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NOTES ON MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY RAPID OPERATOR

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Abstract. In this paper, we introduce and study a new class $M_n(\alpha, \beta, \gamma, \mu, \theta)$ of meromorphic univalent functions defined in $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$. We obtain coefficients inequalities, distortion theorems, extreme points, closure theorems, radius of convexity estimates and modified Hadamard products.

1. Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $U = \{z \in \mathcal{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $f(z)$ which are all univalent in U. A function $f \in A$ is a starlike function by the order $\alpha, 0 \leq \alpha < 1$, if it satisfy

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (z \in U). \tag{2}$$

We denote this class with $S^*(\alpha)$.

A function $f \in A$ is a convex function by the order $\alpha, 0 \leq \alpha < 1$, if it satisfy

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, (z \in U). \tag{3}$$

We denote this class with $K(\alpha)$.

Let Σ^* denote the class of meromorphic function of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \ (a_n \geq 0) \tag{4}$$

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which are analytic in the punctured unit disc $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = Un\{0\}$. Let $g(z) \in \Sigma^*$ be given by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \ (b_n \geq 0) \tag{5}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z). \tag{6}$$

A function $f \in \Sigma^*$ is meromorphic starlike of order $\alpha(0 \leq \alpha < 1)$, if

$$-\,Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \ (z \in U). \tag{7}$$

The class of such functions is denoted by $\Sigma^*(\alpha)$. A function $f \in \Sigma^*$ is meromorphic convex of order $\alpha(0 \leq \alpha < 1)$, if

$$-\,Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ (z \in U). \tag{8}$$

The class of such functions is denoted by $\Sigma_k^*(\alpha)$. The classes $\Sigma^*(\alpha)$ and $\Sigma_k^*(\alpha)$ were introduced and studied by Pommerenke [5], Miller [3], Mogra et al. [4], Cho [2], Venkateswarlu et al. [10].

In [1], Atshan and Kulkarni introduced Rapid-operator for analytic functions and Rosy and Sunil Varma [6] modified their operator to meromorphic functions as follows:

Lemma 1.1. *For $f \in \Sigma^*$ given by (4), $0 \leq \mu \leq 1$ and $0 \leq \theta \leq 1$, if the operator $S_\mu^\theta : \Sigma^* \to \Sigma^*$ is defined by*

$$S_\mu^\theta f(z) = \frac{1}{(1-\mu)^{\theta)}\Gamma(\theta+1)} \int_0^\infty t^{1+\theta} e^{\frac{-t}{1-\mu}} f(zt)dt \tag{9}$$

then

$$S_\mu^\theta f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} L(n, \mu, \theta) a_n z^n, \tag{10}$$

where $L(n, \mu, \theta) = (1-\mu)^{n+1} \frac{\Gamma(n+\theta+2)}{\Gamma(\theta+1)}$ and Γ is the familiar Gamma function.

Motivated by Thirupathi Reddy and Venkateswarlu [8, 9], now we define a new subclass $M_n(\alpha, \beta, \gamma, \mu, \theta)$ of Σ^*.

2. Coefficient Estimates

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\frac{1}{2} \leq \gamma \leq 1$, $0 \leq \mu \leq 1$, $0 \leq \theta \leq 1$, $n \in \mathbb{N}$ and $z \in U^*$.

Theorem 2.1. *The function $f(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta)$ if and only if*

$$\sum_{n=1}^{\infty} [n(1 + 2\beta\gamma - \beta]L(n, \mu, \theta)a_n \leq 2\beta\gamma(1-\alpha). \tag{11}$$

Proof. Suppose (11) holds. So

$$|z^2(S_\mu^\theta f(z))'+1|-\beta|(2\gamma-1)z^2(S_\mu^\theta f(z))'+(2\alpha\gamma-1)|$$
$$=|\sum_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}|-\beta\left|2\gamma(\alpha-1)+\sum_{n=1}^{\infty} n(2\gamma-1)L(n,\mu,\theta)a_n z^{n+1}\right|$$
$$\leq \sum_{n=1}^{\infty} nL(n,\mu,\theta)a_n r^{n+1}|-\beta\left\{2\gamma(\alpha-1)+\sum_{n=1}^{\infty} n(2\gamma-1)L(n,\mu,\theta)a_n r^{n+1}\right\}$$
$$=\sum_{n=1}^{\infty} n(1+2\beta\gamma-\beta)L(n,\mu,\theta)a_n r^{n+1}-2\beta\gamma(1-\alpha).$$

Since the above inequality holds for all $r, 0<r<1$, letting $r\to 1^-$, we have

$$\sum_{n=1}^{\infty} n(1+2\beta\gamma-\beta)L(n,\mu,\theta)a_n-2\beta\gamma(1-\alpha)\leq 0$$

by (11), hence $f(z)\in M_n(\alpha,\beta,\gamma,\mu,\theta)$.

Conversely, suppose that $f(z)$ is in the class $M_n(\alpha,\beta,\gamma,\mu,\theta)$. Then

$$\left|\frac{z^2(S_\mu^\theta f(z))'+1}{(2\gamma-1)z^2(S_\mu^\theta f(z))'+(2\alpha\gamma-1)}\right|=\left|\frac{\sum_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}{2\gamma(1-\alpha)-\sum_{n=1}^{\infty} n(2\gamma-1)L(n,\mu,\theta)a_n z^{n+1}}\right|\leq\beta.$$

Using the fact that $Re(z)\leq|z|$ for all z, we have

$$\left|\frac{z^2(S_\mu^\theta f(z))'+1}{(2\gamma-1)z^2(S_\mu^\theta f(z))'+(2\alpha\gamma-1)}\right|\leq\left\{\frac{\sum_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}{2\gamma(1-\alpha)-\sum_{n=1}^{\infty} n(2\gamma-1)L(n,\mu,\theta)a_n z^{n+1}}\right\}\leq\beta. \tag{12}$$

If we choose z to be real so that $z^2(S_\mu^\theta f(z))'$ is real. Upon cleaning the denominator in (12) and letting $z\to 1^-$ through positive values, we obtain

$$\sum_{n=1}^{\infty} n[1+2\beta\gamma-\beta]L(n,\mu,\theta)a_n\leq 2\beta\gamma(1-\alpha).$$

This completes the proof of the theorem. □

Corollary 2.1. *Let the function $f(z)$ denoted by (1.4) be in the class $M_n(\alpha,\beta,\gamma,\mu,\theta)$. Then*

$$a_n\leq\frac{2\beta\gamma(1-\alpha)}{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}\ (n\geq 1),$$

with equality for the function

$$f(z)=\frac{1}{z}+\frac{2\beta\gamma(1-\alpha)}{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}z^n. \tag{13}$$

3. Distortion Theorems

Theorem 3.1. *Let the function* $f(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta)$. *Then for* $0 < |z| = \gamma < 1$ *, we have*

$$\frac{1}{r} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)} r \leq |f(z)| \leq \frac{1}{r} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)} r \tag{14}$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)} z^n. \tag{15}$$

Proof. Suppose that f is in $M_n(\alpha, \beta, \gamma, \mu, \theta)$. In view of Theorem 2.3, we have

$$\begin{aligned} &(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)\sum_{n=1}^{\infty} a_n \\ \leq &\sum_{n=1}^{\infty} n[1+2\beta\gamma-\beta]L(n,\mu,\theta)a_n \\ \leq &2\beta\gamma(1-\alpha). \end{aligned}$$

Then

$$\sum_{n=1}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)}. \tag{16}$$

Consequently, we obtain

$$\begin{aligned} |f(z)| = \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| &\leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n \\ &\leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n \\ &\leq \frac{1}{r} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)} r. \end{aligned} \tag{17}$$

Also,

$$\begin{aligned} |f(z)| = \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| &\geq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n |z|^n \\ &\geq \frac{1}{r} - r\sum_{n=1}^{\infty} a_n \\ &\geq \frac{1}{r} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)} r. \end{aligned} \tag{18}$$

Hence, (3.1) follows. □

Theorem 3.2. *Let the function $f \in M_n(\alpha, \beta, \gamma, \mu, \theta)$. Then for* $0 < |z| = r < 1$, *we have*

$$\frac{1}{r^2} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2))} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2))} \tag{19}$$

with equality for the function $f(z)$ given by (15).

Proof. From Theorem 2.1 and (3.3), we have,

$$\sum_{n=1}^{\infty} na_n \leq \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)}. \tag{20}$$

The remaining part of the proof is similar to the proof of Theorem 3.1, so we omit the details. □

4. Closure Theorems

Let the functions $f_j(z)$ be defined for $j = 1, 2, ..., m$ by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \ (a_{n,j} \geq 0). \tag{21}$$

Theorem 4.1. *Let $f_j(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta), (j = 1, 2, ...m)$. Then the function*

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{\infty} a_{n,j} \right) z^n \tag{22}$$

is in $M_n(\alpha, \beta, \gamma, \mu, \theta)$.

Proof. Since$f_j(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta), (j = 1, 2, ...m)$, it follows from Theorem 2.1, that

$$\sum_{n=1}^{\infty} n[1+2\beta\gamma-\beta]L(n, \mu, \theta)a_{n,j} \leq 2\beta\gamma(1-\alpha),$$

for every $j = 1, 2, ..., m$. Hence

$$\sum_{n=1}^{\infty} n[1+2\beta\gamma-\beta]L(n, \mu, \theta) \left(\frac{1}{m} \sum_{j=1}^{\infty} a_{n,j} \right)$$
$$= \frac{1}{m} \sum_{j=1}^{\infty} \left[\sum_{n=1}^{\infty} n[1+2\beta\gamma-\beta]L(n, \mu, \theta)a_{n,j} \right] \leq 2\beta\gamma(1-\alpha).$$

From Theorem 2.1, it follows that $h(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta)$.
This completes the proof. □

Theorem 4.2. *The class $M_n(\alpha, \beta, \gamma, \mu, \theta)$ is closed under convex linear combinations.*

Proof. Let $f_j(z), (j = 1, 2)$ defined by (4.1) be in the class $M_n(\alpha, \beta, \gamma, \mu, \theta)$. Then it is sufficient to show that

$$h(z) = \xi f_1(z) + (1 - \xi) f_2(z),\ (0 \leq \xi \leq 1) \tag{23}$$

is in the class $M_n(\alpha, \beta, \gamma, \mu, \theta)$. Since

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} [\xi a_{n,1} + (1 - \xi) a_{n,2}] z^n, \tag{24}$$

then, we have from Theorem 2.1, that

$$\sum_{n=1}^{\infty} n[1 + 2\beta\gamma - \beta] L(n, \mu, \theta)[\xi a_{n,1} + (1 - \xi) a_{n,2}]$$
$$\leq 2\xi\beta\gamma(1 - \alpha) + 2\beta\gamma(1 - \xi)(1 - \alpha) = 2\beta\gamma(1 - \alpha).$$

So, $h(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta)$. □

Theorem 4.3. *Let* $0 \leq \rho < 1$. *Then*

$$M_n(\alpha, \beta, \gamma, \mu, \theta) \leq M_n(\alpha, \beta, 1, \mu, \theta) = M_n(\alpha, \beta, \mu, \theta),$$

where

$$\rho = 1 - \frac{\gamma(1 + \beta)(1 - \alpha)}{(1 + 2\beta\gamma - \beta)}. \tag{25}$$

Proof. Let $f(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta)$. Then

$$\sum_{n=1}^{\infty} \frac{n[1 + 2\beta\gamma - \beta] L(n, \mu, \theta)}{2\beta\gamma(1 - \alpha)} a_n \leq 1. \tag{26}$$

We need to find the value of ρ such that

$$\sum_{n=1}^{\infty} \frac{n(1 + \beta)}{2\beta(1 - \rho)} L(n, \mu, \theta) a_n \leq 1. \tag{27}$$

In view of equations (26) and (27), we have

$$\frac{n[1 + \beta]}{2\beta(1 - \rho)} L(n, \mu, \theta) \leq \frac{n[1 + 2\beta\gamma - \beta] L(n, \mu, \theta)}{2\beta\gamma(1 - \alpha)},$$

that is

$$\rho \leq 1 - \frac{\gamma(1 + \beta)(1 - \alpha)}{(1 + 2\beta\gamma - \beta)},$$

which completes the proof of theorem. □

Theorem 4.4. *Let* $f_0(z) = \frac{1}{z}$ *and*

$$f_n(z) = \frac{1}{z} + \frac{2\beta\gamma(1 - \alpha)}{n[1 + 2\beta\gamma - \beta] L(n, \mu, \theta)} z^n,\ n \geq 1. \tag{28}$$

Then $f(z)$ *is in the class* $M_n(\alpha, \beta, \gamma, \mu, \theta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z), \tag{29}$$

where $\mu_n \geq 0$ *and* $\sum_{n=0}^{\infty} \mu_n = 1$.

Proof. Assume that

$$f(z)=\sum_{n=0}^{\infty}\mu_n f_n(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2\beta\gamma(1-\alpha)}{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}\mu_n z^n. \tag{30}$$

Then it follows that

$$\sum_{n=1}^{\infty}\frac{2\beta\gamma(1-\alpha)}{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}\mu_n.\frac{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}{2\beta\gamma(1-\alpha)}$$
$$=\sum_{n=1}^{\infty}\mu_n=1-\mu_0\leq 1,$$

which implies that $f(z)\in M_n(\alpha,\beta,\gamma,\mu,\theta)$.

Conversely, assume that the function $f(z)$ defined by (1.4) be in the class $M_n(\alpha,\beta,\gamma,\mu,\theta)$. Then

$$a_n\leq\frac{2\beta\gamma(1-\alpha)}{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}.$$

Setting

$$\mu_n=\frac{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}{2\beta\gamma(1-\alpha)},\ n\geq 1$$

and

$$\mu_0=1-\sum_{n=1}^{\infty}\mu_n,$$

we can see that $f(z)$ can be expressed in the form (29).

This completes the proof of the theorem. □

5. Integral Operators

Theorem 5.1. *Let the function $f(z)\in M_n(\alpha,\beta,\gamma,\mu,\theta)$. Then the integral operator*

$$F_c(z)=c\int_0^1 u^c f(uz)dz,\ (0<u\leq 1;c>0) \tag{31}$$

is in the class $\in M_n(\xi,\beta,\gamma,\mu,\theta)$, where

$$\xi=1-\frac{2\beta\gamma c(1-\alpha)}{(1+2\beta\gamma-\beta)(c+2)L(1,\mu,\theta)}. \tag{32}$$

The result is sharp for the function $f(z)$ given by (15).

Proof. Let $f(z)\in M_n(\alpha,\beta,\gamma,\mu,\theta)$. Then

$$F_c(z)=c\int_0^1 u^c f(u,z)dz=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{c}{n+c+1}a_n z^n. \tag{33}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{nc}{(n+c+1)(1-\xi)}a_n\leq 1. \tag{34}$$

Since $f(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta)$, then

$$\sum_{n=1}^{\infty} \frac{n(1+2\beta\gamma-\beta)L(n,\mu,\theta)}{2\beta\gamma(1-\alpha)} a_n \leq 1. \tag{35}$$

From (34) and (35), we have

$$\frac{nc}{(n+c+1)(1-\xi)} \leq \frac{n(1+2\beta\gamma-\beta)L(n,\mu,\theta)}{2\beta\gamma(1-\alpha)}.$$

Then

$$\xi \leq 1 - \frac{2\beta\gamma c(1-\alpha)}{n(1+2\beta\gamma-\beta)(n+c+1)L(n,\mu,\theta)}.$$

Since

$$H(n) = 1 - \frac{2\beta\gamma c(1-\alpha)}{n(1+2\beta\gamma-\beta)(n+c+1)L(n,\mu,\theta)}$$

is an increasing function of n $(n \geq 1)$, we obtain

$$\xi \leq H(1) = 1 - \frac{2\beta\gamma c(1-\alpha)}{(1+2\beta\gamma-\beta)(c+2)L(1,\mu,\theta)}$$

and hence the proof of theorem 5.1 is completed. □

6. Radius of Convexity

Theorem 6.1. *Let the function $f(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta)$. Then $f(z)$ is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $0 < |z| < r$, where*

$$r \leq \left\{ \frac{(1+2\beta\gamma-\beta)(1-\delta)L(n,\mu,\theta)}{2\beta\gamma(n+2-\delta)(1-\alpha)} \right\}^{1/n+1}. \tag{36}$$

The result is sharp.

Proof. We must show that

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq 1-\delta \text{ for } 0 < |z| < r, \tag{37}$$

where r is given by (36). Indeed, we find from (6.2) that

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq \sum_{n=1}^{\infty} \frac{n(n+1)a_n|z|^{n+1}}{1 - \sum\limits_{n=1}^{\infty} na_n|z|^{n+1}}.$$

This will be bounded by $1-\delta$, if

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n r^{n+1} \leq 1. \tag{38}$$

But by using Theorem 2.1, (38) will be true, if

$$\frac{n(n+2-\delta)}{1-\delta} r^{n+1} \leq \frac{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}{2\beta\gamma(1-\alpha)}.$$

Then

$$r \leq \left\{ \frac{(1+2\beta\gamma-\beta)(1-\delta)L(n,\mu,\theta)}{2\beta\gamma(n+2-\delta)(1-\alpha)} \right\}^{1/n+1}.$$

This completes the proof of theorem. □

7. Modified Hadamard Product

For $f_j(z)$ $(j = 1, 2)$ defined by (21), the modified Hadamard product of $f_1(z)$ and $f_2(z)$ defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}z^n = (f_2 * f_1)(z). \tag{39}$$

Theorem 7.1. *Let $f_j(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta)$ $(j = 1, 2)$. Then $(f_1 * f_2)(z) \in M_n(\phi, \beta, \gamma, \mu, \theta)$, where*

$$\phi = 1 - \frac{2\beta\gamma(1-\alpha)^2}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)}. \tag{40}$$

The result is sharp for the function $f_j(z))$ given by

$$f_j(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(1-\mu)^2(\theta+1)(\theta+2)}z \ (j = 1, 2). \tag{41}$$

Proof. Using the technique for Schild and Silverman [7], we need to find the largest ϕ such that

$$\sum_{n=1}^{\infty} \frac{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}{2\beta\gamma(1-\phi)} a_{n,1}a_{n,2} \le 1. \tag{42}$$

Since $f_j(z) \in M_n(\alpha, \beta, \gamma, \mu, \theta)$, $(j = 1, 2)$, we readily see that

$$\sum_{n=1}^{\infty} \frac{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}{2\beta\gamma(1-\alpha)} a_{n,1} \le 1 \tag{43}$$

and

$$\sum_{n=1}^{\infty} \frac{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}{2\beta\gamma(1-\alpha)} a_{n,2} \le 1. \tag{44}$$

By the Cauchy Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{n[1+2\beta\gamma-\beta]}{2\beta\gamma(1-\alpha)} \sqrt{a_{n,1}a_{n,2}} \le 1. \tag{45}$$

Thus it is sufficient to show that

$$\frac{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}{2\beta\gamma(1-\phi)} a_{n,1}a_{n,2} \le \frac{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}{2\beta\gamma(1-\alpha)} \sqrt{a_{n,1}a_{n,2}} \tag{46}$$

or equivalently

$$\sqrt{a_{n,1}a_{n,2}} \le \frac{1-\phi)}{(1-\alpha)}. \tag{47}$$

Connecting with (45), it is sufficient to prove that

$$\frac{2\beta\gamma(1-\alpha)}{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)} \le \frac{(1-\phi)}{(1-\alpha)}. \tag{48}$$

It follows from (48) that

$$\phi \le 1 - \frac{2\beta\gamma(1-\alpha)^2}{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}.$$

Now defining the function $G(n)$ by

$$G(n) = 1 - \frac{2\beta\gamma(1-\alpha)^2}{n[1+2\beta\gamma-\beta]L(n,\mu,\theta)}.$$

We see that G(n) is an increasing function of $n(n \geq 1)$.
Therefore, we conclude that

$$\phi \leq G(1) = 1 - \frac{2\beta\gamma(1-\alpha)^2}{[1+2\beta\gamma-\beta](1-\mu)^2(\theta+1)(\theta+2)}$$

which evidently completes the proof of the theorem. □

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