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COMMON FIXED POINT THEOREMS USING WEAKLY COMPATIBLE MAPS IN MULTIPLICATIVE METRIC SPACES

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Abstract. In this paper, we proved some common fixed point theorems using multiplicative contractive conditions using weak compatible maps in complete and non-complete Multiplicative metric spaces.

1. Introduction

In the past years, many authors work on Banach's fixed point theorem in various spaces such as metric space, Fuzzy metric space, , Partial metric space, probabilistic metric space and generalized metric spaces. A new type of non-Newton calculus, called multiplicative calculus, was developed by Grossman and Katz [6]. In this calculus the operations of subtraction and addition are replaced by division and multiplication. It is well know that the set of positive real numbers R^+ is not complete according to the usual metric. To overcome this problem, by using the ideas of Grossman and Katz[5],in 2008, Bashirov et al. [2] defined a new distance so called a multiplicative distance by using the concept of multiplicative absolute value. Multiplicative metric space was introduced by Bashirov in 2008. After that, a huge number of paper appeared where authors use a various contractive condition used in order to prove a fixed point theorem. But, in the paper [4] on Multiplicative metric space, T. Doenovic proved that various well known fixed point theorems in multiplicative metric spaces have equivalent fixed point theorem in metric space. So, natural question has appeared: Is the multiplicative metric space a generalization of the metric space? Based on that, T. Doenovic, S.Radenovic [5] study fixed point theorems in multiplicative metric space where the contractive condition is complicated (i.e. rational type contractive condition) and at first, they conclude that there is not always equivalent theorem in metric space. Open question is the following one: Is it possible to find a better condition in metric space without additional conditions? If answer is negative, we realize that in some cases multiplicative metric space is useful. Inspired by multiplicative calculus, zavsar and evikel [8] defined and developed the topological properties of the multiplicative metric space by using the same idea of multiplicative distance as follows:

Definition 1.1[2] Let X be a non-empty set. A multiplicative metric is a mapping

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$\mathrm{d}: \mathrm{X} \times \mathrm{X} \rightarrow \mathbb{R}^{+}$ satisfying the following conditions:
(i) $\mathrm{d}(\mathrm{x}, \mathrm{y}) \geq 1$ for all $\mathrm{x}, \mathrm{y} \in \mathrm{X}$ and $\mathrm{d}(\mathrm{x}, \mathrm{y})=1$ if and only if $\mathrm{x}=\mathrm{y}$
(ii) $\mathrm{d}(\mathrm{x}, \mathrm{y})=\mathrm{d}(\mathrm{y}, \mathrm{x})$ for all $\mathrm{x}, \mathrm{y} \in \mathrm{X}$
(iii) $\mathrm{d}(\mathrm{x}, \mathrm{y}) \leq \mathrm{d}(\mathrm{x}, \mathrm{z}) \cdot \mathrm{d}(\mathrm{z}, \mathrm{y})$ for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$ (multiplicative triangle inequality).
Then mapping d together with X i.e., (X, d) is a multiplicative metric space.
Example 1.2 [8] Let R_{+}^{n} be the collection of all n-tuples of positive real numbers. Let $d^{*}: \mathbb{R}_{+}^{\mathrm{n}} \times \mathbb{R}_{+}^{\mathrm{n}} \longrightarrow \mathbb{R}$ be defined as follows:

$$d^{*}(\mathrm{x}, \mathrm{y})=\left(\left|\frac{x_{1}}{y_{1}}\right|^{*} \cdot\left|\frac{x_{2}}{y_{2}}\right|^{*} \quad \ldots\left|\frac{x_{n}}{y_{n}}\right|^{*}\right)$$

where $\mathrm{x}=\left(x_{1}, \ldots, x_{n}\right), \mathrm{y}=\left(y_{1}, \ldots, y_{n}\right) \in \mathbb{R}^{\mathrm{n}}+$ and $|.|: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ is defined by

$$|a|^{*}=\begin{cases} a & \text{if } a \geq 1 \\ \frac{1}{a} & \text{if } a<1 \end{cases}$$

Then it is obvious that all conditions of multiplicative metric are satisfied.

Example 1.3 [9] Let $\mathrm{d}: \mathbb{R} \times \mathbb{R} \rightarrow[1, \infty)$ be defined as $\mathrm{d}(\mathrm{x}, \mathrm{y})=a^{|x-y|}$, where $\mathrm{x}, \mathrm{y} \in \mathbb{R}$ and $\mathrm{a}>1$. Then $\mathrm{d}(\mathrm{x}, \mathrm{y})$ is multiplicative metric and (X, d) is called multiplicative metric spaces. We may call it usual multiplicative metric spaces.

Example 1.4 [9] Let (X, d) be a metric space. Define a mapping d_{a} on X by $\mathrm{d}_{\mathrm{a}}(\mathrm{x}, \mathrm{y})=a^{d(x, y)}$ where $\mathrm{a}>1$ is a real number and $\mathrm{d}_{\mathrm{a}}(\mathrm{x}, \mathrm{y})=a^{d(x, y)}=\begin{cases} 1 \text{ if } x=y \\ a \text{ if } x \neq y \end{cases}$
The metric $\mathrm{d}_{\mathrm{a}}(\mathrm{x}, \mathrm{y})$ is called discrete multiplicative metric and X together with d_{a} i.e., $(\mathrm{X}, \mathrm{d}_{\mathrm{a}})$ is known as discrete multiplicative metric space.

Example 1.5 [1] Let $X=C^{*}[a, b]$ be the collection of all real-valued multiplicative continuous functions over $[a, b] \subseteq R^{+}$. Then (X, d) is a multiplicative metric space with d defined by
$\mathrm{d}(f, g)=\sup_{\mathrm{x} \in[\mathrm{a}, \mathrm{b}]}\left|\frac{f(x)}{g(x)}\right|^{*}$ for arbitrary $\mathrm{f}, \mathrm{g} \in X$
Remark 1.6 [9] We note that multiplicative metric and metric spaces are independent structures. The mapping d^{*} defined above is multiplicative metric but it is not a metric as it doesn't satisfy triangular inequality.
For this we consider
$d^{*}\left(\frac{1}{3}, \frac{1}{2}\right)+d^{*}\left(\frac{1}{2}, 3\right)=\frac{3}{2}+6=7.5<9=d^{*}\left(\frac{1}{3}, 3\right)$, where d^{*} is defined as in example 1.2.
On the other hand, the usual metric on R is not a multiplicative metric as it doesn't satisfy multiplicative triangular inequality,
i.e., $\mathrm{d}(2,3) \cdot \mathrm{d}(3,6)=3<4=\mathrm{d}(2,6)$.
One can refer to [1] for detailed multiplicative metric topology.
Definition 1.7 [8] Let (X, d) be a multiplicative metric space. A sequence $\left\{x_{n}\right\}$ in X said to be a
(i) multiplicative convergent sequence to x, if for every multiplicative open ball $B_{\epsilon}(\mathrm{x})=\{\mathrm{y} \mid \mathrm{d}(\mathrm{x}, \mathrm{y})<\epsilon\}, \epsilon>1$, there exists a natural number N such that $x_{n} \in B_{\epsilon}(\mathrm{x})$ for all $\mathrm{n} \geq \mathrm{N}$, i. e , $\mathrm{d}\left(x_{n}, x\right) \rightarrow 1$ as $\mathrm{n} \rightarrow \infty$
(ii) multiplicative Cauchy sequence if for all $\epsilon>1$, there exists $\mathrm{N} \in \mathbb{N}$ such that $\mathrm{d}\left(x_{n}, x_{m}\right)<\epsilon$ for all $\mathrm{m}, \mathrm{n}>\mathrm{N}$ i. e , $\mathrm{d}\left(x_{n}, x_{m}\right) \rightarrow 1$ as $\mathrm{n} \rightarrow \infty$
A multiplicative metric space is called complete if every multiplicative Cauchy sequence in X is multiplicative converging to $\mathrm{x} \in \mathrm{X}$

In 2012, Ozavsar gave the concept of multiplicative contraction mapping and proved some fixed point theorem for these maps in complete multiplicative metric spaces.
Definition 1.8 [8] Let (X, d) be a multiplicative metric space. The map $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0, 1)$ such that

$$\mathrm{d}\left(\mathrm{f}\left(x_1\right), \mathrm{f}\left(x_2\right)\right) \leq \mathrm{d}\left(x_1, x_2\right)^\lambda \text{ for all x, y} \in \mathrm{X}$$

In 1996, Jungck [7] introduced the concept of weakly compatible mappings and prove fixed point Theorems using these mappings on metric spaces.
Definition 1.9 [7] Two maps f and g are said to be weakly compatible if they commute at coincidence points, that is, if fx = gx implies fgx = gfx for x $\in$ X.
In similar mode, we use weakly compatible in multiplicative metric spaces.

In 2009, S. Young Cho and M. J. Yoo[3] gives the following results in metric spaces using compatible maps: Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions:
(i) $\mathrm{A(X)} \subseteq \mathrm{T(X)}, \mathrm{B(X)} \subseteq \mathrm{S(X)}$
(ii) $\mathrm{d(Ax, By)} \leq p\left(\max\left\{\begin{array}{c}\mathrm{d(Ax, Sx), d(By, Ty)},\\ \left[\frac{\mathrm{d(Ax,Ty)+d(By,Sx)}}{2}\right., \mathrm{d(Sx, Ty)}\end{array}\right\}\right) + \mathrm{q}(\max\{\mathrm{d(Ax, Sx), d(By, Ty)}\}) +$ $\mathrm{r}(\max\{\mathrm{d(Ax, Ty), d(By, Sx)}\})$, for all $\mathrm{x, y} \in \mathrm{X}$, where $0 < \mathrm{h} = \mathrm{p + q + 2r} < 1$(p, q and r are non-negative real numbers).
(iii) one of A, B, S and T is continuous,
(iv) the pairs A, S and B, T are compatible on X. Then A, B, S and T have a unique common fixed point in X.

2. Main Results

Now we prove above result of S. Young Cho and M. J. Yoo[3] for weakly compatible mappings in setting of complete and non-complete multiplicative metric spaces as follow:
Theorem 2.1 Let (X, d) be a complete multiplicative metric space. Let A, B, S and T be self-mappings of X into itself satisfying the following conditions
(C_1) $\mathrm{A(X)} \subseteq \mathrm{T(X)}, \mathrm{B(X)} \subseteq \mathrm{S(X)}$
(C_2) one of the subspace AX or BX or SX or TX is complete
(C_3) the pairs (A, S) and (B, T) are weakly compatible
(C_4) $\mathrm{d(Ax, By)} \leq \left(\begin{array}{c}\mathrm{d(Ax, Sx), d(By, Ty)},\\ \max\{\sqrt{[\mathrm{d(Ax, Ty)} \cdot \mathrm{d(By, Sx)}]}, \mathrm{d(Sx, Ty)}\end{array}\right\}\Bigg)^p \cdot$
$\left(\max\left\{\mathrm{d}\left(\mathrm{A}x, \mathrm{S}x\right), \mathrm{d}\left(\mathrm{B}y, \mathrm{T}y\right)\right\}\right)^q \cdot \left(\max\left\{\mathrm{d}\left(\mathrm{A}x, \mathrm{T}y\right), \mathrm{d}\left(\mathrm{B}y, \mathrm{S}x\right)\right\}\right)^r$
for all $\mathrm{x, y} \in \mathrm{X}$, where $0 < \mathrm{h} = \mathrm{p + q + 2r} < 1$(p, q and r are non-negative real numbers). Then A, B, S and T have a unique common fixed point in X.
Proof. Let $x_0 \in X$ be an arbitrary point $\cdot$ Since $\mathrm{B(X)} \subseteq \mathrm{S(X)}$, therefore, for $\mathrm{x_0} \in \mathrm{X}$, there exists $\mathrm{x_1} \in \mathrm{X}$ such that $\mathrm{Tx_1} = \mathrm{Ax_0} = y_1 \cdot$ Now for this x_1 there exists $x_2 \in \mathrm{X}$ such that $\mathrm{S}x_2 = Bx_1 = y_2$ Similarly, we can inductively define $\mathrm{B}x_{2n-1} = Sx_{2n} = y_{2n}; \mathrm{A}x_{2n} = Tx_{2n+1} = y_{2n+1}$ for $n = 0, 1, 2, \ldots$ Now we prove $\{\mathrm{y_n}\}$ is a Cauchy sequence in X.

For this we consider

$$\begin{aligned}&\mathrm{d}\left(y_{2n+1}, y_{2n+2}\right) = \mathrm{d}\left(\mathrm{A}x_{2n}, \mathrm{B}x_{2n+1}\right) \leq \\ &\left(\max\left\{\begin{array}{c}\mathrm{d}\left(\mathrm{A}x_{2n}, \mathrm{Sx}_{2n}\right), \mathrm{d}\left(\mathrm{B}x_{2n+1}, \mathrm{T}x_{2n+1}\right), \\ \sqrt{[\mathrm{d}\left(\mathrm{A}x_{2n}, \mathrm{T}x_{2n+1}\right) \cdot \mathrm{d}\left(\mathrm{B}x_{2n+1}, \mathrm{S}x_{2n}\right)]}, \mathrm{d}\left(\mathrm{S}x_{2n}, \mathrm{T}x_{2n+1}\right)\end{array}\right\}\right)^{p} \cdot \\ &\left(\max\left\{\mathrm{d}\left(\mathrm{A}x_{2n}, \mathrm{S}x_{2n}\right), \mathrm{d}\left(\mathrm{B}x_{2n+1}, \mathrm{T}x_{2n+1}\right)\right\}\right)^{q} \cdot \left(\max\left\{\mathrm{d}\left(\mathrm{A}x_{2n}, \mathrm{T}x_{2n+1}\right), \mathrm{d}\left(\mathrm{B}x_{2n+1}, \mathrm{S}x_{2n}\right)\right\}\right)^{r}\end{aligned}$$

$$\begin{aligned}&\mathrm{d}\left(y_{2n+1}, y_{2n+2}\right) \leq \\ &\left[\begin{array}{c}\left(\max\left\{\mathrm{d}\left(y_{2n+1}, y_{2n}\right), \mathrm{d}\left(y_{2n+2}, y_{2n+1}\right), \sqrt{[\mathrm{d}\left(y_{2n+1}, y_{2n+1}\right) \cdot \mathrm{d}\left(y_{2n+2}, y_{2n}\right)]}, \mathrm{d}\left(y_{2n}, y_{2n+1}\right)\right\}\right)^{p} \\ \left(\max\left\{\mathrm{d}\left(y_{2n+1}, y_{2n}\right), \mathrm{d}\left(y_{2n+2}, y_{2n+1}\right)\right\}\right)^{q} \cdot \left(\max\left\{\mathrm{d}\left(y_{2n+1}, y_{2n+1}\right), \mathrm{d}\left(y_{2n+2}, y_{2n}\right)\right\}\right)^{r}\end{array}\right]\end{aligned}$$

$$\begin{aligned}&\mathrm{d}\left(y_{2n+1}, y_{2n+2}\right) \leq \\ &\left[\begin{array}{c}\left(\max\left\{\begin{array}{c}\mathrm{d}\left(y_{2n+1}, y_{2n}\right), \mathrm{d}\left(y_{2n+2}, y_{2n+1}\right), \sqrt{[\mathrm{d}\left(y_{2n+1}, y_{2n+1}\right) \cdot \mathrm{d}\left(y_{2n+1}, y_{2n}\right) \cdot \mathrm{d}\left(y_{2n+1}, y_{2n+2}\right)]}, \\ \mathrm{d}\left(y_{2n}, y_{2n+1}\right)\end{array}\right\}\right)^{p} \\ \left(\max\left\{\mathrm{d}\left(y_{2n+1}, y_{2n}\right), \mathrm{d}\left(y_{2n+2}, y_{2n+1}\right)\right\}\right)^{q} \\ \left(\max\left\{\mathrm{d}\left(y_{2n+1}, y_{2n+1}\right), \mathrm{d}\left(y_{2n+1}, y_{2n}\right) \cdot \mathrm{d}\left(y_{2n+1}, y_{2n+2}\right)\right\}\right)^{r}\end{array}\right].\end{aligned}$$

$$\mathrm{d}\left(y_{2n+1}, y_{2n+2}\right) \leq \left[\begin{array}{c}\left(\max\left\{\begin{array}{c}\mathrm{d}\left(y_{2n+1}, y_{2n}\right), \mathrm{d}\left(y_{2n+2}, y_{2n+1}\right), \\ \sqrt{[1.\mathrm{d}\left(y_{2n+2}, y_{2n}\right) \cdot \mathrm{d}\left(y_{2n+1}, y_{2n+2}\right)]}, \mathrm{d}\left(y_{2n}, y_{2n+1}\right)\end{array}\right.\right)^{p} \\ \left(\max\left\{\mathrm{d}\left(y_{2n+1}, y_{2n}\right), \mathrm{d}\left(y_{2n+2}, y_{2n+1}\right)\right\}\right)^{q} \\ \left(\max\left\{1, \mathrm{d}\left(y_{2n+2}, y_{2n}\right) \cdot \mathrm{d}\left(y_{2n+1}, y_{2n+2}\right)\right\}\right)^{r}\end{array}\right] \tag{2.1}$$

In(2.1), if $\mathrm{d}\left(y_{2n+2}, y_{2n+1}\right) > \mathrm{d}\left(y_{2n+1}, y_{2n}\right)$ for some positive integer n, then we have $\mathrm{d}\left(y_{2n+1}, y_{2n+2}\right) \leq \left(\mathrm{d}\left(y_{2n+1}, y_{2n+2}\right)\right)^{h}$, where $\mathrm{h} = \mathrm{p} + \mathrm{q} + 2\mathrm{r} < 1$, which is a contradiction.
Thus we have $\mathrm{d}\left(y_{2n+2}, y_{2n+1}\right) \leq \left(\mathrm{d}\left(y_{2n}, y_{2n+1}\right)\right)^{h}$.
Similarly, we have

$$\mathrm{d}\left(y_{2n}, y_{2n+1}\right) \leq \left(\mathrm{d}\left(y_{2n-1}, y_{2n}\right)\right)^{h}$$

Thus for every $\mathrm{n} \in \mathbb{N}, \mathrm{d}\left(y_{n}, y_{2n+1}\right) \leq \left(\mathrm{d}\left(y_{2n-1}, y_{2n}\right)\right)^{h}$.
Continue like this, we have

$$\mathrm{d}\left(y_{n}, y_{2n+1}\right) \leq \left(\mathrm{d}\left(y_{n-1}, y_{n}\right)\right)^{h} \leq \left(\mathrm{d}\left(y_{n-2}, y_{n-1}\right)\right)^{h^{2}} \leq \ldots \leq \left(\mathrm{d}\left(y_{0}, y_{1}\right)\right)^{h^{n}}$$

Let $\mathrm{m}, \mathrm{n} \in \mathrm{N}$ such that $\mathrm{m} > \mathrm{n}$, we get

$$\begin{aligned}\mathrm{d}\left(y_{m}, y_{n}\right) \leq &\mathrm{d}\left(y_{m}, y_{m-1}\right) \ldots \mathrm{d}\left(y_{n+1}, y_{n}\right) \\ &\leq \left(\mathrm{d}\left(y_{1}, y_{0}\right)\right)^{h^{m-1}+\cdots h^{n}} \\ &\leq \left(\mathrm{d}\left(y_{1}, y_{0}\right)\right)^{\frac{h^{n}}{1-h}} \to 1 \text{ as } \mathrm{m}, \mathrm{n} \to \infty\end{aligned}$$

Hence $\{y_n\}$ is a multiplicative Cauchy sequence. since X is complete so $\{y_n\} \to z$ in X.
Therefore, subsequence $\{\mathrm{S}x_{2n}\}, \{Bx_{2n-1}\}, \{Ax_{2n}\}, \{\mathrm{T}x_{2n-1}\}$ of $\{y_n\}$ also converges to z in X.
Now suppose S(X) is complete, therefore, let $\mathrm{w} \in S^{-1}\mathrm{z}$ then Sw = z First we claim that Aw = z.
Let if possible Aw $\neq$ z.
On putting $\mathrm{x} = \mathrm{w}, \mathrm{y} = x_{2n+1}$ in inequality (C_4), we get
$\mathrm{d}\left(\mathrm{Aw}, \mathrm{B}x_{2n+1}\right) \leq$

$$\left[\begin{array}{c}\left(\max\left\{d(Aw,Sw),d\left(Bx_{2n+1},Tx_{2n+1}\right),\sqrt{\left[d\left(Aw,Tx_{2n+1}\right)\cdot d\left(Bx_{2n+1},Sw\right)\right]},d\left(Sw,Tx_{2n+1}\right)\right\}\right)^{p}\\ \left(\max\left\{d(Aw,Sw),d\left(Bx_{2n+1},Tx_{2n+1}\right)\right\}\right)^{q}\cdot\left(\max\left\{d\left(Aw,Tx_{2n+1}\right),d\left(Bx_{2n+1},Sw\right)\right\}\right)^{r}\end{array}\right]$$

Letting $n\to\infty$, we have

$$d(Aw,z)\leq\left[\begin{array}{c}(\max\{d(Aw,z),d(z,z),\sqrt{[d(Aw,z)\cdot d(z,z)]},d(z,z)\})^{p}\\ (\max\{d(Aw,z),d(z,z)\})^{q}\cdot(\max\{d(Aw,z),d(z,z)\})^{r}\end{array}\right]$$

$d(Aw,z)\leq d(Aw,z)^{p+q+r}$, a contradiction. Hence $Aw=z$.

This implies, $z=Sw=Aw$.

Therefore, w is coincidence point of A and S.

$$\tag{2.2}$$

since $z=Aw\in A(X)\subseteq T(X)$, therefore, there exists $v\in X$ such that $z=Tv$

Next we claim that $Bv=z$.

Let if possible $Bv\neq z$. On putting $x=x_{2n}, y=v$ in inequality (C_4), we have

$$d\left(Ax_{2n},Bv\right)\leq\left[\begin{array}{c}\left(\max\left\{d\left(Ax_{2n},Sx_{2n}\right),d(Bv,Tv),\sqrt{\left[d\left(Ax_{2n},Tv\right)\cdot d\left(Bv,Sx_{2n}\right)\right]},d\left(Sx_{2n},Tv\right)\right\}\right)^{p}\\ \left(\max\left\{d\left(Ax_{2n},Sx_{2n}\right),d(Bv,Tv)\right\}\right)^{q}\cdot\left(\max\left\{d\left(Ax_{2n},Tv\right),d\left(Bv,Sx_{2n}\right)\right\}\right)^{r}\end{array}\right]$$

Letting $n\to\infty$, we have

$$d(z,Bv)=d\left(Ax_{2n},Bv\right)\leq\left[\begin{array}{c}(\max\{d(z,z),d(Bv,z),\sqrt{[d(z,z)\cdot d(Bv,z)]},d(z,z)\})^{p}\\ (\max\{d(z,z),d(Bv,z)\})^{q}\cdot(\max\{d(z,z),d(Bv,z)\})^{r}\end{array}\right]$$

$d(Bv,z)\leq d(Bv,z)^{p+q+r}$, a contradiction.

Therefore, $z=Tv=Bv$.

Hence v is coincidence point of B and T.

$$\tag{2.3}$$

Since the pairs (A,S) and (B,T) are weakly compatible and u sing(2.2), (2.3), we have

$$Sz=SAw=ASw=Az \tag{2.4}$$

and

$$Tz=TBv=BTv=Bz. \tag{2.5}$$

Next we claim that $Az=z$.

Let if possible $Az\neq z$ then using inequality (C_4) and on putting $x=z, y=x_{2n+1}$, we have

$$d\left(Az,Bx_{2n+1}\right)\leq$$

$$\left[\begin{array}{c}\left(\max\left\{d(Az,Sz),d\left(Bx_{2n+1},Tx_{2n+1}\right),\sqrt{\left[d\left(Az,Tx_{2n+1}\right)\cdot d\left(Bx_{2n+1},Sz\right)\right]},d\left(Sz,Tx_{2n+1}\right)\right\}\right)^{p}\\ \left(\max\left\{d(Az,Sz),d\left(Bx_{2n+1},Tx_{2n+1}\right)\right\}\right)^{q}\cdot\left(\max\left\{d\left(Az,Tx_{2n+1}\right),d\left(Bx_{2n+1},Sz\right)\right\}\right)^{r}\end{array}\right].$$

Letting $n\to\infty$, we have $d(Az,z)\leq\left[\begin{array}{c}(\max\{1,1,\sqrt{[d(Az,z)\cdot d(z,Az)]},d(Az,z)\})^{p}\\ (\max\{1,1\})^{q}\cdot(\max\{d(Az,z),d(z,Az)\})^{r}\end{array}\right]$

$d(Av,z)\leq d(Av,z)^{p+r}$, a contradiction.

So ,we have $Az=z$

Now using (2.4) , we have

$$Sz=Az=z \tag{2.6}$$

Next, we claim that $Bz=z$

Let if possible $Bz\neq z$ then on putting $x=x_{2n}, y=z$ in inequality (C_4), we have

$$d\left(Ax_{2n},Bz\right)\leq\left[\begin{array}{c}\left(\max\left\{d\left(Ax_{2n},Sx_{2n}\right),d(Bz,Tz),\sqrt{\left[d\left(Ax_{2n},Tz\right)\cdot d\left(Bz,Sx_{2n}\right)\right]},d\left(Sx_{2n},Tz\right)\right\}\right)^{p}\\ \left(\max\left\{d\left(Ax_{2n},Sx_{2n}\right),d(Bz,Tz)\right\}\right)^{q}\cdot\left(\max\left\{d\left(Ax_{2n},Tz\right),d\left(Bz,Sx_{2n}\right)\right\}\right)^{r}\end{array}\right].$$

Letting $n \to \infty$, we have

$$\mathrm{d(z,Bz)} \leq \left[\begin{array}{c} (\max\{1,1,\sqrt{[\mathrm{d(z,Bz)}\cdot \mathrm{d(Bz,z)}]},\mathrm{d(z,Bz)}\})^p \\ (\max\{1,1\})^q \cdot (\max\{\mathrm{d(z,Bz)},\mathrm{d(Bz,z)}\})^r \end{array} \right]$$

$\mathrm{d(Bz,z)} \leq \mathrm{d(Bz,z)^{p+r}}$, a contradiction.
Therefore, $Bz = z$.
Now, from (2.5) we conclude that

$$Tz = Bz = z. \tag{2.7}$$

Thus, from (2.6) and (2.7), z is the common fixed point of A, B, S and T.
The proofs for cases in which $\mathrm{A(X), B(X), T(X)}$, are assumed complete are similar to completeness of S(X), therefore omitted.
Uniqueness can easily follows from inequality (C_4).
Cor. 2.2 On putting $S = T$ then Theorem 2.1 can be written as
Theorem 2.3 Let (X, d) be a complete multiplicative metric space.
Let A, B, S be self-mappings of X into itself satisfying the following conditions
(C_5) $\mathrm{A(X) \subseteq S(X), B(X) \subseteq S(X)}$
(C_6) one of the subspace AX or BX or SX is complete
(C_7) the pairs (A, S) and (B, S) are weakly compatible
(C_8) $\mathrm{d(Ax,By)} \leq \left[\begin{array}{c} \mathrm{d(Ax,Sx),d(By,Sy)}, \\ (\max\{\sqrt{[\mathrm{d(Ax,Sy)}\cdot \mathrm{d(By,Sx)}]},\mathrm{d(Sx,Sy)} \end{array} \right\}\Bigg)^p$ for all $\mathrm{x, y} \in$ X, where $0 < \mathrm{h = p + q + 2r} < 1$(p, q and r are non-negative real numbers).
Then A, B, S have a unique common fixed point in X.
Cor. 2.4 On putting $\mathrm{A = B = I}$ then Theorem 2.1 can be written as
Theorem 2.5 Let (X, d) be a complete multiplicative metric space . Let S and T be self-mappings of X into itself satisfying the following conditions
(C_9) one of the subspace SX or TX is complete
(C_{10}) the pairs (S, T) is weakly compatible
(C_{11}) $\mathrm{d(Ax,By)} \leq \left[\begin{array}{c} \mathrm{d(Ax,Sx),d(By,Ty)}, \\ (\max\{\sqrt{[\mathrm{d(Ax,Ty)}\cdot \mathrm{d(By,Sx)}]},\mathrm{d(Sx,Ty)} \end{array} \right\}\Bigg)^p$ for all $\mathrm{x, y} \in$ X, where $0 < \mathrm{h = p + q + 2r} < 1$(p, q and r are non-negative real numbers).
Then A, B, S and T have a unique common fixed point in X.
Now we prove common fixed points for weakly compatible mappings on multiplicative metric spaces without completeness of space X as follow:
Theorem 2.6 Let A, B, S and T be self-mappings of a multiplicative metric space (X, d) satisfying the conditions
$(C_1), (C_2), (C_3)$ and (C_4)
Then A, B, S and T have unique common fixed point.
Proof. From Theorem 2.1, $\{y_n\}$ is a multiplicative Cauchy sequence.
Suppose S(X) is complete there exists $\mathrm{u} \in \mathrm{S(X)}$ such that
$y_{2n+1} = Sx_{2n+2} = \mathrm{B}x_{2n+1} \to u$ as $n \to \infty$
Consequently, we can find $v \in X$ such that

$$\mathrm{Sv = u.} \tag{2.8}$$

Since $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore, the sequence $\{y_n\}$ also converges, implying thereby, the convergence of $\{y_{2n}\}$ being a subsequence of the convergent sequence $\{y_n\}$.
Thus we have $y_{2n} = Ax_{2n+2} = \mathrm{T}x_{2n+1} \to u$ as $n \to \infty$
We claim $\mathrm{Av = u}$.

Let if possible $\mathrm{Av} \neq \mathrm{u}$ then putting $\mathrm{x}=\mathrm{v}, \mathrm{y}=x_{2n+1}$ in inequality (C_4) we get
$\mathrm{d}(\mathrm{Av}, y_{2n+1}) = \mathrm{d}(\mathrm{Av}, \mathrm{B}x_{2n+1}) <$

$$\left[\begin{array}{c} \left(\max\left\{\mathrm{d}(\mathrm{Av},\mathrm{Sv}), \mathrm{d}(\mathrm{B}x_{2n+1}, \mathrm{T}x_{2n+1}), \sqrt{[\mathrm{d}(\mathrm{Av}, \mathrm{T}x_{2n+1}) \cdot \mathrm{d}(\mathrm{B}x_{2n+1}, \mathrm{Sv})]}, \mathrm{d}(\mathrm{Sv}, \mathrm{T}x_{2n+1})\right\}\right)^p \\ (\max\{\mathrm{d}(\mathrm{Av},\mathrm{Sv}), \mathrm{d}(\mathrm{B}x_{2n+1}, \mathrm{T}x_{2n+1})\})^q \cdot (\max\{\mathrm{d}(\mathrm{Av}, \mathrm{T}x_{2n+1}), \mathrm{d}(\mathrm{B}x_{2n+1}, \mathrm{Sv})\})^r \end{array}\right]$$

Letting $\mathrm{n} \to \infty$, we have

$$\mathrm{d}(\mathrm{Av},\mathrm{u}) \leq \left[\begin{array}{c} (\max\{\mathrm{d}(\mathrm{Av},\mathrm{u}), \mathrm{d}(\mathrm{u},\mathrm{u}), \sqrt{[\mathrm{d}(\mathrm{Av},\mathrm{u}) \cdot \mathrm{d}(\mathrm{u},\mathrm{u})]}, \mathrm{d}(\mathrm{u},\mathrm{u})\})^p \\ (\max\{\mathrm{d}(\mathrm{Av},\mathrm{u}), \mathrm{d}(\mathrm{u},\mathrm{u})\})^q \cdot (\max\{\mathrm{d}(\mathrm{Av},\mathrm{u}), \mathrm{d}(\mathrm{u},\mathrm{u})\})^r \end{array}\right]$$

$\mathrm{d}(\mathrm{Av},\mathrm{u}) \leq \mathrm{d}(\mathrm{Av},\mathrm{u})^{\mathrm{p+q+r}}$, a contradiction. Hence

$$\mathrm{Sv} = \mathrm{Av} = \mathrm{u}. \tag{2.9}$$

Hence v is coincidence point of A and S.
Since $\mathrm{u} = \mathrm{Av} \in \mathrm{AX} \subseteq \mathrm{TX}$ there exists $\mathrm{w} \in \mathrm{X}$ such that

$$\mathrm{u} = \mathrm{Tw}. \tag{2.10}$$

We claim $\mathrm{Bw} = \mathrm{u}$.
Let if possible $\mathrm{Bw} \neq \mathrm{u}$ then on putting $\mathrm{x}=\mathrm{v}, \mathrm{y}=\mathrm{w}$ in inequality (C_4) we have
$\mathrm{d}(\mathrm{u},\mathrm{Bw}) = \mathrm{d}(\mathrm{Av},\mathrm{Bw}) \leq$

$$\left[\begin{array}{c} (\max\{\mathrm{d}(\mathrm{Av},\mathrm{Sv}), \mathrm{d}(\mathrm{Bw},\mathrm{Tw}), \sqrt{[\mathrm{d}(\mathrm{Av},\mathrm{Tw}) \cdot \mathrm{d}(\mathrm{Bw},\mathrm{Sv})]}, \mathrm{d}(\mathrm{Sv},\mathrm{Tw})\})^p \\ (\max\{\mathrm{d}(\mathrm{Av},\mathrm{Sv}), \mathrm{d}(\mathrm{Bw},\mathrm{Tw})\})^q \cdot (\max\{\mathrm{d}(\mathrm{Av},\mathrm{Tw}), \mathrm{d}(\mathrm{Bw},\mathrm{Sv})\})^r \end{array}\right]$$

$$\mathrm{d}(\mathrm{u},\mathrm{Bw}) = \mathrm{d}(\mathrm{Av},\mathrm{Bw}) \leq \left[\begin{array}{c} (\max\{\mathrm{d}(\mathrm{u},\mathrm{u}), \mathrm{d}(\mathrm{Bw},\mathrm{u}), \sqrt{[\mathrm{d}(\mathrm{u},\mathrm{u}) \cdot \mathrm{d}(\mathrm{Bw},\mathrm{u})]}, \mathrm{d}(\mathrm{u},\mathrm{u})\})^p \\ (\max\{\mathrm{d}(\mathrm{u},\mathrm{u}), \mathrm{d}(\mathrm{Bw},\mathrm{u})\})^q \cdot (\max\{\mathrm{d}(\mathrm{u},\mathrm{u}), \mathrm{d}(\mathrm{Bw},\mathrm{u})\})^r \end{array}\right]$$

$\mathrm{d}(\mathrm{u},\mathrm{Bw}) \leq \mathrm{d}(\mathrm{Bw},\mathrm{u})^{\mathrm{p+q+r}}$, a contradiction implies,

$$\mathrm{u} = \mathrm{Bw}. \tag{2.11}$$

Using(2.8) and (2.9), we get

$$\mathrm{u} = \mathrm{Av} = \mathrm{Sv} \tag{2.12}$$

i.e., v is coincidence point of A and S.
From (2.10) and (2.11), we have

$$\mathrm{u} = \mathrm{Bw} = \mathrm{Tw} \tag{2.13}$$

i.e., w is coincidence point of B and T
i.e., $\mathrm{Av} = \mathrm{Sv} = \mathrm{Bw} = \mathrm{Tw} = \mathrm{u}$ since the pairs (A,S) and (B,T) are weakly compatible then from (2.8),(2.9),(2.10) and (2.11) we have
$\mathrm{Su} = \mathrm{S}(\mathrm{Av}) = \mathrm{A}(\mathrm{Sv}) = \mathrm{Au} = \mathrm{w}_1(\text{ say })$ $\mathrm{Tu} = \mathrm{T}(\mathrm{Bw}) = \mathrm{B}(\mathrm{Tw}) = \mathrm{Bu} = \mathrm{w}_2(\text{ say })$
From inequality (C_4), we have
From (2.10) and (2.11), we have $\mathrm{u} = \mathrm{Bw} = \mathrm{Tw}$ i.e., w is coincidence point of B and T i.e., $\mathrm{Av} = \mathrm{Sv} = \mathrm{Bw} = \mathrm{Tw} = \mathrm{u}$ since the pairs (A,S) and (B,T) are weakly compatible then from (2.8),(2.9),(2.10) and (2.11) we have $\mathrm{Su} = \mathrm{S}(\mathrm{Av}) = \mathrm{A}(\mathrm{Sv}) = \mathrm{Au} = \mathrm{w}_1(\text{ say })$ and $\mathrm{Tu} = \mathrm{T}(\mathrm{Bw}) = \mathrm{B}(\mathrm{Tw}) = \mathrm{Bu} = \mathrm{w}_2(\text{ say })$
From inequality (C_4), we have
$\mathrm{d}(\mathrm{w}_1, \mathrm{w}_2) = \mathrm{d}(\mathrm{Au},\mathrm{Bu}) \leq$

$$\left[\begin{array}{c} (\max\{\mathrm{d}(\mathrm{Au},\mathrm{Su}), \mathrm{d}(\mathrm{Bu},\mathrm{Tu}), \sqrt{[\mathrm{d}(\mathrm{Au},\mathrm{Tu}) \cdot \mathrm{d}(\mathrm{Bu},\mathrm{Su})]}, \mathrm{d}(\mathrm{Su},\mathrm{Tu})\})^p \\ (\max\{\mathrm{d}(\mathrm{Au},\mathrm{Su}), \mathrm{d}(\mathrm{Bu},\mathrm{Tu})\})^q \cdot (\max\{\mathrm{d}(\mathrm{Au},\mathrm{Tu}), \mathrm{d}(\mathrm{Bu},\mathrm{Su})\})^r \end{array}\right].$$

$\mathrm{d}(\mathrm{w}_1, \mathrm{w}_2) = \mathrm{d}(\mathrm{Au},\mathrm{Bu}) \leq$

$$\left[\begin{array}{c} (\max\{\mathrm{d}(\mathrm{Au},\mathrm{Su}), \mathrm{d}(\mathrm{Bu},\mathrm{Tu}), \sqrt{[\mathrm{d}(\mathrm{Au},\mathrm{Tu}) \cdot \mathrm{d}(\mathrm{Bu},\mathrm{Su})]}, \mathrm{d}(\mathrm{Su},\mathrm{Tu})\})^p \\ (\max\{\mathrm{d}(\mathrm{Au},\mathrm{Su}), \mathrm{d}(\mathrm{Bu},\mathrm{Tu})\})^q \cdot (\max\{\mathrm{d}(\mathrm{Au},\mathrm{Tu}), \mathrm{d}(\mathrm{Bu},\mathrm{Su})\})^r \end{array}\right].$$

$$\mathrm{d}\left(\mathrm{w}_1, \mathrm{w}_2\right) \leq \left[\begin{array}{c} \left(\max\left\{\mathrm{d}\left(w_1, w_1\right), \mathrm{d}\left(w_2, w_2\right), \sqrt{\left[\mathrm{d}\left(w_1, w_2\right) \cdot \mathrm{d}\left(w_2, w_1\right)\right]}, \mathrm{d}\left(w_1, w_2\right)\right\}\right)^p \\ \left(\max\left\{\mathrm{d}\left(w_1, w_1\right), \mathrm{d}\left(w_2, w_2\right)\right\}\right)^q \cdot \left(\max\left\{\mathrm{d}\left(w_1, w_2\right), \mathrm{d}\left(w_2, w_1\right)\right\}\right)^r \end{array}\right]$$

$\mathrm{d}\left(w_1, w_2\right) \leq \mathrm{d}\left(w_1, w_2\right)^{\mathrm{p+r}}$, a contradiction. i.e., $w_1 = w_2$.
Therefore, we have

$$\mathrm{Su} = \mathrm{Au} = \mathrm{Tu} = \mathrm{Bu} \tag{2.14}$$

Again using inequality (C_4) and u sing(2.14), we have on putting $\mathrm{x} = \mathrm{v}, \mathrm{y} = \mathrm{u}$ in inequality (C_4), we have

$$\mathrm{d(Av, Bu)} \leq \left[\begin{array}{c} (\max\{\mathrm{d(Av, Sv)}, \mathrm{d(Bu, Tu)}, \sqrt{[\mathrm{d(Av, Tu)} \cdot \mathrm{d(Bu, Sv)}]}, \mathrm{d(Sv, Tu)}\})^p \\ (\max\{\mathrm{d(Av, Sv)}, \mathrm{d(Bu, Tu)}\})^q \cdot (\max\{\mathrm{d(Av, Tu)}, \mathrm{d(Bu, Sv)}\})^r \end{array}\right]$$

$$\mathrm{d(Av, Bu)} \leq \left[\begin{array}{c} (\max\{1, 1, \sqrt{[\mathrm{d(Av, Bu)} \cdot \mathrm{d(Bu, Av)}]}, \mathrm{d(Av, Bu)}\})^p \\ (\max\{1, 1\})^q \cdot (\max\{\mathrm{d(Av, Bu)}, \mathrm{d(Bu, Av)}\})^r \end{array}\right]$$

$\mathrm{d(Av, Bu)} \leq \mathrm{d(Av, Bu)}^{\mathrm{p+r}}$, a contradiction.
This implies that $\mathrm{Av} = \mathrm{Bu}$, that is, $\mathrm{u} = \mathrm{Bu}$.
Therefore, we have $u = Tu = Su = Au = Bu$.
Hence u is a common fixed point of $\mathrm{A, B, S}$ and T.
Now we prove that u is the common fixed point of $\mathrm{A, B, S}$ and T.
The proof for cases in which A(X), B(X), T(X) is complete are similar and are therefore omitted.
The following corollary follows immediately from Theorem 2.1 and 2.6.

Corollary 2.7. Let (X, d) be a complete multiplicative metric space. Let $\mathrm{A, B, S}$ and T be self- mappings of X into itself satisfying conditions $(C_1), (C_2), (C_3)$.
Suppose that
(C_{12})

$$\mathrm{d(Ax, By)} \leq \left[\begin{array}{c} (\max\{\mathrm{d(Ax, Sx)}, \mathrm{d(By, Ty)}, \sqrt{[\mathrm{d(Ax, Ty)}]}, \sqrt{[\mathrm{d(By, Sx)}]}, \mathrm{d(Sx, Ty)}\})^p \\ (\max\{\mathrm{d(Ax, Sx)}, \mathrm{d(By, Ty)}\})^q \cdot (\max\{\mathrm{d(Ax, Ty)}, \mathrm{d(By, Sx)}\})^r \end{array}\right]$$

or all $\mathrm{x, y} \in \mathrm{X}$, where $0 < \mathrm{h} = \mathrm{p} + \mathrm{q} + 2\mathrm{r} < 1$($\mathrm{p, q}$ and r are non-negative real numbers).
Then $\mathrm{A, B, S}$ and T have a unique common fixed point in X.

Example 2.8 Let $X = [0, \infty)$ and (X, d) be multiplicative metric space defined by $\mathrm{d(x, y)} = e^{|x-y|}$ for all $\mathrm{x, y} \in \mathrm{X}$. Define self maps $\mathrm{A, B, S}$ and T on X by $\mathrm{Ax} = \frac{1}{2}x, \mathrm{Bx} = \mathrm{x}\ \mathrm{Sx} = 3\mathrm{x}, \mathrm{Tx} = 2\mathrm{x}$.
Then the self maps $\mathrm{A, B, S}$ and T satisfy $(C_1), (C_2), (C_3)$ and (C_4) conditions of Theorem 2.2 and have a unique common fixed point at $x = 0$

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