A NOTE ON \((m; n)\)-PARANORMAL OPERATORS

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Abstract. In this paper, we prove properties of the class of \((m, n)\)-paranormal operators (a generalization of paranormal operators) on Hilbert space. Equality of the approximate point spectrum and the joint approximate point spectrum, for \((m, n)\)-paranormal operators has been proved under certain given conditions. Moreover, the point spectrum coincides with the joint point spectrum for the class of \((m, n)\)-paranormal operators. We also discuss the SVEP, normaloid and subnormality for the same class of operators.

1. Introduction

Throughout this note, \(B(\mathcal{H})\) be the \(C^\ast\)-algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space \(\mathcal{H}\). If \(T \in B(\mathcal{H})\), then we shall write \(N(T)\) and \(R(T)\) for the null space and the range space of \(T\), respectively. In this paper, \(\mathbb{C}\) and \(\mathbb{N}\) denote the set of all complex numbers and the set of all natural numbers, respectively. The orthogonal complement \(S^\perp\) of a subset \(S\) of Hilbert space is defined by \(S^\perp = \{x \in \mathcal{H} : \langle x, y \rangle = 0\ \text{for all} \ y \in S\}\).

For \(T, S\) in \(B(\mathcal{H})\), \(T \otimes S\) denotes the tensor product on the product space \(\mathcal{H} \otimes \mathcal{H}\). If \(T \in B(\mathcal{H})\), then we write \(\sigma(T), \sigma_p(T), \sigma_p(T), \sigma_a(T)\) and \(\sigma_{ja}(T)\) for the spectrum, the point spectrum, the joint point spectrum, the approximate point spectrum and the joint approximate point spectrum of \(T\), respectively. An operator \(T\) in \(B(\mathcal{H})\) is said to be:

1) positive (denoted \(T \geq 0\)) if \(\langle Tx, x \rangle \geq 0\), for all \(x \in \mathcal{H}\).
2) if \(T^*T - TT^* \geq 0\), or equivalently, \(\|Tx\| \geq \|T^*x\|\) for all \(x \in \mathcal{H}\)[15].
3) paranormal if \(\|Tx\|^2 \leq \|T^2x\|\|x\|\), for all \(x \in \mathcal{H}\) [15, 13].
4) \((m, n)\)-paranormal and \((m, n)^*\)-paranormal if \(\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n\) and \(\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n\), respectively for all \(x\) in \(\mathcal{H}\), where \(m\) is a positive real number and \(n\) is a positive integer [10].
5) normaloid, if its spectral radius coincides with its norm, that is, \(r(T) = \|T\|\), or equivalently, \(\|T^n\| = \|T\|^n\) for every positive integer \(n\).

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2. (m, n)-PARANORMAL OPERATORS

We begin this section with the following theorem for the class of (m, n)-paranormal operators.

**Theorem 2.1.** Let \( T \in B(\mathcal{H} \oplus \mathcal{H}) \) be a (m, n)-paranormal operator defined by 2 \times 2 matrix representation \( T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \). Then \( A \) is (m, n)-paranormal.

**Proof.** By [10, Theorem 2.1], the following matrix

\[
m^{\frac{2}{m+1}}T^{n+1}T^{n+1} - (n + 1)a^nT^*T + m^{\frac{2}{m+1}}na^{n+1}I = \begin{bmatrix} Q & R \\ R^* & S \end{bmatrix}
\]

is positive for each \( a > 0 \), where

\[
Q = m^{\frac{2}{m+1}}A^{n+1}A^{n+1} - (n + 1)a^nA^*A + m^{\frac{2}{m+1}}na^{n+1}I
\]

\[
R = m^{\frac{2}{m+1}}A^{n+1}P - (n + 1)a^nA^*C
\]

and

\[
S = m^{\frac{2}{m+1}}(P^*P + B^{n+1}B^{n+1}) - (n + 1)a^n(C^*C + B^*B) + m^{\frac{2}{m+1}}na^{n+1}I
\]

Here, we have

\[
P = A^nC + A^{n-1}CB + A^{n-2}CB^2 + \ldots + ACB^{n-1} + CB^n
\]

Since \( T \) is (m, n)-paranormal, so \( Q \) is positive for each \( a > 0 \). Hence, \( A \) is (m, n)-paranormal.

Now, in the sequel of the above result, we have \( S \) is positive for each \( a > 0 \). Thus, we have

\[
m^{\frac{2}{m+1}}B^{n+1}B^{n+1} - (n + 1)a^nB^*B + m^{\frac{2}{m+1}}na^{n+1}I
\]

\[
\geq (n + 1)a^nC^*C - m^{\frac{2}{m+1}}P^*P.
\]

Therefore, if we take \( (n + 1)a^nC^*C \geq m^{\frac{2}{m+1}}P^*P \) for each \( a > 0 \), then \( B \) is also (m, n)-paranormal. This is our next result.

**Proposition 2.2.** Let \( T \in B(\mathcal{H} \oplus \mathcal{H}) \) be a (m, n)-paranormal operator defined by 2 \times 2 matrix representation \( T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \). Then \( B \) is (m, n)-paranormal provided \( (n + 1)a^nC^*C \geq m^{\frac{2}{m+1}}P^*P \), for each \( a > 0 \).

**Remark 2.3.** It is well known that \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) is positive if and only if \( x \geq 0, z \geq 0 \) and \( y = \frac{1}{2}wz \) for some contraction \( w \). Now, if we choose \( Q = 0 \) in Theorem 2.1, then we have \( R = 0 \), that is,

\[
m^{\frac{2}{m+1}}A^{n+1}(A^nC + A^{n-1}CB + A^{n-2}CB^2 + \ldots + ACB^{n-1} + CB^n)
\]

\[
= (n + 1)a^nA^*C.
\]

**Remark 2.4.** In Theorem 2.1, if we set \( C = 0 \), then \( B \) is always (m, n)-paranormal.

**Remark 2.5.** If \( T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \) on \( \mathcal{H} = M \oplus M^\perp \) is (m, n)-paranormal and \( M \) be a closed invariant subspace of \( \mathcal{H} \) under \( T \), then \( T \) is (m, n)-paranormal on \( M \).
In the following theorem, we show the relationship between \((m, n)\)-paranormal and \((m, n + 1)\)-paranormal operators for \(n \geq 2\).

**Theorem 2.6.** [16, Lemma 1] Let \(T\) be a \((m, n)\)-paranormal operator and for all unit vectors in \(H\), \(\|T^n x\| \|Tx\| \leq \|T^{n+1} x\|\). Then \(T\) is \((m, n + 1)\)-paranormal.

Conversely, if \(T\) is \((m, n + 1)\)-paranormal and \(\|T^{n+1} x\|^n \leq m \|T^n x\|^{n+1}\) for all unit vectors in \(H\), then \(T\) is \((m, n)\)-paranormal.

**Proof.** By using the \((m, n)\)-paranormality of \(T\) and given condition, we have

\[
\|Tx\|^{n+1} \leq m \|T^n x\| \|Tx\| \leq m \|T^{n+1} x\|,
\]

that is,

\[
\|Tx\|^{n+1} \leq m \|T^{n+1} x\|.
\]

Conversely, with \(T\) \((m, n + 1)\)-paranormal and given condition, it follows that

\[
\|Tx\|^{n(n+1)} \leq m \|T^{n+1} x\|^n \leq m^{n+1} \|T^n x\|^{n+1},
\]

that is,

\[
\|Tx\|^n \leq m \|T^n x\|.
\]

Hence, the result holds. \(\square\)

It is a natural question to ask whether an operator \(T\) is normaloid or not. The following example provides an operator which is \((m, n)\)-paranormal but not normaloid for \(m > 1\).

**Example 2.7.** Let \(H = l^2(\mathbb{N}, \mathbb{C})\). Define weighted shift operator \(T\) by \(T(e_k) = w_k e_{k+1}\) for all positive integers \(k\), with non-zero weights \(w_k\) and orthonormal basis \(e_k\), where

\[
w_k = 1 \text{ if } k = 12 \text{ if } k = 23 \text{ if } k \geq 3.
\]

Equivalently, for \(x \in l^2(\mathbb{N}, \mathbb{C})\), we have

\[
T(x_1, x_2, ...) = (0, x_1, 2x_2, 3x_3, 3x_4, ...).
\]

By [10, Theorem 2.9], \(T\) is \((m, n)\)-paranormal if and only if

\[
|w_k|^{n-1} \leq m |w_{k+1}| |w_{k+2}| \cdots |w_{k+n-1}|, \tag{2.1}
\]

for \(n \geq 2\), all positive integers \(k\) and all unit vectors. Note that the inequality (2.1) is satisfied for all \(m \geq 1\) by weighted sequences. Hence, \(T\) is \((m, n)\)-paranormal. Now, \(\|T\| = \sup |w_k|\) and so it is easy to see that \(\|T\| = 3\). It is well known that \(0 \leq r(T) \leq \|T\|\). Thus, \(r(T) \leq 3\).

Now, we claim that \(r(T) < 3\). Suppose if possible, \(r(T) = 3\). Then there exists \(\lambda \in \mathbb{C}\) such that \(|\lambda| = 3\) and \(T - \lambda I\) is not invertible. Note that

\[
(T - \lambda I)(x_1, x_2, ...) = (-\lambda x_1, x_1 - \lambda x_2, 2x_2 - \lambda x_3, 3x_3 - \lambda x_4, 3x_4 - \lambda x_5, ...)
\]

It is easy to see that \(T - \lambda I\) is one one and onto. Hence, \(T - \lambda I\) is invertible, which is a contradiction. Therefore, \(\lambda \notin \sigma(T)\) and \(r(T) \neq 3\). Thus, \(r(T) < 3\). Hence, \(T\) is not normaloid.

To the sequel, we sketch the following theorem which shows that a \((m, n)\)-paranormal operator is normaloid for \(m \leq 1\).

**Theorem 2.8.** [16, Proposition 1] If an operator \(T\) is \((m, n)\)-paranormal for \(m \leq 1\), then \(T\) is normaloid.
The proof of the next theorem is similar to that of [11, Theorem 2.3].

**Theorem 2.9.** Let $\mathcal{H}$ be the direct sum of countably many isomorphic copies of Hilbert spaces $\mathcal{H}_i$. If $T_i$ is $(m, n)$-paranormal operator on $\mathcal{H}_i$ for each $i$, then the direct sum of $T_i$ is also $(m, n)$-paranormal.

**Lemma 2.10.** [10, Theorem 2.9] Let an operator $T: l^2(\mathbb{Z}, \mathbb{C}) \to l^2(\mathbb{Z}, \mathbb{C})$ be defined by $T(e_k) = w_k e_{k-1}$ with non zero weights $(w_k)$, and the orthonormal basis $(e_k)$. Then $T$ is $(m, n)$-paranormal if and only if

$$|w_{k-1}|^{n-1} \leq m|w_{k-2}||w_{k-3}| \cdots |w_{k-n}|$$

holds for all integers $k$, unit vectors and $n \geq 2$.

In the following example, we show that the inverse of $(m, n)$-paranormal operator need not be $(m, n)$-paranormal.

**Example 2.11.** Let $H = l^2(\mathbb{Z}, \mathbb{C})$ and $T$ be a weighted shift operator on $H$ defined by $Te_k = w_k e_{k+1}$ with non zero weights $w_k$, and the orthonormal basis $e_k$ for all integers $k$, where

$$w_k = \begin{cases} 
\frac{1}{2} & \text{if } k \leq 0 \\
0 & \text{if } k = 14 \\
1 & \text{if } k \geq 2 
\end{cases}$$

Equivalently, $T$ is defined by

$$T(..., x_{-1}, x_0, x_1, ...) = (... , \frac{1}{2}x_{-1}, \frac{1}{2}x_0, 2x_1, 4x_2, 4x_3, ...)$$

By [10, Theorem 2.9], $T$ is $(m, n)$-paranormal if and only if

$$|w_k|^n \leq m|w_{k+1}||w_{k+2}| \cdots |w_{k+n-1}|, \quad (2.2)$$

for unit vectors and $n \geq 2$. Thus, (2.2) holds for $m \geq 1$. It is straightforward to see that $T$ is invertible. Also,

$$T^{-1}(..., y_{-1}, y_0, y_1, ...) = (... , 2y_0, 2y_1, y_2, y_3, y_4, ...),$$

that is,

$$T^{-1}e_k = \alpha_{k-1} e_{k-1}$$

with weighted sequence

$$\alpha_k = \begin{cases} 
2 & \text{if } k \leq 0 \\
\frac{1}{2} & \text{if } k = 14 \\
\frac{1}{4} & \text{if } k \geq 2 
\end{cases}$$

Now, we claim that $T^{-1}$ is not $(m, n)$-paranormal. By using Lemma 2.10, $T^{-1}$ is $(m, n)$-paranormal if and only if

$$|\alpha_{k-1}|^{n-1} \leq m|\alpha_{k-2}||\alpha_{k-3}| \cdots |\alpha_{k-n}|, \quad (2.3)$$

for unit vectors, all $k$ and $n \geq 2$. If we choose $n = 3, k = 4$ and $m = \frac{1}{4}$, then (2.3) fails to hold. Hence, the claim holds.

The proof of the following theorem is similar to [11, Theorem 2.6].

**Theorem 2.12.** Let $T$ be an $(m, n)$-paranormal operator. Then $T \otimes I$ and $I \otimes T$ are also $(m, n)$-paranormal.

The following example shows that the tensor product of two $(m, n)$-paranormal operators need not be $(m, n)$-paranormal.
Example 2.13. For each positive integer $k$, assume that $H_k = \mathbb{R} \times \mathbb{R}$. Let $H$ be a Hilbert space such that $H = \bigoplus_{k=1}^{\infty} H_k$. Now, we choose $A$ and $B$ to be positive operators on $H_k$ such that $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $B^4 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$. We simply assign the operator $T$ on $H$ as:

$$T(x_1, x_2, ...) = (0, Ax_1, Ax_2, ..., Ax_n, Bx_{n+1}, Bx_{n+2}, ...).$$

Thus, the adjoint of $T$ is given by

$$T^*(x_1, x_2, ...) = (Ax_2, Ax_3, ..., Ax_{n+1}, Bx_{n+2}, Bx_{n+3}, ...).$$

For $x = (...; 0, 0, x_n, 0, 0, ...) \in H$, by [10, Theorem 2.1], an operator $T$ is $(2^\frac{1}{2}, 2)$-paranormal if and only if

$$2AB^4A - 3a^2A^2 + 4a^3I \geq 0,$$

for each $a > 0$. Now,

$$2AB^4A - 3a^2A^2 + 4a^3I = \begin{bmatrix} 4a^3 - 12a^2 + 32 & 8 \\ 8 & 4a^3 - 12a^2 + 24 \end{bmatrix}$$

is positive for each $a > 0$. Thus, $T$ is $(2^\frac{1}{2}, 2)$-paranormal. Similarly, by [10, Theorem 2.1], the operator $T \otimes T$ is $(2^\frac{1}{2}, 2)$-paranormal if and only if

$$2(AB^4A \otimes AB^4A) - 3a^2(A^2 \otimes A^2) + 4a^3(I \otimes I) \geq 0,$$

for each $a > 0$. For $a = 10$, the operator

$$2(AB^4A \otimes AB^4A) - 3a^2(A^2 \otimes A^2) + 4a^3(I \otimes I)$$

is not positive. Hence, our claim holds.

Embry [12] has proved that an operator $T$ is subnormal if and only if $\sum_{i,j=0}^{k} \langle T^i+jx_i, T^i+jx_j \rangle$ is non-negative for all finite collection of vectors $x_0, x_1, \ldots, x_k$. In the following theorem, by using this characterization, we prove that a $(m, n)$ paranormal is subnormal under certain conditions.

Theorem 2.14. If an operator $T$ is $(m, n)$ paranormal and partial isometry with $m \leq 1$ and $\|T^n\|^2 \leq \frac{1}{m+1}$, then $T$ is subnormal.

Proof. As $T$ is $(m, n)$-paranormal, so by [10, Theorem 2.1], we have

$$m^{\frac{1}{m+1}}T^{n+1}T^{n+1} - (n+1)a^nT^nT + m^{\frac{1}{m+1}}na^{n+1}I \geq 0,$$

for each $a > 0$. Also, it follows that

$$T^*T(m^{\frac{2}{m+1}}T^{n+1}T^{n+1} - (n+1)a^nT^nT + m^{\frac{1}{m+1}}na^{n+1}I)T^T \geq 0$$

(2.4)

for each $a > 0$. Since $T$ is a partial isometry, $TT^T = T$ by [14, Corollary 3, Problem 98]. Now, take $a = 1$ in (2.4). Then we have

$$m^{\frac{1}{m+1}}T^{n+1}T^{n+1} - ((n+1) - m^{\frac{1}{m+1}}n)T^*T \geq 0,$$
that is,
\[ T^*T \leq \frac{m^{\frac{2}{n+1}}}{(n+1) - nm^{\frac{2}{n+1}}} T^{n+1}T^{n+1}, \]
equivalently,
\[ \|Tx\|^2 \leq \frac{m^{\frac{2}{n+1}}}{(n+1) - nm^{\frac{2}{n+1}}} \|T^{n+1}x\|^2 \]
\[ \leq m^{\frac{2}{n+1}} \|T^{n+1}x\|^2 \]
\[ \leq m^{\frac{2}{n+1}} \|T^n\|^2 \|Tx\|^2 \]
\[ \leq \|Tx\|^2. \]

Therefore, we have
\[ T^*T = m^{\frac{2}{n+1}} T^{n+1}T^{n+1} \text{ for all } n. \] (2.5)

Further, let \( x_0, x_1, \ldots, x_k \) be a finite collection of vectors, then by using (2.5) we get
\[
m^4 \sum_{i,j=0}^{k} \langle T^i T_j x_i, T^i T_j x_j \rangle = m^4 \left( \langle x_0, x_0 \rangle + \langle T^* T_0 x_0, x_1 \rangle + \langle T^* T_1 x_0, x_0 \rangle \right) \]
\[ + \sum_{i,j=0}^{k} m^4 m^{\frac{2}{n+1}} \left( m^{\frac{2}{n+1}} T^i T_j x_i, x_j \right) \]
\[ = m^4 \left( \langle x_0, x_0 \rangle + \langle T^* T_0 x_0, x_1 \rangle + \langle T^* T_1 x_0, x_0 \rangle \right) \]
\[ + \sum_{i,j=0}^{k} m^4 m^{\frac{2}{n+1}} \langle T^* T_1 x_i, x_j \rangle. \]

Since \( T^*T \) is a projection by [14, Problem 98], we have
\[
m^4 \sum_{i,j=0}^{k} \langle T^i T_j x_i, T^i T_j x_j \rangle = m^4 \left( \langle x_0, x_0 \rangle + \langle T^* T_0 x_0, T^* T_1 x_1 \rangle + \langle T^* T_1 x_0, T^* T_0 x_0 \rangle \right) \]
\[ + \sum_{i,j=0}^{k} m^4 m^{\frac{2}{n+1}} \langle (T^* T)^{i+j} x_i, (T^* T)^{i+j} x_j \rangle \]
\[ = m^4 \left( \langle x_0, x_0 \rangle + \langle T^* T_0 x_0, T^* T_1 x_1 \rangle + \langle T^* T_1 x_0, T^* T_0 x_0 \rangle \right) \]
\[ + \sum_{i+j=2} m^3 \langle (T^* T)^2 x_i, (T^* T)^2 x_j \rangle + \cdots \]
\[ + \sum_{i+j=2k-1} m^{2(\frac{k-1}{k+1})} \langle (T^* T)^{2k-1} x_i, (T^* T)^{2k-1} x_j \rangle \]
\[ + \sum_{i+j=2k} m^{(\frac{4k-1}{k+1})} \langle (T^* T)^{2k} x_i, (T^* T)^{2k} x_j \rangle. \]

As \( m \leq 1 \), we obtain the following relation
\[ m^3 \geq \cdots \geq m^{2(\frac{k-1}{k+1})} \geq m^{(\frac{4k-1}{k+1})} \geq m^4. \]
By using the above relation we obtain
\[ m^4 \sum_{i,j=0}^{k} \langle T^{i+j}x_i, T^{i+j}x_j \rangle \geq m^4 \sum_{i,j=0}^{k} \langle (T^*T)^{i+j}x_i, (T^*T)^{i+j}x_j \rangle. \]

Since \( T^*T \) is self-adjoint, \( T^*T \) is subnormal and so we have
\[ \sum_{i,j=0}^{k} \langle T^{i+j}x_i, T^{i+j}x_j \rangle \geq 0. \]

Hence, the result holds. \( \square \)

In the sequel, we give the following example to show that there also exists a \((m, n)\)-paranormal operator, which is not subnormal.

**Example 2.15.** Let \( T \) be an operator defined by \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) as a \( 2 \times 2 \) matrix
\[ T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}. \]
By [10, Theorem 2.1], \( T \) is \((2^2, 2)\) paranormal if and only if \( 2T^{*3}T^3 - 3a^2T^*T + 4a^3I \geq 0 \), for each \( a \geq 0 \).

It is easy to see that the matrix
\[ 2T^{*3}T^3 - 3a^2T^*T + 4a^3I = \begin{bmatrix} 6a^2 + 4a^3 & 12 - 6a^2 \\ 12 - 6a^2 & 75 - 15a^2 + 4a^3 \end{bmatrix} \]
is positive for each \( a > 0 \). Hence, \( T \) is \((2^2, 2)\) paranormal.

We next move to show that \( T \) is not subnormal. Consider
\[ \sum_{i,j=0}^{1} \langle T^{i+j}x_i, T^{i+j}x_j \rangle = \langle x_0, x_0 \rangle + \langle Tx_0, Tx_1 \rangle + \langle Tx_1, Tx_0 \rangle + \langle T^2x_1, T^2x_1 \rangle. \]
Choose \( x_0 = \begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), \( x_1 = \begin{bmatrix} x_1^1 \\ x_1^2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \). Then
\[ \sum_{i,j=0}^{1} \langle T^{i+j}x_i, T^{i+j}x_j \rangle = \frac{-55}{16} \]
is not positive. Hence, \( T \) is not subnormal.

### 3. Spectral properties

In this section we prove some spectral properties on \((m, n)\)-paranormal operators. The spectrum set of an operator \( T \in B(\mathcal{H}) \), denoted by \( \sigma(T) \), is the set of complex number \( \lambda \) such that \( T - \lambda I \) is not invertible. A complex number \( \lambda \) is said to be in the point spectrum \( \sigma_p(T) \) of \( T \) if there is a nonzero \( x \in \mathcal{H} \) such that \( (T - \lambda)x = 0 \). If in addition, \( (T^* - \bar{\lambda})x = 0 \), then \( \lambda \) is said to be in the joint point spectrum \( \sigma_{jp}(T) \) of \( T \).

Analogously, a complex number \( \lambda \) is said to be in the approximate point spectrum \( \sigma_a(T) \) of \( T \) if there is a sequence \( \langle x_n \rangle \) of unit vectors in \( \mathcal{H} \) such that \( (T - \lambda)x_n \rightarrow 0 \). If in addition, \( (T^* - \bar{\lambda})x_n \rightarrow 0 \), then \( \lambda \) is said to be in the joint approximate point spectrum \( \sigma_{ja}(T) \) of \( T \). In general, \( \sigma_p(T) \neq \sigma_{jp}(T), \sigma_a(T) \neq \sigma_{ja}(T) \).

Some researchers showed that, for some classes of nonnormal operators \( T \), the nonzero points of its point spectrum and joint point spectrum are identical, the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical [8, 9, 19, 20, 21]. The reader can refer to the recent papers
[1, 2, 3, 4, 5, 6, 7, 17, 18, 22, 23] for the spectrum and fine spectrum of certain linear operators represented by a triangle matrix over some sequence spaces.

The proof of the following theorem is similar to [11, Theorem 3.1].

**Theorem 3.1.** If \( T \in B(\mathcal{H}) \) is a \((m, n)\)-paranormal and hyponormal operator, then \( \sigma_a(T) = \sigma_{ja}(T) \) for unit vectors.

If an operator \( T \) is \((m, n)\)-paranormal but not hyponormal, then the above result does not holds. We prove it in the following example.

**Example 3.2.** Let \( T : l^2(\mathbb{N}, \mathbb{C}) \rightarrow l^2(\mathbb{N}, \mathbb{C}) \) be weighted shift operator defined by \( T(e_k) = w_k e_{k-1}, \) that is,
\[
T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)
\]
with weighted sequence \( (w_k) \) such that \( w_k = 1 \) for all positive integers \( k \) and adjoint of \( T \) is given by
\[
T^*(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).
\]

By [10, Theorem 2.9], \( T \) is \((m, n)\)-paranormal for unit vectors and \( m \geq 1 \). Since it is easy to check that \( \|Tx\| \geq \|T^*x\| \) for some \( x \) in \( \mathcal{H}, T \) is not hyponormal. Next, we move to prove that \( \sigma_a(T) \neq \sigma_{ja}(T) \). Now, we choose \( 0 = \lambda \in \sigma_a(T) \), a unit vector \( x = (1, 0, 0, \ldots) \) and a sequence \( \{x_n\}_{n=1}^{\infty} = \{x, x, \ldots\}. \) Then
\[
\|(T - \lambda I)x_n\| = \|Tx_n\| = 0 \text{ as } n \rightarrow \infty
\]
but \( \|(T - \lambda f^*)x_n\| = \|T^*x_n\| = \|(0, 1, 0, 0, \ldots)\| \rightarrow 0 \text{ as } n \rightarrow \infty. \)

This shows that \( 0 \notin \sigma_{ja}(T). \)

**Definition 3.3.** An operator \( T \) is said to have single valued extension property (abbreviated as SVEP) at \( \gamma_0 \in \mathbb{C} \), if for every open neighborhood \( G \) of \( \gamma_0 \), the only analytic function \( f : G \rightarrow \mathcal{H} \) which satisfies the equation \( (T - \gamma I)f(\gamma) = 0 \) for all \( \gamma \in G \) is the function \( f = 0. \)

An operator \( T \) has SVEP if \( T \) has SVEP at every \( \gamma \in \mathbb{C}. \)

We can prove the following theorem in a similar fashion as that of [11, Theorem 3.4].

**Theorem 3.4.** Let \( T \in B(\mathcal{H}) \) be a \((m, n)\)-paranormal and hyponormal operator. Then \( T \) has SVEP.

The proof of following proposition is similar to [11, Proposition 3.5].

**Proposition 3.5.** Let \( T \in B(\mathcal{H}) \) be a \((m, n)\)-paranormal and hyponormal operator. Then \( N(T - \lambda I) \subseteq N(T^* - \lambda I) \) for unit vectors and for all \( \lambda \in \mathbb{C}. \)

The proof of the following theorem is similar to Theorem 3.1.

**Theorem 3.6.** If \( T \) is a \((m, n)\)-paranormal and hyponormal operator, then
\[
\sigma_p(T) = \sigma_{jp}(T)
\]
for all unit vectors.

**Proposition 3.7.** If \( T \) is a \((m, n)\)-paranormal operator for \( m \leq 1 \) and \( \{x_k\} \) is a sequence of unit vectors in \( \mathcal{H} \), which satisfies \( \lim_{k \rightarrow \infty} \|Tx_k\| = \|T\|, \) then \( \lim_{k \rightarrow \infty} \|T^{n+1}x_k\| = \|T\|^{n+1}. \)
Proof. As $T$ is $(m, n)$-paranormal, so we have the inequality
$$\|Tx\|^{n+1} \leq m\|T^{n+1}x\| \|x\|^n.$$ 
For $m \leq 1$, we have
$$\|Tx_k\|^{n+1} \leq m\|T^{n+1}x_k\| \leq \|T\|\|T\| \cdots \|T\| = \|T\|^{n+1}.$$ 
Equivalently,
$$\|Tx_k\|^{n+1} \leq \|T^{n+1}x_k\| \leq \|T\|^{n+1}.$$ 
As $\lim_{k \to \infty} \|Tx_k\| = \|T\|$, therefore by using squeeze principal, we have
$$\lim_{k \to \infty} \|T^{n+1}x_k\| = \|T\|^{n+1}.$$

References


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