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HYPERCYCLIC TUPLES OF MATRICES ON $\mathbb{C}^n$

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Abstract. This paper extends the results due to Costakis et al. [7] by showing the existence of Hypercyclic (non diagonalizable) n-tuple of matrices in Jordan form on $\mathbb{C}^n$. In doing so, we modify some lemmas from 2×2 matrices in Jordan form to 2×2 matrices in lower triangular form and to 3×3 matrices in Jordan form and some propositions and theorems from $\mathbb{R}$ to $\mathbb{C}$. Some examples are provided to illustrate the results.

1. Introduction and Preliminaries

The study of dynamics of linear operators is done by considering dynamical systems with dense orbits. In the presence of linearity, such systems are called “Hypercyclic”. The theory of Hypercylicity originated from the study of cyclic vectors. The cyclic vectors in turn have played a great role in the study of a long standing famous problem known as the Invariant subspace problem. In this paper, we will consider the dynamics of non-diagonal matrices. The following note is important about the characteristics of non-diagonal matrices: There are non diagonalizable matrices for which the number of linearly independent eigenvectors is less than the dimension of the matrix. This means that algebraic multplicity is greater than the geometric multiplicity for such matrices and thus they fail to obey the condition of $S^{-1}AS$, where A is the given matrix and S is the matrix formed by eigenvectors of A. Also, a non-diagonalizable matrix exists if the matrix is upper triangular (Jordan form) or lower triangular with repeated eigenvalues.

For more literature on this topic one can see the papers due to Ansari [1], Ayadi [3], Bayart and Costakiis [4], Bayart and Matheron [5], Bourdon and Feldman [6], Costakis and Hadjiloucas [8] Costackis and Peris [9], Peris [11], Feldman [13], Grosse-Erdmann [14], Javeheri [15] and Leon- Saavedra [16].

In 1969, Rolewicz [12] gave the concepts of orbit and dense orbit in his work. The definitions are as follows :

Definition 1.1. [12] *Suppose that T is continuous linear operator on a topological vector space X over a field. For an element $x \in X$, the Orbit of x under T is $Orb(T;x) = \{x, Tx, T^2x, \ldots\}$ where $x \in X$ is a fixed vector.*

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Definition 1.2. [2] *A continuous linear operator T on topological vector space X is said to be Hypercyclic if there is a vector $x \in X$ whose orbit under T, $Orb(T, x) = \{x, T^1x, T^2x, ...\}$ is dense in X. In this case $x \in X$ is said to be Hypercyclic vector for T.*

For example, if $A_1, A_2, \ldots, A_n$ are 2×2 matrices, then the sequence $(A_1, A_2, ..., A_n)$ is n-tuple of 2×2 matrices.

Following are different forms of 2×2 non-diagonalizable matrices which will be useful in this study:

$$A_1 = \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \quad A_2 = \begin{pmatrix} a_1 & 0 \\ b & a_2 \end{pmatrix} \quad \text{where } a_1 = a_2 \neq 0 \text{ and}$$
$$A_3 = \begin{pmatrix} a & -\lambda_1 \\ \lambda_2 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_2 & a \end{pmatrix}$$
$$A_5 = \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_2 & a \end{pmatrix} \quad A_6 = \begin{pmatrix} a & \lambda_1 \\ -\lambda_2 & 0 \end{pmatrix}$$

where $a \neq 0$, $\lambda_1 = \lambda_2$ and $\lambda_1 + \lambda_2 = a$.

The following result is due to Costakis *et al.* [7] which gives the necessary and sufficient conditions for $m-$ tuple to be hypercylic.

Lemma 1.1. [7]
Let m be a positive integer and for each $j = 1, 2, ..., m$, let A_j be a 2×2 matrix in Jordan form over a field $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, i.e. $A_j = \begin{pmatrix} a_j & 1 \\ 0 & a_j \end{pmatrix}$

for $a_1, a_2, ..., a_m \in \mathbb{F}$. Then $(A_1, A_2, ..., A_m)$ over $\mathbb{C}$ (respectively $\mathbb{R}$) is hypercyclic if and only if the sequence

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + ... + \frac{k_m}{a_m} \\ a_1^{k_1} a_2^{k_2} ... a_m^{k_m} \end{pmatrix} : k_1, k_2, ..., k_m \in \mathbb{N} \right\}.$$

is dense in $\mathbb{C}^2$ (respectively $\mathbb{R}^2$).

The following theorem is due to Costakis *et al.* [7] and describes the existence of a hypercyclic 4-tuple of 2×2 matrices in Jordan form over $\mathbb{R}$.

Theorem 1.1. [7]
There exist 2×2 matrices $A_j, j = 1, 2, 3, 4$, in Jordan form over $\mathbb{R}$ such that (A_1, A_2, A_3, A_4) is hypercyclic. In particular

$$HC(A_1, A_2, A_3, A_4) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} :\in \mathbb{R}^2 : x_2 \neq 0 \right\}.$$

The following proposition is due to Costakis *et al.* [7] which shows the existence of hypercyclic 3-tuple over $\mathbb{R}$.

Proposition 1.1. [7]
There exist 3-tuples (A_1, A_2, A_3) of 3×3 matrices over $\mathbb{R}$ such that (A_1, A_2, A_3) is hypercyclic.

2. Main Results

In this section, we will extend Lemma 1.1 of [7] and Theorem 1.1 of [7] above from $\mathbb{R}$ to $\mathbb{C}$. Our results are primarily divided into two parts: the extension of the results due to Costakis *et al.* [7]: (Lemma 1.1 above and Theorem 1.1 above) to other forms of matrices (2×2 matrices in lower triangular form; 3×3 matrix in Jordan form (upper triangular matrix)) from $\mathbb{R}$ to $\mathbb{C}$.

The extension of Lemma 1.1 stated as above to 2×2 matrices in lower triangular form is stated below:

Lemma 2.1. *Let m be a positive integer and for each $j = 1, 2, ..., m$, let A_j be a 2×2 matrix. Let $A_j^{k_j} = \begin{pmatrix} a_j & 0 \\ 1 & a_j \end{pmatrix}$ for $a_1, a_2, ..., a_m \in \mathbb{F}$ be lower triangular matrices, then the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + ... + \frac{k_m}{a_m} \\ a_1^{k_1} a_2^{k_2} ... a_m^{k_m} \end{pmatrix} : k_1, k_2, ..., k_m \in \mathbb{N} \right\}.$$

is dense in $\mathbb{C}^2$ (respectively $\mathbb{R}^2$) whenever $(A_1^{k_1}, A_2^{k_2}, ..., A_m^{k_m})$ over $\mathbb{C}$ (respectively $\mathbb{R}$) is hypercyclic.

Proof. We consider the case when the field $\mathbb{F} = \mathbb{C}$; the other case is dealt similarly. Let $A_j^{k_j} = \begin{pmatrix} a_j^{k_j} & 0 \\ k_j a_j^{k_j - 1} & a_j^{k_j} \end{pmatrix}$, where $k_j \in \mathbb{N}$.

Then,

$$A_1^{k_1} A_2^{k_2} = \begin{pmatrix} a_1^{k_1} & 0 \\ k_1 a_1^{k_1-1} & a_1^{k_1} \end{pmatrix} \times \begin{pmatrix} a_2^{k_2} & 0 \\ k_2 a_2^{k_2-1} & a_2^{k_2} \end{pmatrix} = \begin{pmatrix} a_1^{k_1} a_2^{k_2} & 0 \\ k_1 a_1^{k_1-1} a_2^{k_2} + k_2 a_2^{k_2-1} a_1^{k_1} & a_1^{k_1} a_2^{k_2} \end{pmatrix}.$$

In general, one has

$$A_1^{k_1} A_2^{k_2} A_3^{k_3} ... A_m^{k_m} = \begin{pmatrix} \prod_{j=1}^{m} a_j^{k_j} & 0 \\ \prod_{j=1}^{m} a_j^{k_j} \sum_{j=1}^{m} \frac{k_j}{a_j} & \prod_{j=1}^{m} a_j^{k_j} \end{pmatrix}.$$

Now, assume that $(A_1, A_2, ..., A_m)$ is hypercyclic m-tuple and let $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$ be hypercyclic vector for $(A_1, A_2, ..., A_m)$, then the sequence

$$({A_1}^{k_1} {A_2}^{k_2} ... {A_m}^{k_m} : k_1, k_2, ... k_m \in \mathbb{N}) \times \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= \begin{pmatrix} \prod_{j=1}^{m} a_j^{k_j} & 0 \\ \prod_{j=1}^{m} a_j^{k_j} \sum_{j=1}^{m} \frac{k_j}{a_j} & \prod_{j=1}^{m} a_j^{k_j} \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= \begin{pmatrix} z_1 \prod_{j=1}^{m} a_j^{k_j} \\ z_1 \prod_{j=1}^{m} a_j^{k_j} \sum_{j=1}^{m} \frac{k_j}{a_j} + z_2 \prod_{j=1}^{m} a_j^{k_j} \end{pmatrix} : k_1, k_2, ..., k_m \in \mathbb{N} \text{ is dense in } \mathbb{C}^2.$$

We know that $z_2 \neq 0$. Hence, upon dividing the element in the first row by that in the second row, we obtain that the sequence

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + ... + \frac{k_m}{a_m} \\ a_1^{k_1} a_2^{k_2} ... a_m^{k_m} \end{pmatrix} : k_1, k_2, ..., k_m \in \mathbb{N} \right\}.$$

is dense in $\mathbb{C}^2$. □

Now, we will extend Lemma 1.1 to 3×3 matrices in Jordan form (upper triangular matrix).

Consider the case when $\mathbb{F} = \mathbb{C}$. Observe that,

$$A_1^{k_1} = \begin{pmatrix} a_1^{k_1} & k_1 a_1^{k_1-1} & k_1(k_1-1) a_1^{k_1-2} \\ 0 & a_1^{k_1} & k_1 a_1^{k_1-1} \\ 0 & 0 & a_1^{k_1} \end{pmatrix}$$

for $k_i \in \mathbb{N}$,

$$\Rightarrow A_1^{k_1} A_2^{k_2} = \begin{pmatrix} a_1^{k_1} & k_1 a_1^{k_1-1} & k_1(k_1-1) a_1^{k_1-2} \\ 0 & a_1^{k_1} & k_1 a_1^{k_1-1} \\ 0 & 0 & a_1^{k_1} \end{pmatrix} \times \begin{pmatrix} a_2^{k_2} & k_2 a_2^{k_2-1} & k_2(k_2-1) a_2^{k_2-2} \\ 0 & a_2^{k_2} & k_2 a_2^{k_2-1} \\ 0 & 0 & a_2^{k_2} \end{pmatrix}$$

$$= \begin{pmatrix} a_1^{k_1} a_2^{k_2} & k_2 a_1^{k_1} a_2^{k_2-1} + k_1 a_1^{k_1-1} a_2^{k_2} & k_2(k_2-1) a_1^{k_1} a_2^{k_2-2} + k_1 k_2 a_1^{k_1-1} a_2^{k_2-1} + k_1(k_1-1) a_1^{k_1-2} a_2^{k_2} \\ 0 & a_1^{k_1} a_2^{k_2} & k_2 a_1^{k_1} a_2^{k_2-1} + k_1 a_1^{k_1-1} a_2^{k_2} \\ 0 & 0 & a_1^{k_1} a_2^{k_2} \end{pmatrix}$$

$$= \begin{pmatrix} {a_1}^{k_1}{a_2}^{k_2} & {a_1}^{k_1}{a_2}^{k_2}(\frac{k_2}{a_2}+\frac{k_1}{a_1}) & {a_1}^{k_1}{a_2}^{k_2}(\frac{k_1(k_1-1)}{{a_1}^2}+\frac{k_2(k_2-1)}{{a_2}^2}+\frac{k_1k_2}{a_1a_2} \\ 0 & {a_1}^{k_1}{a_2}^{k_2} & {a_1}^{k_1}{a_2}^{k_2}(\frac{k_2}{a_2}+\frac{k_1}{a_1}) \\ 0 & 0 & {a_1}^{k_1}{a_2}^{k_2} \end{pmatrix}.$$

In general, we obtain

$$A_1^{k_1}A_2^{k_2}A_3^{k_3}...A_m^{k_m} = \begin{pmatrix} \prod_{j=1}^m a_j^{k_j} & \prod_{j=1}^m a_j^{k_j}\sum_{j=1}^m \frac{k_j}{a_j} & \prod_{j=1}^m a_j^{k_j}\sum_{j=1,s=2}^m \frac{k_j!}{{a_j}^s}+\prod_{j=1}^m \frac{k_j}{a_j} \\ 0 & \prod_{j=1}^m a_j^{k_j} & \prod_{j=1}^m a_j^{k_j}\sum_{j=1}^m \frac{k_j}{a_j} \\ 0 & 0 & \prod_{j=1}^m a_j^{k_j} \end{pmatrix}.$$

Now, let $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{C}^3$ be hypercyclic vector for $(A_1, A_2, ..., A_m)$, then the sequence

$({A_1}^{k_1}{A_2}^{k_2}...{A_m}^{k_m} : k_1, k_2, ...k_m \in \mathbb{N}) \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ becomes

$$\begin{pmatrix} \prod_{j=1}^m a_j^{k_j} & \prod_{j=1}^m a_j^{k_j}\sum_{j=1}^m \frac{k_j}{a_j} & \prod_{j=1}^m a_j^{k_j}\sum_{j=1,s=2}^m \frac{k_j!}{{a_j}^s}+\prod_{j=1}^m \frac{k_j}{a_j} \\ 0 & \prod_{j=1}^m a_j^{k_j} & \prod_{j=1}^m a_j^{k_j}\sum_{j=1}^m \frac{k_j}{a_j} \\ 0 & 0 & \prod_{j=1}^m a_j^{k_j} \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$= \begin{pmatrix} z_1\prod_{j=1}^m a_j^{k_j} + z_2\prod_{j=1}^m a_j^{k_j}\sum_{j=1}^m \frac{k_j}{a_j} + z_3(\prod_{j=1}^m a_j^{k_j}\sum_{j=1,s=2}^m \frac{k_j!}{{a_j}^s}+\prod_{j=1}^m \frac{k_j}{a_j}) \\ z_2\prod_{j=1}^m a_j^{k_j} + z_3\prod_{j=1}^m a_j^{k_j}\sum_{j=1}^m \frac{k_j}{a_j} \\ z_3\prod_{j=1}^m a_j^{k_j} \end{pmatrix}$$

such that $k_1, k_2, ..., k_m \in \mathbb{N}$ is dense in $\mathbb{C}^3$.

This implies that $z_3 \neq 0$, hence dividing the elements in the first row and the second row and by that in the third row, we conclude that the sequence

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + ... + \frac{k_m}{a_m} + \prod_{j=1}^{m} \frac{k_j}{a_j} \\ \\ \frac{k_1}{a_1} + \frac{k_2}{a_2} + ... + \frac{k_m}{a_m} \\ \\ a_1^{k_1} a_2^{k_2} ... a_m^{k_m} \end{pmatrix} : k_1, k_2, ..., k_m \in \mathbb{N} \right\}.$$

is dense in $\mathbb{C}^3$.

Remark 2.1. *Let m be a positive integer and for each $j = 1, 2, ..., m$ let A_j be a 3×3 matrix in Jordan form over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. By the Lemma 2.1 due to Costakis et al.* [7] *it is immediate that whenever $(A_1, A_2, ..., A_m)$ over $\mathbb{C}$ (respectively $\mathbb{R}$) is hypercyclic, one can completely describe the set of hypercyclic vectors as*

$$HC(A_1, A_2, A_3, A_4) = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} :\in \mathbb{C}^3 : z_3 \neq 0 \right\}.$$

Next is the extension of Theorem 2.1 from the field $\mathbb{R}$ to the field of $\mathbb{C}$ as stated below:

Theorem 2.1. *There exist 2×2 matrices $A_j, j = 1, 2, 3, 4$, in Jordan form over $\mathbb{C}$ such that (A_1, A_2, A_3, A_4) is hypercyclic. In particular*

$$HC(A_1, A_2, A_3, A_4) = \left\{ \begin{pmatrix} z_1 \\ \\ z_2 \end{pmatrix} :\in \mathbb{C}^2 : z_2 \neq 0 \right\}.$$

Proof. Let $A_1 = \begin{pmatrix} a_1 & b_1 \\ \\ 0 & a_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ \\ 0 & a_2 \end{pmatrix}$, $A_3 = \begin{pmatrix} a_3 & b_3 \\ \\ 0 & a_3 \end{pmatrix}$, $A_4 = \begin{pmatrix} a_4 & b_4 \\ \\ 0 & a_4 \end{pmatrix}$ and suppose that $z = \begin{pmatrix} z_1 \\ \\ z_2 \end{pmatrix} \in \mathbb{C}^2$ is the hypercyclic vector for (A_1, A_2, A_3, A_4),

then

$$A_1 A_2 A_3 A_4 \begin{pmatrix} z_1 \\ \\ z_2 \end{pmatrix} =$$

$$\begin{pmatrix} a_1 & b_1 \\ \\ 0 & a_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 \\ \\ 0 & a_2 \end{pmatrix} \times \begin{pmatrix} a_3 & b_3 \\ \\ 0 & a_3 \end{pmatrix} \times \begin{pmatrix} a_4 & b_4 \\ \\ 0 & a_4 \end{pmatrix} \times \begin{pmatrix} z_1 \\ \\ z_2 \end{pmatrix}$$

$$= \begin{pmatrix} z_1 a_1 a_2 a_3 a_4 + z_2 [a_1 a_2 a_3 b_4 + a_1 a_3 a_4 b_2 + a_2 a_3 a_4 b_1] \\ \\ z_2 a_1 a_2 a_3 a_4 \end{pmatrix} \in \mathbb{C}^2,$$

where $z_2 \neq 0$, is dense in $\mathbb{C}^2$, hence (A_1, A_2, A_3, A_4) is hypercyclic.

In general, for an n-tuple of 2×2 matrices in Jordan form (upper triangular matrix) we have,

$$A_1^{k_1}A_2^{k_2}A_3^{k_3}...A_n^{k_n} = \begin{pmatrix} \prod_{j=1}^{m} a_j^{k_j} & \prod_{j=1}^{m} a_j^{k_j} \sum_{j=1}^{m} \frac{k_j}{a_j} \\ 0 & \prod_{j=1}^{m} a_j^{k_j} \end{pmatrix}.$$

Hence, by letting $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$, be hypercyclic vector for $(A_1, A_2, \ldots, A_n)$. Then we have, $A_1^{k_1}A_2^{k_2}A_3^{k_3}...A_n^{k_n}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \prod_{j=1}^{m} a_j^{k_j} & \prod_{j=1}^{m} a_j^{k_j} \sum_{j=1}^{m} \frac{k_j}{a_j} \\ 0 & \prod_{j=1}^{m} a_j^{k_j} \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$= \begin{pmatrix} z_1 \prod_{j=1}^{m} a_j^{k_j} + z_2 \prod_{j=1}^{m} a_j^{k_j} \sum_{j=1}^{m} \frac{k_j}{a_j} \\ z_2 \prod_{j=1}^{m} a_j^{k_j} \end{pmatrix}$$

□

where, $z_2 \neq 0$ is dense in $\mathbb{C}^2$, thus $(A_1, A_2, A_3....A_n)$ is hypercyclic.

Remark 2.2. *While the above result holds for matrices in Jordan form, the result remains true for general matrices over $\mathbb{R}$. The next result demonstrates this observation.*

We extend proposition 1.1, stated as above from the field $\mathbb{R}$ to $\mathbb{C}$.

Theorem 2.2. *There exist 3-tuples (A_1, A_2, A_3) of 3×3 matrices over $\mathbb{C}$ such that (A_1, A_2, A_3) is hypercyclic..*

Proof. We let $a \in \mathbb{C}$ and $b, c, d \in \mathbb{R}$ such that the sequence

$$\left\{ \begin{pmatrix} a^n b^m \\ c^n d^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}.$$

is dense in $\mathbb{C}$. Write $a = |a|e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} |a|cos\theta & 0 & 0 \\ |a|sin\theta & |a|cos\theta & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$A_2 = \begin{pmatrix} b & 0 & 0 \\ 1 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and } A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

$$A_1A_2A_3 = \begin{pmatrix} |a|cos\theta & 0 & 0 \\ |a|sin\theta & |a|cos\theta & 0 \\ 0 & 0 & c \end{pmatrix} \times \begin{pmatrix} b & 0 & 0 \\ 1 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

then we have

$$A_1^n A_2^m A_3^l = \begin{pmatrix} b^m|a|^n cos(n\theta) & 0 & 0 \\ b^m|a|^n sin(n\theta) + |a|^n(1+b^m)cos(n\theta) & b^m|a|^n cos(n\theta) & 0 \\ 0 & 0 & c^n d^l \end{pmatrix} \text{ which}$$

in turn gives

$$A_1^n A_2^m A_3^l \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b^m|a|^n cos(n\theta) & 0 & 0 \\ b^m|a|^n sin(n\theta) + |a|^n(1+b^m)cos(n\theta) & b^m|a|^n cos(n\theta) & 0 \\ 0 & 0 & c^n d^l \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$A_1^n A_2^m A_3^l \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b^m|a|^n cos(n\theta) \\ b^m|a|^n sin(n\theta) + |a|^n(1+b^m)cos(n\theta) \\ c^n d^l \end{pmatrix}$$

is dense in $\mathbb{C}^3$. Choosing the appropriate choices of a, b, c, d implies that (A_1, A_2, A_3) is hypercyclic with $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ being a hypercyclic vector for (A_1, A_2, A_3). □

To verify our results we give the following examples of hypercyclic n-tuples on $\mathbb{C}^n$. We only consider the case when n = 2, 3 since the general case is treated similarly.

Example 2.1. *Consider three non-diagonalizable* 2×2 *matrices in Lower triangular form over* $\mathbb{R}$,

$A = \begin{pmatrix} a_1 & 0 \\ b_1 & a_2 \end{pmatrix}$ *where* $a_1 = a_2 \neq 0$ *and* $b \neq 0$.

Let $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$.

By using Lemma 2.1 we can test the existence of hypercyclic tuples.

Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} i \\ 2i \end{pmatrix} \in \mathbb{C}^2$ *be hypercyclic vector for* (A_1, A_2, A_3). *Then,*

$$A_1A_2A_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

Hence, we have $A_1A_2A_3 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} i \\ 2i \end{pmatrix}$

$= \begin{pmatrix} 2i \\ 11i \end{pmatrix}$ *is dense in* $\mathbb{C}^2$. *Since* $11i \neq 0$, *then the tuple* (A_1, A_2, A_3) *is hypercyclic.*

Example 2.2. *Consider three* 2×2 *matrices in lower triangular form over* $\mathbb{C}$

$$A_1 = \begin{pmatrix} 2 & i \\ 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix} \quad \text{and}$$

let $\begin{pmatrix} i \\ 2i \end{pmatrix} \in \mathbb{C}^2$ *be hypercyclic vector for* (A_1, A_2, A_3). *Then, we have*

$$A_1A_2A_3 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2 & i \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} i \\ 2i \end{pmatrix}$$

$= \begin{pmatrix} -14 + 2i \\ 4i \end{pmatrix} \in \mathbb{C}^2$ *is dense in* $\mathbb{C}^2$, *hence* (A_1, A_2, A_3) *is hypercyclic.*

Example 2.3. *Consider 3-tuple* (A_1, A_2, A_3) *of* 3×3 *non-diagonalizable matrices where, two are in upper triangular form and one which is neither upper nor lower triangular matrix as follows:*

$$A_1 = \begin{pmatrix} -4 & 5 & 5 \\ -5 & 6 & 5 \\ -5 & 5 & 6 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} i & 2i & 3i \\ 0 & 2i & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Remark 2.3. A_1 *is non-diagonalizable matrix because it has repeated eigenvalues.*

Now, Let $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ *be hypercyclic vector for* (A_1, A_2, A_3)

then, $A_1A_2A_3 = \begin{pmatrix} -4 & 5 & 5 \\ -5 & 6 & 5 \\ -5 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix} \times \begin{pmatrix} i & 2i & 3i \\ 0 & 2i & 0 \\ 0 & 0 & i \end{pmatrix} = \begin{pmatrix} -4i & 2i & 15i \\ -5i & 2i & 14i \\ -5i & -6i & 13i \end{pmatrix}.$

$$So,\ A_1A_2A_3\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}-4i & 2i & 15i\\-5i & 2i & 14i\\-5i & -6i & 13i\end{pmatrix}\times\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}11i\\9i\\8i\end{pmatrix}\quad is\ dense\ in\ \mathbb{C}^3.$$

Since the third coordinate, $8i \neq 0$, this implies that (A_1, A_2, A_3) is hypercyclic.

For the last example, we will need the following proposition.

Proposition 2.1. [10] *Let $\mathbb{D}$ denote the open unit disk centered at 0 in the complex plane. If $b \in \mathbb{D}\backslash\{0\}$, then there exists a dense set $\triangle \subset \mathbb{D}$ such that for every $a \in \triangle$ the sequence $a^n b^m : n, m \in \mathbb{N}$ dense in $\mathbb{C}$.*

Example 2.4. *The above proposition implies that there exist $a \in \mathbb{R}\backslash\mathbb{Q}$ and $b \in \mathbb{C}$ such that the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$is dense in $\mathbb{C}$. Indeed, write $b = |b|e^{i\theta}$ and set*

$$A_1=\begin{pmatrix}a & 0\\1 & a\end{pmatrix},\ A_2=\begin{pmatrix}|b|cos\theta & -|b|sin\theta\\|b|sin\theta & |b|cos\theta\end{pmatrix}\ where\ a\neq 0.$$

Then we have another form of non diagonalizable matrices. Now,

$$A_1^nA_2^m=\begin{pmatrix}a^n|b|^m cos(m\theta) & -a^n|b|^m sin(m\theta)\\|b|^m cos(m\theta)+a^n|b|^m sin(m\theta) & -|b|^m sin(m\theta)+a^n|b|^m cos(m\theta)\end{pmatrix}. \tag{1}$$

Multiply (2.1) by vector $\begin{pmatrix}i\\0\end{pmatrix}$ *and taking into account that sequence*

$\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$, we have that sequence

$$\left\{A_1^nA_2^m\begin{pmatrix}i\\0\end{pmatrix}: n,m\in\mathbb{N}\right\}=\left\{\begin{pmatrix}a^n|b|^m cos(m\theta)\\|b|^m cos(m\theta)+a^n|b|^m sin(m\theta)\end{pmatrix}i : n,m\in\mathbb{N}\right\}$$

is dense in $\mathbb{C}^2$. Hence (A_1, A_2) is hypercyclic.

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