Theorem and The Four Value Theorem. The five value theorem of Nevanlinna may be stated as follows:

**Theorem 1**[4, p. 48] Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions and \( a_j \in \mathbb{C} \cup \{ \infty \} \) be distinct for \( j = 1, 2, ..., 5 \). If \( E_\infty (a_j; f) = E_\infty (a_j; g) \) for \( j = 1, 2, ..., 5 \), then \( f(z) \equiv g(z) \).

Gopalakrishna and S. S. Bhoosnurmath [3] improved the above theorem in the following manner.

**Theorem 2**[3] Let \( f, g \) be distinct non-constant meromorphic functions. If there exist distinct elements \( a_1, a_2, ..., a_k \in \mathbb{C} \cup \{ \infty \} \) such that \( E_{p_j} (a_j; f) = E_{p_j} (a_j; g) \) for \( j = 1, 2, ..., k \), where \( p_1, p_2, ..., p_k \) are positive integers or \( \infty \) with \( p_1 \geq p_2 \geq ... \geq p_k \), then

\[
\sum_{j=2}^{k} \frac{p_j}{1 + p_j} \leq 2 + \frac{p_1}{1 + p_1}.
\]

C. C. Yang [6, Theorem 3.2, p. 157] improved Theorem 1 by considering partial sharing of values and proved the following theorem.

**Theorem 3**[6] Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions such that \( E_\infty (a_j; f) \subseteq E_\infty (a_j; g) \) for five distinct elements \( a_1, a_2, ..., a_5 \) of \( \mathbb{C} \cup \{ \infty \} \).

If

\[
\liminf_{r \to \infty} \frac{1}{5} \sum_{j=1}^{5} N(r, a_j; f) < \frac{1}{2},
\]

then \( f(z) \equiv g(z) \).

In 2007 Chen, Chen and Tsai [1] extended Theorem 3 by considering \( f(z) \) and \( g(z) \) partially sharing more than five values proved the following theorem.

**Theorem 4**[1] Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions such that \( E_\infty (a_j; f) \subseteq E_\infty (a_j; g) \) for \( k \geq 5 \) distinct elements \( a_1, a_2, ..., a_k \) of \( \mathbb{C} \cup \{ \infty \} \).

If

\[
\liminf_{r \to \infty} \frac{1}{k} \sum_{j=1}^{k} N(r, a_j; f) > \frac{1}{k - 3},
\]

then \( f(z) \equiv g(z) \).

In 2012 R. S. Dyavanal [2] improved Theorem 3 and Theorem 4 by considering uniqueness of \( n \)-th derivatives of meromorphic functions and proved the following theorem.

**Theorem 5**[2] Let \( f \) and \( g \) be two non-constant meromorphic functions and \( a_j \in \mathbb{C} \cup \{ \infty \} \) be distinct for \( j = 1, 2, ..., k \geq 5 \) and for a non-negative integer
n, if \( E_\infty(a_j, f^{(n)}) \subseteq E_\infty(a_j, g^{(n)}) \) for \( 1 \leq j \leq k \), \( E_\infty(0, f) \subseteq E_\infty(0, f^{(n)}) \), \( E_\infty(0, g) \subseteq E_\infty(0, g^{(n)}) \) and

\[
\liminf_{r \to \infty} \frac{\sum_{j=1}^{k} N(r, a_j; f^{(n)})}{\sum_{j=1}^{k} N(r, a_j; g^{(n)})} > \frac{n + 1}{k - (n + 3)},
\]

then \( f^{(n)} \equiv g^{(n)} \).

I. Lahiri and R. Pal\[5\] prove the following uniqueness theorem of meromorphic functions sharing \( k (\geq 5) \) small functions.

**Theorem 6**\[5\] Let \( f \) and \( g \) be two non-constant meromorphic functions and \( a_j = a_j(z) \in S(f) \cap S(g) \) be distinct for \( j = 1, 2, \ldots, k (k \geq 5) \). Suppose that \( p_1 \geq p_2 \geq \ldots \geq p_k \) are positive integers or infinity and \( \delta (\geq 0) \) is such that

\[
\frac{1}{p_1} + \frac{1}{p_1} \sum_{j=2}^{k} \frac{1}{1 + p_j} + 1 + \delta < (k - 2)(1 + \frac{1}{p_1}).
\]

Let \( A_j = E_{p_j}(a_j; f) \setminus E_{p_j}(a_j; g) \) for \( j = 1, 2, \ldots, k \). If \( \sum_{j=1}^{k} N_{A_j}(r, a_j; f) \leq \delta T(r, f) \) and

\[
\liminf_{r \to \infty} \frac{\sum_{j=1}^{k} N_{p_j}(r, a_j; f)}{\sum_{j=1}^{k} N_{p_j}(r, a_j; g)} > \frac{p_1}{(k - 2)(1 + p_1) - (1 + p_1) \sum_{j=2}^{k} \frac{1}{1 + p_j} - 1 - (1 + \delta)p_1},
\]

then \( f \equiv g \).

In the paper we prove the following theorems:

**Theorem 7** Let \( f \) and \( g \) be two non-constant meromorphic functions and \( a_j (j = 1, 2, \ldots, k) \) be \( k (\geq 5) \) distinct complex numbers. For a non-negative integer \( n \), let \( A_j = E(a_j; f^{(n)}) \setminus E(a_j; g^{(n)}) \) and \( \sum_{j=1}^{k} N_{A_j}(r, a_j; f^{(n)}) \leq \delta T(r, f^{(n)}) \), for some \( \delta \) such that \( 0 \leq \delta \leq \frac{k n}{k n + k - 1} \). If

\[
\liminf_{r \to \infty} \frac{\sum_{j=1}^{k} N(r, a_j; f^{(n)})}{\sum_{j=1}^{k} N(r, a_j; g^{(n)})} > \frac{1}{k - 3 + \frac{k n}{k n + k - 1} - \delta},
\]

then \( f^{(n)}(z) \equiv g^{(n)}(z) \).

**Theorem 8** Let \( f_1 \) and \( f_2 \) be two non-constant meromorphic functions and \( a_j = a_j(z) \in S(f) \cap S(g) \) be distinct for \( j = 1, 2, \ldots, k (k \geq 5) \). Suppose that \( m (1 \leq m \leq k) \) is an integer; \( p_1 \geq p_2 \geq \ldots \geq p_k \) are positive integers or infinity and \( \delta (\geq 0) \) is such that

\[
(1 + \frac{1}{p_m}) \sum_{j=m}^{k} \frac{1}{1 + p_j} + 2 + \delta < (k - m - 1)(1 + \frac{1}{p_m}) + m.
\]

Let \( A_j = E_{p_j}(a_j; f_1) \setminus E_{p_j}(a_j; f_2) \) for \( j = 1, 2, \ldots, k \). If \( \sum_{j=1}^{k} N_{A_j}(r, a_j; f_1) \leq \delta T(r, f_1) \) and

\[
\liminf_{r \to \infty} \frac{\sum_{j=1}^{k} N_{p_j}(r, a_j; f_1)}{\sum_{j=1}^{k} N_{p_j}(r, a_j; f_2)} > \frac{p_m}{(1 + p_m) \sum_{j=m}^{k} \frac{p_j}{1 + p_j} + (m - 2 - \delta)p_m - 2(1 + p_m)},
\]

then \( f_1 \equiv f_2 \).
Theorem 9 Let \( f_1, f_2 \) be two non-constant meromorphic functions and \( a_j \in \mathbb{C} \cup \{\infty\} \) be distinct for \( j = 1, 2, \ldots, k \) \((k \geq 5)\). Suppose that \( p_1 \geq p_2 \geq \ldots \geq p_k \) are positive integers or infinity and \( \delta \geq 0 \) is such that
\[
\frac{1}{p_1} + (1 + \frac{1}{p_1}) \sum_{j=2}^{k} \frac{1}{1 + p_j} + 1 + \delta < \frac{k - 2}{n + 1} (1 + \frac{1}{p_1})
\]
for a non-negative integer \( n \). Let \( A_j = \mathcal{E}_{p_j}(a_j; f_1^{(n)}) \setminus \mathcal{E}_{p_j}(a_j; f_2^{(n)}) \) for \( j = 1, 2, \ldots, k \) and \( E(0; f_i) \subset E(0; f_i^{(n)}) \) for \( i = 1, 2 \). If \( \sum_{j=1}^{k} N_{A_j}(r, a_j; f_1^{(n)}) \leq \delta T(r, f_1^{(n)}) \) and
\[
\liminf_{r \to \infty} \frac{\sum_{j=1}^{k} \mathcal{N}_{p_j}(r, a_j; f_1^{(n)})}{\sum_{j=1}^{k} \mathcal{N}_{p_j}(r, a_j; f_2^{(n)})} > \frac{(n+1)p_1}{(k-2)(1+p_1)-(n+1)(1+p_1) \sum_{j=2}^{k} \frac{1}{1+p_j} - (n+1)(1+\delta)p_1 + 1},
\]
then \( f_1^{(n)}(z) \equiv f_2^{(n)}(z) \).

2. Lemma

In this section we prove some lemmas which is needed in the sequel.

Lemma 1[7] Let \( f \) be a non-constant meromorphic function and \( a_j \in S(f) \) be distinct for \( j = 1, 2, \ldots, k \). Then for any \( \epsilon > 0 \)
\[
(k - 2 - \epsilon)T(r, f) \leq \sum_{j=1}^{k} \mathcal{N}(r, a_j; f) + S(r, f).
\]

Lemma 2 ([6], Theorem 1.35, p-49) Let \( f(z) \) be a transcendental meromorphic function in the complex plane and \( a_1, a_2, \ldots, a_k \) be \( k \geq 2 \) distinct finite complex numbers. Then for any positive integer \( n \), we have
\[
\left( k - 1 - \frac{k - 1}{kn + k - 1} \right) T(r, f^{(n)}) < \sum_{j=1}^{k} N \left( r, a_j; f^{(n)} \right) + \epsilon T(r, f^{(n)}) + S(r, f^{(n)}),
\]
where \( \epsilon \) is any positive number.

Lemma 3 Let \( f \) be a non-constant meromorphic function and \( a_1, a_2, \ldots, a_k \) be \( k \geq 3 \) distinct complex numbers. If for a non-negative integer \( n \), \( E(0; f) \subset E(0; f^{(n)}) \), then \( (k - 2 + o(1))T(r, f) \leq \sum_{j=1}^{k} \mathcal{N}(r, a_j; f^{(n)}) \).

Proof. By the Nevanlinna’s first fundamental theorem, we have
\[
T(r, f) = T(r, \frac{1}{f}) + O(1)
\]
\[
\leq N(r, 0; f) + m(r, \frac{f}{f}) + m(r, \frac{1}{f}) + O(1)
\]
\[
\leq N(r, 0; f) + T(r, f^{(n)}) - N(r, 0; f^{(n)}) + S(r, f)
\]
(1)

By the Nevanlinna’s second fundamental theorem, we get
\[
(k - 1)T(r, f^{(n)}) \leq \mathcal{N}(r, \infty; f^{(n)}) + \sum_{j=1}^{k-1} \mathcal{N}(r, a_j; f^{(n)}) + \mathcal{N}(r, 0; f^{(n)}) + S(r, f).
\]
Without loss of generality, we may assume that \(a_k = 0\). Otherwise a suitable linear transformation is done. Then the above inequality reduces to

\[
(k - 1)T(r, f^n) \leq N(r, \infty; f^{(n)}) + \sum_{j=1}^k N(r, a_j; f^{(n)}) + S(r, f) \tag{2}
\]

Using (2) in (1), we obtain

\[
(k - 1)T(r, f) \leq (k - 1)N(r, 0; f) + N(r, \infty; f^n) + \sum_{j=1}^k N(r, a_j; f^{(n)}) - (k - 1)N(r, 0; f^{(n)}) + S(r, f)
\]

\[
\Rightarrow (k - 1)T(r, f) \leq (k - 1)N(r, 0; f) + N(r, \infty; f^n) + \sum_{j=1}^k N(r, a_j; f^{(n)}) - (k - 1)N(r, 0; f^{(n)}) + S(r, f). \tag{3}
\]

Since \(E(0; f) \subseteq E(0; f^{(n)})\), we have from (3)

\[
(k - 1)T(r, f) \leq N(r, \infty; f^n) + \sum_{j=1}^k N(r, a_j; f^{(n)}) + S(r, f)
\]

\[
\Rightarrow (k - 2 + o(1))T(r, f) \leq \sum_{j=1}^k N(r, a_j; f^{(n)}).
\]

This complete the proof of the lemma. \(\square\)

3. Proof of Main Theorems

Proof of Theorem 7:

**Proof.** Let us assume that \(f^{(n)}(z) \neq g^{(n)}(z)\). By Lemma 2, we have

\[
(k - 1 - \frac{k - 1}{kn + k - 1} - \epsilon)T(r, f^{(n)}) < \sum_{j=1}^k N(r, a_j; f^{(n)}) + S(r, f^{(n)}),
\]

\[
\Rightarrow \left( k - 1 - \frac{k - 1}{kn + k - 1} - \epsilon + o(1) \right) T(r, f^{(n)}) < \sum_{j=1}^k N(r, a_j; f^{(n)}) \tag{4}
\]

Similarly,

\[
\left( k - 1 - \frac{k - 1}{kn + k - 1} - \epsilon + o(1) \right) T(r, g^{(n)}) < \sum_{j=1}^k N(r, a_j; g^{(n)}) \tag{5}
\]

Now, let \(B_j = E(a_j; f^{(n)}) \setminus A_j\), for \(j = 1, 2, ..., k\). Then,

\[
\sum_{j=1}^k N(r, a_j; f^n) = \sum_{j=1}^k N_{A_j}(r, a_j; f^{(n)}) + \sum_{j=1}^k N_{B_j}(r, a_j; f^{(n)})
\]

\[
\leq \delta T(r, f^{(n)}) + N(r, 0; f^{(n)} - g^{(n)}),
\]

\[
\leq (1 + \delta)T(r, f^{(n)}) + T(r, g^{(n)}).
\]
Using (4) and (5) we have,
\[ (k - 1 - \frac{k - 1}{kn + k - 1} - \epsilon + o(1)) \sum_{j=1}^{k} N(r, a_j; f^{(n)}) \leq (1 + \delta) \sum_{j=1}^{k} N(r, a_j; f^{(n)}) + \sum_{j=1}^{k} N(r, a_j; g^{(n)}). \]

Therefore,
\[ \{k - 1 - \frac{k - 1}{kn + k - 1} - \epsilon - (1 + \delta) + o(1)\} \sum_{j=1}^{k} N(r, a_j; f^{(n)}) \leq \sum_{j=1}^{k} N(r, a_j; g^{(n)}) \]
\[ \Rightarrow \sum_{j=1}^{k} N(r, a_j; f^{(n)}) \leq \sum_{j=1}^{k} N(r, a_j; g^{(n)}) \]
\[ \Rightarrow \frac{\sum_{j=1}^{k} N(r, a_j; f^{(n)})}{\sum_{j=1}^{k} N(r, a_j; g^{(n)})} \leq \frac{1}{k - 1 - \frac{k - 1}{kn + k - 1} - (1 + \delta)}. \]

Since \( \epsilon \) is arbitrary, taking limit as \( r \to \infty \), we have
\[ \liminf_{r \to \infty} \frac{\sum_{j=1}^{k} N(r, a_j; f^{(n)})}{\sum_{j=1}^{k} N(r, a_j; g^{(n)})} \leq \frac{1}{k - 3 + \frac{kn}{kn + k - 1} - \delta}. \]

which is a contradiction.
Hence \( f^{(n)}(z) \equiv g^{(n)}(z) \).

\[ \square \]

**Corollary 1** In Theorem 7 if \( E_{\infty}(a_j; f^{(n)}) \subseteq E_{\infty}(a_j; g^{(n)}) \), for \( j = 1, 2, \ldots, k \), then \( A_j = \phi \). So we can choose \( \delta = 0 \). Then
\[ \liminf_{r \to \infty} \frac{\sum_{j=1}^{k} N(r, a_j; f^{(n)})}{\sum_{j=1}^{k} N(r, a_j; g^{(n)})} \geq \frac{1}{k - 3 + \frac{kn}{kn + k - 1}}. \]

Since \( \frac{1}{k - 3 + \frac{kn}{kn + k - 1}} \leq \frac{1}{k - (n + 1)} \), therefore Theorem 7 is an improvement of Theorem 5.

Proof of Theorem 8.

**Proof.** Suppose \( f_1 \not\equiv f_2 \). Then by Lemma 1 we have
\[ (k - 2 - \epsilon) T(r, f_1) \leq \sum_{j=1}^{k} \overline{N}(r, a_j; f_1) + S(r, f_1) \]
\[ \leq \sum_{j=1}^{k} \{ \overline{N}(p_j) r, a_j; f_1 \} + \overline{N}(p_{j+1})(r, a_j; f_1) \} + S(r, f_1) \]
\[ \leq \sum_{j=1}^{k} \{ p_j \overline{N}(p_j)(r, a_j; f_1) + \frac{1}{1 + p_j} \overline{N}(p_{j+1})(r, a_j; f_1) \} + S(r, f_1) \]
\[ \leq \sum_{j=1}^{k} \frac{p_j}{1 + p_j} \overline{N}(p_j)(r, a_j; f_1) + \sum_{j=1}^{k} \frac{1}{1 + p_j} T(r, f_1) + S(r, f_1)(6) \]
Since $1 \geq \frac{p_1}{1 + p_1} \geq \frac{p_2}{1 + p_2} \geq ... \geq \frac{p_k}{1 + p_k} \geq \frac{1}{2}$, we get from (6)

\[
(k - 2 - \epsilon)T(r, f_1) \leq \sum_{j=1}^{m-1} \left\{ \frac{p_j}{1 + p_j} - \frac{p_m}{1 + p_m} \right\} N_{p_j}(r, a_j; f_1) + \sum_{j=1}^{k} \frac{1}{1 + p_j} T(r, f_1)
\]

\[
+ \sum_{j=1}^{k} \frac{p_m}{1 + p_m} N_{p_j}(r, a_j; f_1) + S(r, f_1)
\]

\[
\leq \sum_{j=1}^{k} \frac{p_m}{1 + p_m} N_{p_j}(r, a_j; f_1)
\]

\[
+ \left( m - 1 - \frac{(m - 1)p_m}{1 + p_m} + \sum_{j=m}^{k} \frac{1}{1 + p_j} \right) T(r, f_1) + S(r, f_1)
\]

i.e.,

\[
\left( \sum_{j=m}^{k} \frac{p_j}{1 + p_j} + \frac{(m - 1)p_m}{1 + p_m} - 2 - \epsilon + o(1) \right) T(r, f_1) \leq \frac{p_m}{1 + p_m} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_1)
\]

Similarly, we get

\[
\left( \sum_{j=m}^{k} \frac{p_j}{1 + p_j} + \frac{(m - 1)p_m}{1 + p_m} - 2 - \epsilon + o(1) \right) T(r, f_2) \leq \frac{p_m}{1 + p_m} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_2)
\]

(7)

Let $B_j = E_{p_j}(a_j; f_1) \setminus A_j$ for $j = 1, 2, ..., k$ and using (7), (8) we have

\[
\sum_{j=1}^{k} N_{p_j}(r, a_j; f_1) = \sum_{j=1}^{k} N_{A_j}(r, a_j; f_1) + \sum_{j=1}^{k} N_{B_j}(r, a_j; f_1)
\]

\[
\leq \delta T(r, f_1) + N(r, 0; f_1 - f_2)
\]

\[
\leq (1 + \delta)T(r, f_1) + T(r, f_2) + O(1)
\]

(8)

i.e.,

\[
\left( \sum_{j=m}^{k} \frac{p_j}{1 + p_j} + \frac{(m - 1)p_m}{1 + p_m} - 2 - \epsilon + o(1) \right) \sum_{j=1}^{k} N_{p_j}(r, a_j; f_1)
\]

\[
\leq (1 + \delta) \frac{p_m}{1 + p_m} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_1) + \{1 + o(1)\} \frac{p_m}{1 + p_m} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_2)
\]

\[
\left( \sum_{j=m}^{k} \frac{p_j}{1 + p_j} + \frac{(m - 1)p_m}{1 + p_m} - (1 + \delta) \frac{p_m}{1 + p_m} - 2 - \epsilon + o(1) \right) \sum_{j=1}^{k} N_{p_j}(r, a_j; f_1)
\]

\[
\leq \{1 + o(1)\} \frac{p_m}{1 + p_m} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_2))
\]
Since $\epsilon (>0)$ is arbitrary, we have
\[
\liminf_{r \to \infty} \frac{\sum_{j=1}^{k} N_{p_j}(r, a_j; f_1)}{\sum_{j=1}^{k} N_{p_j}(r, a_j; f_2)} \leq \left(\sum_{j=m}^{k} p_j + \frac{(m-1)p_m}{1+p_m} - (1+\delta)\frac{p_m}{1+p_m} - 2\right),
\]
which is a contradiction. Therefore $f_1(z) \equiv f_2(z)$. This completes the proof. \qed

**Corollary 2** For $m = 1$ in Theorem 8 we get Theorem 6. Hence Theorem 8 is a generalization of Theorem 6.

**Proof of Theorem 9.**

By Lemma 3, we have
\[
(k - 2 + o(1))T(r, f_1) < \sum_{j=1}^{k} N(r, a_j; f_1^{(n)}) \tag{9}
\]
and
\[
(k - 2 + o(1))T(r, f_2) < \sum_{j=1}^{k} N(r, a_j; f_2^{(n)}). \tag{10}
\]

From (9) we have
\[
(k - 2 + o(1))T(r, f_1) \leq \sum_{j=1}^{k} \left\{ N_{p_j}(r, a_j; f_1^{(n)}) + N_{p_j+1}(r, a_j; f_1^{(n)}) \right\}
\leq \sum_{j=1}^{k} \left\{ \frac{p_j}{1+p_j} N_{p_j}(r, a_j; f_1^{(n)}) + \frac{1}{1+p_j} N(r, a_j; f_1^{(n)}) \right\}
\leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} N_{p_j}(r, a_j; f_1^{(n)}) + \sum_{j=1}^{k} \frac{1}{1+p_j} T(r, f_1^{(n)})
\leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} N_{p_j}(r, a_j; f_1^{(n)}) + (n + 1) \sum_{j=1}^{k} \frac{1}{1+p_j} T(r, f_1)
\]
i.e.,
\[
\{(k - 2) - (n + 1) \sum_{j=1}^{k} \frac{1}{1+p_j} + o(1)\}T(r, f_1) \leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} N_{p_j}(r, a_j; f_1^{(n)})
\]
Similarly from (10) we get
\[
\{(k - 2) - (n + 1) \sum_{j=1}^{k} \frac{1}{1+p_j} + o(1)\}T(r, f_2) \leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} N_{p_j}(r, a_j; f_2^{(n)})
\]
Let $B_j = \overline{E}_{p_j}(a_j; f_1^{(n)}) \setminus A_j$ for $j = 1, 2, \ldots, k$. 

Now
\[
\sum_{j=1}^{k} N_{p_j}(r, a_j; f_1^{(n)}) = \sum_{j=1}^{k} N_{A_j}(r, a_j; f_1^{(n)}) + \sum_{j=1}^{k} N_{B_j}(r, a_j; f_1^{(n)}) \\
\leq \delta T(r, f_1^{(n)}) + N(r, 0; f_1^{(n)} - f_2^{(n)}) \\
\leq (1 + \delta)(n + 1)T(r, f_1) + (n + 1)T(r, f_2)
\]
i.e.,
\[
\{(k - 2) - (n + 1) \sum_{j=1}^{k} \frac{1}{1 + p_j} + o(1)\} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_1^{(n)}) \\
\leq (1 + \delta)(n + 1) \sum_{j=1}^{k} \frac{p_j}{1 + p_j} N_{p_j}(r, a_j; f_1^{(n)}) + (n + 1) \sum_{j=1}^{k} \frac{p_j}{1 + p_j} N_{p_j}(r, a_j; f_2^{(n)})
\]
Since \(1 \geq \frac{p_1}{1 + p_1} \geq \frac{p_2}{1 + p_2} \geq \ldots \geq \frac{p_k}{1 + p_k} \geq \frac{1}{2}\), we get from above inequality
\[
\{(k - 2) - (n + 1) \sum_{j=1}^{k} \frac{1}{1 + p_j} + o(1)\} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_1^{(n)}) \\
\leq (1 + \delta)(n + 1) \frac{p_1}{1 + p_1} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_1^{(n)}) + (n + 1) \frac{p_1}{1 + p_1} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_2^{(n)})
\]
i.e.,
\[
\{(k - 2) - (n + 1) \sum_{j=1}^{k} \frac{1}{1 + p_j} - (1 + \delta)(n + 1) \frac{p_1}{1 + p_1} + o(1)\} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_1^{(n)}) \\
\leq (n + 1) \frac{p_1}{1 + p_1} \sum_{j=1}^{k} N_{p_j}(r, a_j; f_2^{(n)})
\]
Therefore
\[
\lim_{r \to \infty} \inf \frac{\sum_{j=1}^{k} N_{p_j}(r, a_j; f_1^{(n)})}{\sum_{j=1}^{k} N_{p_j}(r, a_j; f_2^{(n)})} \\
\leq \frac{(n + 1)p_1}{(k - 2)(1 + p_1) - (n + 1)(1 + p_1) \sum_{j=1}^{k} \frac{1}{1 + p_j} - (n + 1)(1 + \delta)p_1} \\
= \frac{(n + 1)p_1}{(k - 2)(1 + p_1) - (n + 1)(1 + p_1) \sum_{j=2}^{k} \frac{1}{1 + p_j} - (n + 1)\{1 + \delta\}p_1 + 1}
\]
which is a contradiction.
Therefore \(f_1^{(n)}(z) \equiv f_2^{(n)}(z)\). This complete the proof. \(\Box\)

**Corollary 3** Let \(p_k = \infty\) and \(L = \liminf_{r \to \infty} \frac{\sum_{j=1}^{k} N_{p_j}(r, a_j; f_1^{(n)})}{\sum_{j=1}^{k} N_{p_j}(r, a_j; f_2^{(n)})} > \frac{n + 1}{k - (n + 3)}\).

If \(\sum_{j=1}^{k} N_{A_j}(r, a_j; f_1^{(n)}) \leq \delta T(r, f_1^{(n)})\), for some \(\delta\) with \(0 \leq \delta < \frac{k - (n + 3)}{n + 1} - \frac{1}{L}\),
then \(f_1^{(n)}(z) \equiv f_2^{(n)}(z)\).
If we assume $E_\infty(a_j; f_1^{(n)}) \subseteq E_\infty(a_j; f_2^{(n)})$, then $A_j = \phi$ for $j = 1, 2, \ldots, k$ and so we can choose $\delta = 0$. Choosing $n = 0$ we get Theorem 4.

**Corollary 4** For $k = 5$, then Corollary 1 is reduced to Theorem 3.

**Corollary 5** Let $f_1 \neq f_2$. For $n = 0$ and $E_{p_j}(a_j; f_1) = E_{p_j}(a_j; f_2)$ for $j = 1, 2, \ldots, k$, we have $A_j = \phi$, therefore we can choose $\delta = 0$. We have from (11)

$$1 \leq \frac{p_1}{(1 + p_1)(k - 2) - (1 + p_1)\sum_{j=2}^{k} \frac{1}{1 + p_j} - (1 + p_1)}$$

$$\Rightarrow \sum_{j=2}^{k} \frac{p_j}{1 + p_j} \leq \frac{p_1}{(1 + p_1)} + 2.$$ 

hence Theorem 9 reduced to Theorem 2.

**Example 1** Let $f(z) = e^z + a$ and $g(z) = e^z + b$ where $a, b$ $(a \neq b)$ are constants. Then $E(a_j; f') = E(a_j; g')$ so, $A_j = \phi$ for $j = 1, 2, \ldots, 5$, we can choose $\delta = 0$ and

$$\lim_{r \to \infty} \inf \frac{\sum_{j=1}^{5} N(r, a_j; f')}{\sum_{j=1}^{5} N(r, a_j; g')} = 1 > \frac{9}{23}.$$ 

Therefore by Theorem 7 we have $f'(z) \equiv g'(z)$.

**Example 2** Let $f(z) = \frac{i}{e^{z+1}}$ and $g(z) = \frac{-ie^z}{e^{z+1}}$. Clearly, $E(0; f) \subset E(0; f')$ and $E(0; g) \subset E(0; g')$. Since $E(a_j; f') = E(a_j; g')$ so, $A_j = \phi$ for $j = 1, 2, \ldots, 7$, we can choose $\delta = 0$ and

$$\lim_{r \to \infty} \inf \frac{\sum_{j=1}^{7} N(r, a_j; f')}{\sum_{j=1}^{7} N(r, a_j; g')} = 1 > \frac{2}{3}.$$ 

Therefore by Theorem 9 we have $f'(z) \equiv g'(z)$.

**References**


**Dilip Chandra Pramanik**

Department of Mathematics, University of North Bengal, Raja Rammohunpur, Darjeeling-734013, West Bengal, India,

E-mail address: dcpramanik.nbu2012@gmail.com

**Jayanta Roy**

Department of Mathematics, DDE, University of North Bengal, Raja Rammohunpur, Darjeeling-734013, West Bengal, India,

E-mail address: jayantaroy983269@yahoo.com