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EXISTENCE OF SOLUTION AND CONVERGENCE OF RESOLVENT ITERATIVE ALGORITHMS FOR A SYSTEM OF NONLINEAR VARIATIONAL INCLUSION PROBLEM

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Abstract. In this paper, we introduce and study a system of variational inclusions called system of nonlinear variational inclusion problem involving A-monotone mappings in real Hilbert spaces. By means of resolvent operator technique, we suggest an resolvent iterative algorithm for finding the approximate solution of this system and discuss the convergence criteria of the sequences generated by the resolvent iterative algorithm under some suitable conditions.

1. Introduction

Variational inequality and variational inclusion problems are of fundamental importance in a wide range of mathematical and applied sciences problems, such as mathematical programming, traffic engineering, economics and equilibrium problems. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas of sciences and are proved to be productive and innovative. It has been shown that this theory provides a simple, natural and unified framework for a general treatment of unrelated problems. The projection and its invariant forms represent an important tool for finding the approximation solution of various types of variational inequalities. For further details of the approximation solvability of variational inclusions, we refer to [1-4,6-8,10-15,17].

It is known that the monotonicity of the underlying operator plays a prominent role in solving different classes of variational inequality problems. In 2003, Fang and Huang [5] introduced and studied a new class of variational inclusions involving H-monotone operators in a Hilbert space. They have obtained a new algorithm for solving the associated class of variational inclusions using resolvent operator technique. A considerable research in approximation solvability and A-monotone operators and H-η-accretive operators has been carried out by He *et al.* [9], Lan *et al.* [15], Verma [19,21]. Fang *et al.* [6] have considered a class of variational inclusions and discussed its solvability using H-η-accretive operators.

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Motivated and inspired by the work going on in this direction, in this paper we give the existence and Lipschitz continuity of the resolvent operators. As an application, we consider a system of nonlinear variational inclusion problem involving A-monotone operator in Hilbert spaces. Further, using resolvent operator, we suggest a resolvent iterative algorithm for approximating the solution of this system and discuss the convergence analysis of the sequences generated by the resolvent iterative algorithm.

2. Resolvent Operator and Formulation of Problem

We need the following definitions and results from the literature.

Let X be a real Hilbert space equipped with norm $\|.\|$ and an inner product $<.,.>$, respectively. Let 2^X denote the family of all non-empty subsets of X.

Definition 2.1. Let $A : X \to X$ be a single-valued mapping. Then A is said to be

(i) monotone if

$$\langle Au - Av, u - v\rangle \geq 0, \ \forall u, v \in X;$$

(ii) strictly monotone if, A is monotone and

$$\langle Au - Av, u - v\rangle = 0,$$

if and only if $u = v$.

(iii) δ-strongly monotone if there is a constant $\delta > 0$ such that

$$\langle Au - Av, u - v\rangle \geq \delta\|u - v\|^2, \ \forall u, v \in X.$$

Definition 2.2. Let $p, g : X \to X$ $S : X \times X \to X$ be a single-valued mappings. Then S is said to be

(i) p-monotone in the first argument, if

$$\langle S(p(u), z) - S(p(v), z), u - v\rangle \geq 0, \ \forall u, v, z \in X;$$

(ii) g-monotone in the second argument, if

$$\langle S(z, g(u)) - S(z, g(v)), u - v\rangle \geq 0, \ \forall u, v, z \in X;$$

(iii) p-monotone with respect to A in the first argument, if

$$\langle S(p(u), z) - S(p(v), z), A(u) - A(v)\rangle \geq 0, \ \forall u, v, z \in X;$$

(iv) g-monotone with respect to A in the second argument, if

$$\langle S(z, g(u)) - S(z, g(v)), A(u) - A(v)\rangle \geq 0, \ \forall u, v, z \in X;$$

Definition 2.3. Let $A, H : X \to X$ be single-valued mappings and $M : X \to 2^X$ be a multi-valued mapping. Then M is said to be

(i) monotone if

$$\langle x - y, u - v\rangle \geq 0, \ \forall u, v \in X, x \in Mu, y \in Mv;$$

(ii) monotone with respect to A if

$$\langle x - y, Au - Av\rangle \geq 0, \ \forall u, v \in X, x \in Mu, y \in Mv;$$

(iii) ν-strongly monotone if there is a constant $\nu > 0$ such that

$$\langle x - y, u - v\rangle \geq \nu\|u - v\|^2,\ \forall u, v \in X, x \in Mu, y \in Mv;$$

(iv) maximal monotone, if M is monotone and $(I + \rho M)(X) = X\ \forall\rho > 0$, where I denotes the identity mapping on H;
(v) relaxed monotone if there exists a positive constant $\mu > 0$ such that

$$\langle x - y, u - v\rangle \geq -\mu\|u - v\|^2,\ \forall u, v \in X, x \in Mu, y \in Mv;$$

(vi) H-monotone, if M is monotone and $(H + \rho M)(X) = X\ \forall\rho > 0$
(vii) strongly H-monotone, if M is strongly monotone and $(H+\rho M)(X) = X\ \forall\rho > 0$
(viii) A-monotone, if M is relaxed monotone and $(A + \rho M)(X) = X\ \forall\rho > 0$.
The relation between A-monotone mapping, H-monotone mapping and strongly H-monotone mapping can be denoted as

$\{A$-monotone mapping$\} \supset \{H$-monotone mapping$\} \supset \{$strongly H-monotone mapping$\}$.
Theorem 2.4. Let a single-valued mapping $A : X \to X$ be δ-strongly monotone. Let $M : X \to 2^X$ be A-monotone operator. If for all $(v, y) \in \text{Graph}(M)$, $\langle x - y, u - v\rangle \geq 0$ holds, where $\text{Graph}(M) = \{(a, b) \in X \times X : b \in M(a)\}$, then $(u, x) \in \text{Graph}(M)$.

Proof. Since M is A-monotone, we know that $(A + \rho M)X = X$ holds $\forall\rho > 0$ and so there exists $(u_1, x_1) \in \text{Graph}(M)$ such that

$$A(u) + \rho x = A(u_1) + \rho x_1.$$

Since A is δ-strongly monotone, we have

$$\begin{aligned}0 &\leq \rho\langle x - x_1, u - u_1\rangle \\ &= -\langle A(u) - A(u_1), u - u_1\rangle \\ &\leq -\delta||u - u_1||^2 \leq 0.\end{aligned}$$

This implies that $u = u_1$ and $x = x_1$. Thus $(u, x) = (u_1, x_1) \in$ Graph(M). This completes the proof.
Theorem 2.5. Let a single-valued mapping $A : X \to X$ be δ-strongly monotone. Let $M : X \to 2^X$ be A-monotone operator. Then the operator $(A + \rho M)^{-1}$ is single-valued for $0 < \rho < \dfrac{\delta}{\mu}$, where δ, ρ and μ are positive constants.
Proof. For any given $u^\star \in X$, let $\forall u, v \in (A + \rho M)^{-1}(u^\star)$. It follows that

$$-A(u) + u^\star \in \rho M(u) \text{ and } -A(v) + u^\star \in \rho M(v).$$

Since A is δ-strongly monotone, $M : X \to 2^X$ is an A-monotone operator, that is, M is μ-relaxed monotone, we have

$$-\mu||u - v||^2 \leq \frac{1}{\rho}\langle(-A(u) + u^\star) - (-A(v) + u^\star), u - v\rangle$$

$$= -\frac{1}{\rho}\langle A(u) - A(v), u - v\rangle$$

$$\leq -\frac{1}{\rho}\ \delta||u - v||^2$$

$$= -\frac{\delta}{\rho}||u - v||^2.$$

This implies

$$\mu\rho||u - v||^2 \geq \delta||u - v||^2.$$

If $u \neq v$, then $\rho \geq \dfrac{\delta}{\mu}$ contradicts with $0 < \rho < \dfrac{\delta}{\mu}$. Thus $u = v$, that is, $(A + \rho M)^{-1}$ is single-valued.

Definition 2.6[18]. Let $A : X \to X$ be a strictly monotone mapping and $M : X \to 2^X$ be A-monotone mapping. The resolvent operator is defined by $J^M_{\rho,A} : X \to X$ is defined by

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u),\ \forall u \in X.$$

Definition 2.7[5]. Let $H : X \to X$ be a strictly monotone mapping and $M : X \to 2^X$ be H-monotone mapping. The resolvent operator is defined by $J^M_{\rho,H} : X \to X$ is defined by

$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u),\ \forall u \in X.$$

Lemma 2.8. If $A : X \to X$ be δ-strongly monotone and $M : X \to 2^X$ be A-monotone. Then the resolvent operator $J^M_{\rho,A} : X \to X$ is $\dfrac{1}{\delta - \rho\mu}$-Lipschitz continuous for $0 < \rho < \dfrac{\delta}{\mu}$, where δ, ρ and μ are positive constants.

Proof. For any $u, v \in X$, we have

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u),$$

$$J^M_{\rho,A}(v) = (A + \rho M)^{-1}(v).$$

This implies that

$$\frac{1}{\rho}\Big(u - A(J^M_{\rho,A}(u))\Big) \in M(J^M_{\rho,A}(u)),$$

$$\frac{1}{\rho}\Big(v - A(J^M_{\rho,A}(v))\Big) \in M(J^M_{\rho,A}(v)).$$

M is A-monotone, implies, M is μ-relaxed monotone. Hence we have

$$\frac{1}{\rho}\Big\langle (u - A(J^M_{\rho,A}(u))) - (v - A(J^M_{\rho,A}(v))), J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\Big\rangle$$

$$= \frac{1}{\rho}\Big\langle u - v - (AJ^M_{\rho,A}(u) - AJ^M_{\rho,A}(v)), J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\Big\rangle$$

$$\geq (-\mu)||J^M_{\rho,A}(u) - J^M_{\rho,A}(v)||^2.$$

Now we have

$$||u - v||\ ||J^M_{\rho,A}(u) - J^M_{\rho,A}(v)|| \geq \Big\langle u - v, J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\Big\rangle$$

$$= \Big\langle u - v - (AJ^M_{\rho,A}(u) - AJ^M_{\rho,A}(v)), J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\Big\rangle$$

$$+\Big\langle AJ^M_{\rho,A}(u) - AJ^M_{\rho,A}(v), J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\Big\rangle$$

$$\geq -\rho\mu||J^M_{\rho,A}(u) - J^M_{\rho,A}(v)||^2 + \delta||J^M_{\rho,A}(u) - J^M_{\rho,A}(v)||^2$$

$$= (\delta - \rho\mu)||J^M_{\rho,A}(u) - J^M_{\rho,A}(v)||^2.$$

This implies that

$$||u - v|| \geq (\delta - \rho\mu)||J^M_{\rho,A}(u) - J^M_{\rho,A}(v)||,$$

or,

$$||J^M_{\rho,A}(u) - J^M_{\rho,A}(v)|| \leq \frac{1}{\delta - \rho\mu}||u - v||,\ 0 < \rho < \frac{\delta}{\mu}.$$

Taking $A = I$, the identity operator, we immediately have the following corollary:

Corollary 2.9. Let $M : X \to 2^X$ be μ-relaxed monotone. Then the resolvent operator $J^M_{\rho,I} = (I+\rho M)^{-1} : X \to X$ is $\dfrac{1}{1-\rho\mu}$-Lipschitz continuous for $0 < \rho < \dfrac{1}{\mu}$, where ρ and μ are positive constants and I is the identity mapping.

Lemma 2.10[16]. Let C be a nonempty closed convex subset of X. Then

$$v = P_C[t]$$

$$\Longleftrightarrow \langle v - t, u - v\rangle \geq 0,\ \forall\ t \in X \text{ and } u \in C,$$

where, $P_C[t]$ denotes the projection of t onto C, that is, $P_C[t]$ is such that

$$\|z - P_C[t]\| = dist(t, C),$$

where $dist(t, C)$ is defined by

$$dist(t, C) = \ \inf\ _{z\in C}\|t - z\|.$$

Now, we formulate our main problem.

Let X be a real Hilbert space with norm $\|.\|$. Let $p, g : X \to X, S : X \times X \to X$ be single-valued mapping, let $M : X \to 2^X$ be A-monotone mappings. We consider the following system of nonlinear variational inclusion problem (in short, SNVP): Find $u, v \in X$, such that

$$0 \in S(p(u), g(u)) + M(u), \tag{2.1}$$

$$0 \in S(p(v), g(v)) + M(v). \tag{2.2}$$

Some Special Cases:

I. For $S \equiv 0, u = v,\ \forall u, v \in X$. Then above problem SNVP(2.1)-(2.2) reduces to the following problem:

$$0 \in M(u) \tag{2.3}$$

Problem (2.3) is the general inclusion problem considered and studied by Verma [20].

II. If $S : X \to X$ such that $S(p(u), g(u)) = S(p(v), g(v)) = S(p(u)) - S(g(u))\ \forall u \in X$ and $M(u) = M(v) = \partial\varphi(u)\ \forall u \in X$ where $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semicontinuous function $\varphi : X \to R \cup \{+\infty\}$. Then problem SNVP(2.1)-(2.2) reduces to the following problem: Find $u \in X$ and

$$\langle S(p(u)) - S(g(u)), v - u\rangle \geq \varphi(u) - \varphi(v),\ \forall v \in X. \ (2.4)$$

Problem (2.4) is called a class of variational inclusions considered and studied by Hassouni and Moudafi [8].

3. Resolvent Iterative Algorithm

Here we consider the following resolvent iterative algorithm for finding an approximate solution of SNVP (2.1)-(2.2) which consists of the following steps:

Resolvent Iterative Algorithm (RIA) 3.1.

Step 1. Initiation Step:

Select $w^0, t^0 \in X$ and set $n = 0$.

Step 2. Resolvent Step:

Find $w^n, t^n \in X$ such that

$$u^n = J^M_{\rho^n, A}\Big\{A(w^n) - \rho^n S(p(u^n), g(u^n))\Big\}. \tag{3.1}$$

$$v^n = J^M_{\gamma^n, A}\Big\{A(t^n) - \gamma^n S(p(v^n), g(v^n))\Big\}. \tag{3.2}$$

where ρ^n, γ^n are such that

$$0 < \rho^n, \gamma^n < \frac{\delta}{\mu}. \tag{3.3}$$

Step 3. Projection step:

Set

$$C = \Big\{w \in X : \Big\langle A(w^n) - A(u^n), w - A(u^n)\Big\rangle \leq 0\Big\}.$$

If $A(w^n) = A(u^n)$, then stop, otherwise, choose w^{n+1} such that

$$A(w^{n+1}) = P_C(A(w^n)). \tag{3.4}$$

Again, set

$$C = \Big\{t \in X : \Big\langle A(t^n) - A(v^n), t - A(v^n)\Big\rangle \leq 0\Big\}.$$

If $A(t^n) = A(v^n)$, then stop, otherwise, choose t^{n+1} such that

$$A(t^{n+1}) = P_C(A(t^n)). \tag{3.5}$$

Step 4. Let $n = n + 1$ and return to Step 1.

Remark 3.2.

From (3.1), (3.2), we can have

$$A(w^n) \in A(u^n) + \rho^n(S(p(u^n), g(u^n)) + M(u^n)),$$

or,

$$\frac{1}{\rho^n}\Big(A(w^n) - A(u^n)\Big) \in (S(p(u^n), g(u^n)) + M(u^n)). \tag{3.6}$$

Similarly, (3.2) implies that

$$A(t^n) \in A(v^n) + \gamma^n(S(p(v^n), g(v^n)) + M(v^n)),$$

or,

$$\frac{1}{\gamma^n}\Big(A(t^n) - A(v^n)\Big) \in (S(p(v^n), g(v^n)) + M(v^n)). \tag{3.7}$$

4. Existence of Solution and Convergence Analysis

Now, we give the following theorem, which gives the existence of solution of SNVP (2.1)-(2.2) and ensures the convergence of the sequences generated by the resolvent iterative algorithm 3.1 for SNVP (2.1)-(2.2).

Theorem 4.1. Let X be a real Hilbert space. Let a single-valued map $S : X\times X \to X$ be continuous such that S is p-monotone and g-monotone with respect to A in the first and second argument, respectively, and S is p-monotone and g-monotone in the first and second argument, respectively. Suppose that a continuous single-valued mapping $A : X \to X$ be δ-strongly monotone. Let an A-monotone mapping $M : X \to 2^X$ be monotone with respect to A. Then the iterative sequences $\{u^n\}$,$\{v^n\}$ generated by resolvent iterative algorithm 3.1 converges weakly to a solution of SNVP (2.1)-(2.2).

Proof. Suppose u' be a solution of (2.1). Therefore, we have

$$0 \in S(p(u'), g(u')) + M(u'), \tag{4.1}$$

Now, we can have

$$\begin{aligned}
&\Big\|A(u') - A(w^{n+1})\Big\|^2 \\
= \;& \Big\|A(u') - A(w^n) - (A(w^{n+1}) - A(w^n))\Big\|^2 \\
= \;& \Big\|A(u') - A(w^n)\Big\|^2 - 2\Big\langle A(u') - A(w^n), A(w^{n+1}) - A(w^n)\Big\rangle \\
&+\Big\|A(w^{n+1}) - A(w^n)\Big\|^2 \\
= \;& \Big\|A(u') - A(w^n)\Big\|^2 - 2\Big\langle A(w^{n+1}) - A(w^n), A(w^{n+1}) - A(w^n)\Big\rangle \\
&-2\Big\langle A(u') - A(w^{n+1}), A(w^{n+1}) - A(w^n)\Big\rangle + \Big\|A(w^{n+1}) - A(w^n)\Big\|^2 \\
\leq \;& \Big\|A(u') - A(w^n)\Big\|^2 - 2\Big\langle A(u') - A(w^{n+1}), A(w^{n+1}) - A(w^n)\Big\rangle \\
&-\Big\|A(w^{n+1}) - A(w^n)\Big\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\Big\|A(u') - A(w^{n+1})\Big\|^2 \\
&\leq \Big\|A(u') - A(w^n)\Big\|^2 - 2\Big\langle A(u') - A(w^{n+1}), A(w^{n+1}) - A(w^n)\Big\rangle \\
&\quad -\Big\|A(w^{n+1}) - A(w^n)\Big\|^2.
\end{aligned} \tag{4.2}$$

Now, since S is p-monotone and g-monotone with respect to A in the first and second argument, respectively, we have

$$\Big\langle S(p(u'), g(u')) - S(p(u^n), g(u^n)), Au' - Au^n\Big\rangle$$

$$\begin{aligned} &= \Big\langle S(p(u'), g(u')) - S(p(u^n), g(u')), Au' - Au^n \Big\rangle \\ &\quad + \Big\langle S(p(u^n), g(u')) - S(p(u^n), g(u^n)), Au' - Au^n \Big\rangle \\ &\geq 0. \end{aligned}$$

This implies that

$$\Big\langle S(p(u'), g(u')) - S(p(u^n), g(u^n)), Au' - Au^n \Big\rangle \geq 0. \tag{4.3}$$

Also as M is monotone with respect to A, we have

$$\Big\langle M(u') - M(u^n), Au' - Au^n \Big\rangle \geq 0. \tag{4.4}$$

On adding (4.3) and (4.4), we have

$$\Big\langle S(p(u'), g(u')) + M(u') - \Big(S(p(u^n), g(u^n)) + M(u^n)\Big), Au' - Au^n \Big\rangle \geq 0.$$

Using (3.6) and (4.1), it follows that

$$\Big\langle 0 - \frac{1}{\rho^n}\Big(A(w^n) - A(u^n)\Big), Au' - Au^n \Big\rangle \geq 0,$$

or,

$$\Big\langle A(w^n) - A(u^n), Au' - Au^n \Big\rangle \leq 0. \tag{4.5}$$

Therefore for $A(u') \in C = w \in X$, (4.5) can be rewritten as

$$C = \Big\{ w \in X : \Big\langle A(w^n) - A(u^n), w - Au^n \Big\rangle \leq 0 \Big\}. \tag{4.6}$$

Since by resolvent iterative algorithm 3.1 $A(w^{n+1}) = P_C(A(w^n))$, it follows from Lemma 2.10 that

$$\Big\langle A(w^{n+1}) - A(w^n), A(u') - A(w^{n+1}) \Big\rangle \geq 0. \tag{4.7}$$

Using (4.7) in (4.2), it follows that

$$\begin{aligned} &\Big\| A(u') - A(w^{n+1}) \Big\|^2 \\ &\quad \leq \Big\| A(u') - A(w^n) \Big\|^2 - \Big\| A(w^{n+1}) - A(w^n) \Big\|^2. \end{aligned} \tag{4.8}$$

Therefore, from (4.8) we have

$$\Big\| A(u') - A(w^{n+1}) \Big\| \leq \Big\| A(u') - A(w^n) \Big\|, \ \forall n \geq 0. \tag{4.9}$$

From (4.9), it follows that $\Big\{ \Big\| A(u') - A(w^n) \Big\| \Big\}$ is a convergent sequence.

Again since A is δ-strongly monotone, we have

$$\Big\langle A(u') - A(w^n), u' - w^n \Big\rangle \geq \delta \|u' - w^n\|^2,$$

or,

$$\|u' - w^n\| \leq \frac{1}{\delta} \Big\| A(u') - A(w^n) \Big\|. \tag{4.10}$$

Thus it follows from (4.10) that $\{w^n\}$ is a bounded sequence.
From (4.8), it follows that

$$\begin{aligned} 0 &\leq \Big\| A(w^{n+1}) - A(w^n) \Big\|^2 \\ &\leq \Big\| A(u') - A(w^n) \Big\|^2 - \Big\| A(u') - A(w^{n+1}) \Big\|^2. \end{aligned}$$

Taking limits on both sides as $n \to \infty$, we have

$$\lim_{n\to\infty} \Big\| A(w^{n+1}) - A(w^n) \Big\|^2$$
$$\leq \lim_{n\to\infty} \Big\{ \Big\| A(u') - A(w^n) \Big\|^2 - \Big\| A(u') - A(w^{n+1}) \Big\|^2 \Big\} = 0.$$

Therefore, it follows that

$$\lim_{n\to\infty} \Big\| A(w^{n+1}) - A(w^n) \Big\| = 0.$$

Now, from $A(w^{n+1}) = P_C(A(w^n)) \in C$ and $A(u^n) \in C$, we have

$$\Big\langle A(w^n) - A(u^n), A(w^{n+1}) - A(u^n) \Big\rangle \leq 0,$$

and

$$\begin{aligned} \Big\| A(u^n) - A(w^n) \Big\|^2 &= \Big\langle A(u^n) - A(w^n), A(u^n) - A(w^n) \Big\rangle \\ &\leq \Big\langle A(w^{n+1}) - A(w^n), A(u^n) - A(w^n) \Big\rangle. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \Big\| A(w^n) - A(u^n) \Big\| = 0. \tag{4.11}$$

Again, from the δ-strongly monotonicity of A, we have

$$\begin{aligned} \|A(u^n) - A(w^n)\| \, \|u^n - w^n\| &\geq \Big\langle A(u^n) - A(w^n), u^n - w^n \Big\rangle \\ &\geq \delta \|u^n - w^n\|^2. \end{aligned}$$

This implies that $\lim\limits_{n\to\infty} \|w^n - u^n\| = 0$, and hence $\lim\limits_{n\to\infty} (w^n - u^n) = 0$. Thus it follows from the boundedness of $\{w^n\}$ that $\{u^n\}$ is also a bounded sequence.
Thus, both the sequences $\{u^n\}$ and $\{w^n\}$ have same weak limit points.
Now, by letting v' to be a solution of (2.2), we can have

$$0 \in S(p(v'), g(v')) + M(v'). \tag{4.12}$$

Following the similar procedure as from (4.2) to (4.11), we can show that $\{t^n\}$ is a bounded sequence and $\lim\limits_{n\to\infty} (t^n - v^n) = 0$ and therefore from the boundedness of $\{t^n\}$ it follows that $\{v^n\}$ is a bounded sequence and thus both the bounded sequences have same weak limit points.
Next, we claim that each weak limit point of the sequences $\{u^n\}$ and $\{v^n\}$ is a solution of SNVP (2.1)-(2.2).
Let l be the weak limit point of $\{u^n\}$, that is, $\lim\limits_{n\to\infty} \{u^n\} = l$ (weakly).
This implies that $\lim\limits_{n\to\infty} \{w^n\} = l$ (weakly).
Suppose for the fixed elements $y, y' \in X$, we consider an arbitrary elements $x, x' \in X$ such that

$$x \in S(p(y), g(y)) + M(y). \tag{4.13}$$

$$x' \in S(p(y'), g(y')) + M(y'). \tag{4.14}$$

Therefore, we can find elements $z \in M(y), z' \in M(y')$ such that

$$x = S(p(y), g(y)) + z. \tag{4.15}$$

$$x' = S(p(y'), g(y')) + z'. \tag{4.16}$$

Since, S is p-monotone in the first argument and g-monotone in the second argument, we have

$$\begin{aligned}&\Big\langle u^n - y, S(p(u^n), g(u^n)) - S(p(y), g(y))\Big\rangle \\ &= \Big\langle u^n - y, S(p(u^n), g(u^n)) - S(p(y), g(u^n))\Big\rangle \\ &\quad + \Big\langle u^n - y, S(p(y), g(u^n)) - S(p(y), g(y))\Big\rangle \\ &\geq 0.\end{aligned}$$

Therefore, we have

$$\Big\langle u^n - y, S(p(u^n), g(u^n)) - S(p(y), g(y))\Big\rangle \geq 0. \tag{4.17}$$

Also, since M is A-monotone, implies, M is μ-relaxed monotone, we have

$$\Big\langle u^n - y, M(u^n) - M(y)\Big\rangle \geq -\mu\|u^n - y\|^2.$$

Using (3.6) and as $z \in M(y)$, it follows that

$$\Big\langle u^n - y, \Big\{\frac{1}{\rho^n}(A(w^n) - A(u^n)) - S(p(u^n), g(u^n))\Big\} - z\Big\rangle \geq -\mu\|u^n - y\|^2. \tag{4.18}$$

On adding (4.17), (4.18), we have

$$\Big\langle u^n - y, \frac{1}{\rho^n}(A(w^n) - A(u^n)) - (S(p(y), g(y)) + z)\Big\rangle \geq -\mu\|u^n - y\|^2. \tag{4.19}$$

Using (4.15) in (4.19), we have

$$\Big\langle u^n - y, \frac{1}{\rho^n}(A(w^n) - A(u^n)) - x\Big\rangle \geq -\mu\|u^n - y\|^2,$$

or

$$\Big\langle u^n - y, -x\Big\rangle \geq \Big\langle u^n - y, \frac{1}{\rho^n}(A(w^n) - A(u^n))\Big\rangle - \mu\|u^n - y\|^2. \tag{4.20}$$

Using (4.11) and the boundedness of $\{u^n\}, \{\rho^n\}$, we can have

$$\Big\langle u^n - y, \frac{1}{\rho^n}(A(w^n) - A(u^n))\Big\rangle \to 0. \tag{4.21}$$

Combining (4.20) and (4.21), we have

$$\Big\langle u^n - y, -x\Big\rangle \geq -\mu\|u^n - y\|^2.$$

Therefore, by taking limits as $n \to \infty$, we have

$$\Big\langle l - y, 0 - x\Big\rangle = \lim_{n\to\infty}\Big\langle u^n - y, 0 - x\Big\rangle \geq -\mu\|u^n - y\|^2. \tag{4.22}$$

Since by (4.13) $(y, x) \in Graph(S(p(.), g(.)) + M(.))$. Using Theorem 2.4, (4.22) implies that $(l, 0) \in Graph(S(p(.), g(.)) + M(.))$, that is,

$$0 \in S(p(l), g(l)) + M(l).$$

This implies that l is a solution of (2.1).
Following the similar steps as in (4.13)-(4.22), we can show that for a weak limit point l' of $\{v^n\}$,

$$0 \in S(p(l'), g(l')) + M(l').$$

That is l' is a solution of (2.2).
Lastly, we show that there is a unique weak limit point of $\{u^n\}$ and $\{v^n\}$.

If possible let w_1, w_2 be two weak limit points of $\{w^n\}$ and $\{w^{n_j}\}$,$\{w^{n_i}\}$ be two subsequences of $\{w^n\}$ that converges weakly to w_1, w_2, respectively.
Then, it follows that $\{\|A(w^n) - A(w_1)\|^2\}, \{\|A(w^n) - A(w_2)\|^2\}$ are convergent sequences.
Suppose, that

$$\alpha_1 = \lim_{n\to\infty} \|A(w^n) - A(w_1)\|^2, \tag{4.23}$$

$$\alpha_2 = \lim_{n\to\infty} \|A(w^n) - A(w_2)\|^2, \tag{4.24}$$

$$\alpha_3 = \lim_{n\to\infty} \|A(w_1) - A(w_2)\|^2. \tag{4.25}$$

Therefore, we can have

$$\begin{aligned} \|A(w^{n_j}) - A(w_2)\|^2 = {}& \|A(w^{n_j}) - A(w_1)\|^2 + \|A(w_1) - A(w_2)\|^2 \\ & +2\Big\langle A(w^{n_j}) - A(w_1), A(w_1) - A(w_2)\Big\rangle. \end{aligned} \tag{4.26}$$

$$\begin{aligned} \|A(w^{n_i}) - A(w_1)\|^2 = {}& \|A(w^{n_i}) - A(w_2)\|^2 + \|A(w_1) - A(w_2)\|^2 \\ & +2\Big\langle A(w^{n_i}) - A(w_2), A(w_2) - A(w_1)\Big\rangle. \end{aligned} \tag{4.27}$$

Letting $j \to \infty$ in (4.26) and $i \to \infty$ in (4.27), using the continuity of A and noting that w_1, w_2 are two weak limit points of $\{w^{n_j}\}$,$\{w^{n_i}\}$, we can get the third term on R.H.S of (4.26) and (4.27) converges to zero.
Now, from (4.23),(4.24) and (4.25), it follws that

$$\alpha_1 = \alpha_2 + \alpha_3,$$

or,

$$\alpha_1 - \alpha_2 = \alpha_3, \tag{4.28}$$

and

$$\alpha_2 = \alpha_1 + \alpha_3,$$

or,

$$\alpha_2 - \alpha_1 = \alpha_3. \tag{4.29}$$

On adding (4.28) and (4.29), we have $\alpha_3 = 0$. This in turn implies that $A(w_1) = A(w_2)$.
Again by using the δ-strongly monotonicity of A, we have

$$\begin{aligned} \delta\|w_1 - w_2\|^2 &\le \Big\langle A(w_1) - A(w_2), w_1 - w_2\Big\rangle \\ &\le \|A(w_1) - A(w_2)\| \, \|w_1 - w_2\| \end{aligned} \tag{4.30}$$

Since $A(w_1) = A(w_2)$, (4.30) implies that $w_1 = w_2$.
Thus it follows that all the weak limit points of $\{w^n\}$ are equal. That is, $\{u^n\}$ is weakly convergent to a solution of (2.1).

Similarly, following the same procedure as in (4.23)-(4.30), for any two weak limit points t_1, t_2 of $\{t^n\}$ we can show that $t_1 = t_2$. That is, $\{v^n\}$ is weakly convergent to a solution of (2.2). This completes the proof.

Similar results can be obtained for H-monotone operators. For the sake of completeness, we state the following result for H-monotone operators.
Corollary 4.2. Let X be a real Hilbert space. Let a single-valued map $S : X \times X \to X$ be continuous such that S is p-monotone and g-monotone with respect to H in the first and second argument, respectively, and S is p-monotone and g-monotone

in the first and second argument, respectively. Suppose that a continuous single-valued mapping $H : X \to X$ be δ-strongly monotone. Let an H-monotone mapping $M : X \to 2^X$ be monotone with respect to H. If the iterative sequences $\{u^n\}$,$\{v^n\}$ are generated by the following resolvent iterative algorithm
Step 1. Initiation Step:
Select $w^0, t^0 \in X$ and set $n = 0$.
Step 2. Resolvent Step:
Find $w^n, t^n \in X$ such that

$$u^n = J^M_{\rho^n,H}\Big\{H(w^n) - \rho^n S(p(u^n), g(u^n))\Big\}. \tag{4.31}$$

$$v^n = J^M_{\gamma^n,H}\Big\{H(t^n) - \gamma^n S(p(v^n), g(v^n))\Big\}. \tag{4.32}$$

where ρ^n, γ^n are such that

$$\inf_{n\geq 0}\rho^n, \gamma^n > 0. \tag{4.33}$$

Step 3. Projection step:
Set

$$C = \Big\{w \in X : \Big\langle H(w^n) - H(u^n), w - H(u^n)\Big\rangle \leq 0\Big\}.$$

If $H(w^n) = H(u^n)$, then stop, otherwise, choose w^{n+1} such that

$$H(w^{n+1}) = P_C(H(w^n)). \tag{4.34}$$

Again, set

$$C = \Big\{t \in X : \Big\langle H(t^n) - H(v^n), t - H(v^n)\Big\rangle \leq 0\Big\}.$$

If $H(t^n) = H(v^n)$, then stop, otherwise, choose t^{n+1} such that

$$H(t^{n+1}) = P_C(H(t^n)). \tag{4.35}$$

Step 4. Let $n = n + 1$ and return to Step 1.
then the sequences converges weakly to a solution of SNVP (2.1)-(2.2).

5. Conclusion

A system of nonlinear variational inclusion problem involving A-monotone mappings has been introduced in real Hilbert spaces. Using A-monotone mappings, a resolvent iterative algorithm has been constructed to solve the proposed system, and the convergence analysis of the resolvent iterative algorithm has been investigated. Moreover the obtained results are generalized to solve the system of variational inclusions involving A-monotone mappings. The obtained results generalize most of the results investigated in the literature, and offer a wide range of applications to future research on the sensitivity analysis, variational inclusion problems, variational inequality problems in Banach spaces. Researchers can use the proposed work in the future for research work. Proposed system of nonlinear variational inclusion problem in real Hilbert spaces finds and also will find greater applicability in various fields of real life in the future.

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