Electronic Journal of Mathematical Analysis and Applications
Vol. 9(2) July 2021, pp. 60-76.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

CERTAIN SUBCLASS OF ANALYTIC FUNCTION ASSOCIATED WITH A GENERALIZATION OF $q-$SALAGEAN OPERATOR WITH NEGATIVE COEFFICIENTS

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Abstract. In this paper, we study different properties for the new class $TY_q^{\lambda}(n,\alpha,\beta,\gamma)$ of analytic starlike and convex functions associated with a generalization of $q-$Salagean operator.

1. Introduction

Denote by $\mathcal{S}$ be the class of analytic and univalent functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k\ , z\in\mathbb{U}=\{z:z\in\mathbb{C}:|z|<1\}. \tag{1}$$

Also by $\mathcal{S}^*(\alpha)$ and $C(\alpha)$ the subclasses of $\mathcal{S}$ which are, respectively, starlike and convex functions of order $\alpha(0\leq\alpha<1)$, satisfying (see Robertson [22])

$$\mathcal{S}^*(\alpha)=\left\{f:f\in\mathcal{S}\ \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha\right\}, \tag{2}$$

and

$$C(\alpha)=\left\{f:f\in\mathcal{S}\ \text{ and } \Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\right\}. \tag{3}$$

It is follows from (2) and (3) that

$$f(z)\in C(\alpha)\Leftrightarrow zf'(z)\in S^*(\alpha).$$

For $f(z)\in\mathcal{S}$, given by (1) and $0<q<1$, the Jackson's $q-$derivative of a function f is given by [19] (see also [2], [7], [12], [14], [17], [26], [27], [34] and [35]):

$$D_qf(z)=\frac{f(z)-f(qz)}{(1-q)\,z},(z\in\mathbb{U},0<q<1,z\neq0), \tag{4}$$

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic functions, coefficient estimates, distortion, $q-$Al-Oboudi type difference operator, closure, modified Hadamard products, integral means, partial sums.

Submitted Aug. 27, 2020.

$D_q f(0) = f^{'}(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (4) we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1}, \tag{5}$$

where

$$[k]_q = \frac{1-q^k}{1-q} \quad (0 < q < 1). \tag{6}$$

As $q \to 1^-, [k]_q \to k$ and, so $D_q f(z) = f^{'}(z)$.

For $f(z) \in \mathcal{S}, \lambda \geq 0$ and $0 < q < 1$. Let

$$\begin{aligned} D^0_{\lambda,q} f(z) &= f(z), \\ D^1_{\lambda,q} f(z) &= (1-\lambda) f(z) + \lambda z D_q f(z) = D_{\lambda,q} f(z) \\ &= z + \sum_{k=2}^{\infty} \left[1 + \lambda([k]_q - 1)\right] a_k z^k \\ &\vdots \\ D^n_{\lambda,q} f(z) &= D_{\lambda,q}(D^{n-1}_{\lambda,q} f(z)) \ (n \in \mathbb{N}, \mathbb{N} = \{1,2,...\}). \end{aligned} \tag{7}$$

It follows from (1) and (7) that

$$\begin{aligned} D^n_{\lambda,q} f(z) &= z + \sum_{k=2}^{\infty} \left[1 + \lambda([k]_q - 1)\right]^n a_k z^k \\ (n &\in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \end{aligned} \tag{8}$$

where $[k]_q$ is given by (6).

We note that

(i) $\lim\limits_{q \longrightarrow 1^-} D^n_{\lambda,q} f(z) = D^n_\lambda f(z)$ (see Al-Oboudi [1] , Aouf and Mostafa [8] and Aouf et al. [13]);

(ii) $D^n_{1,q} f(z) = D^n_q f(z)$ (see Govindaraj and Sivasubramanian [18] and Aouf et al. [9]);

(iii) $\lim\limits_{q \longrightarrow 1^-} D^n_{1,q} f(z) = D^n f(z)$ see Salagean [25] (also see [3], [4], [5] and [6]).

Definition 1. For $0 < q < 1, -1 \leq \alpha \leq 1, \beta \geq 0, n \in \mathbb{N}_0, 0 \leq \gamma < 1$ and $\lambda \geq 0$, let $Y^\lambda_q(n,\alpha,\beta,\gamma)$ be the class consisting of functions $f \in \mathcal{S}$ satisfying

$$\Re\left\{\frac{zD_q(D^n_{\lambda,q} f(z))}{(1-\gamma)D^n_{\lambda,q} f(z) + \gamma z D_q(D^n_{\lambda,q} f(z))} - \alpha\right\} > \beta \left|\frac{zD_q(D^n_{\lambda,q} f(z))}{(1-\gamma)D^n_{\lambda,q} f(z) + \gamma z D_q(D^n_{\lambda,q} f(z))} - 1\right|, z \in \mathbb{U}. \tag{9}$$

Let $T \subset \mathcal{S}$ the class of functions

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \ , a_k \geq 0, \tag{10}$$

and

$$TY^\lambda_q(n,\alpha,\beta,\gamma) = Y^\lambda_q(n,\alpha,\beta,\gamma) \cap T. \tag{11}$$

We note that

(i) $\lim\limits_{q \to 1^-} TY^\lambda_q(n,\alpha,\beta,\gamma) = TY^\lambda(n,\alpha,\beta,\gamma)$:

$\{f \in T : \Re\left\{\frac{z(D^n_\lambda f(z))^{'}}{(1-\gamma)D^n_\lambda f(z) + \gamma z (D^n_\lambda f(z))^{'}} - \alpha\right\} > \beta \left|\frac{z(D^n_\lambda f(z))^{'}}{(1-\gamma)D^n_\lambda f(z) + \gamma z (D^n_\lambda f(z))^{'}} - 1\right|, z \in \mathbb{U}\}$;

(ii) $\lim_{q\to 1^-} TY_q^{\lambda}(n,\alpha,\beta,0) = TS_{\lambda}(n,\alpha,\beta)$ (see Aouf and Mostafa [8] and [13]);

(iii) $\lim_{q\to 1^-} TY_q^{1}(n,\alpha,\beta,0) = TS(n,\alpha,\beta)$ (see Rosy and Murugusundaramoorthy [23]);

(iv) $TY_q^{\lambda}(n,\alpha,\beta,0) = TY_q^{\lambda}(n,\alpha,\beta)$:
$\{f \in T : \Re\left\{\frac{zD_q(D^n_{\lambda,q}f(z))}{D^n_{\lambda,q}f(z)} - \alpha\right\} > \beta\left|\frac{zD_q(D^n_{\lambda,q}f(z))}{D^n_{\lambda,q}f(z)} - 1\right|, z \in \mathbb{U}\}$;

(v) $TY_q^{1}(n,\alpha,\beta,0) = T_q(n,\alpha,\beta)$:
$\{f \in T : \Re\left\{\frac{zD_q(D^n_{q}f(z))}{D^n_{q}f(z)} - \alpha\right\} > \beta\left|\frac{zD_q(D^n_{q}f(z))}{D^n_{q}f(z)} - 1\right|, z \in \mathbb{U}\}$;

(vi) $TY_q^{\lambda}(0,\alpha,\beta,\gamma) = TY_q^{\lambda}(\alpha,\beta,\gamma)$:
$\{f \in T : \Re\left\{\frac{zD_q(f(z))}{(1-\gamma)f(z)+\gamma zD_q(f(z))} - \alpha\right\} > \beta\left|\frac{zD_q(f(z))}{(1-\gamma)f(z)+\gamma zD_q(f(z))} - 1\right|, z \in \mathbb{U}\}$.

In [29], Silverman found that $f_2(z) = z - \frac{z^2}{2}$ is often extermal over the family T. He applied this function to resolve his integral means inequality, conjectured in [30] and settled in [31], that

$$\int_0^{2\pi} |f(re^{i\theta})|^{\zeta} d\theta \le \int_0^{2\pi} |f_2(re^{i\theta})|^{\zeta} d\theta,$$

for all $f \in T, \zeta > 0$ and $0 < r < 1$.

Lemma 1. [20] If f and g are analytic in $\mathbb{U}$ with $g \prec f$ where $\prec$ denotes subordination, then $\zeta > 0$ and $0 < r < 1$,

$$\int_0^{2\pi} |g(re^{i\theta})|^{\zeta} d\theta \le \int_0^{2\pi} |f(re^{i\theta})|^{\zeta} d\theta.$$

2. COEFFICIENT ESTIMATES

Unless indicated, we assume that $-1 \le \alpha < 1$, $\beta \ge 0, \lambda \ge 0$, $0 < q < 1, n \in \mathbb{N}_0, 0 \le \gamma < 1$, $f(z) \in T$ and $z \in \mathbb{U}$.

Theorem 1. A function $f(z) \in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$ if and only if

$$\sum_{k=2}^{\infty} \{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n a_k \le 1-\alpha. \quad (12)$$

Proof. Assume that (12) holds. Then it is suffices to show that

$$\begin{aligned} &\beta\left|\frac{zD_q(D^n_{\lambda,q}f(z))}{(1-\gamma)D^n_{\lambda,q}f(z)+\gamma zD_q(D^n_{\lambda,q}f(z))} - 1\right| \\ &-\Re\left\{\frac{zD_q(D^n_{\lambda,q}f(z))}{(1-\gamma)D^n_{\lambda,q}f(z)+\gamma zD_q(D^n_{\lambda,q}f(z))} - 1\right\} \\ \le\ & 1-\alpha. \end{aligned}$$

We have

$$\begin{aligned}
&\beta\left|\frac{zD_q(D^n_{\lambda,q}f(z))}{(1-\gamma)D^n_{\lambda,q}f(z)+\gamma zD_q(D^n_{\lambda,q}f(z))}-1\right| \\
&-\Re\left\{\frac{zD_q(D^n_{\lambda,q}f(z))}{(1-\gamma)D^n_{\lambda,q}f(z)+\gamma zD_q(D^n_{\lambda,q}f(z))}-1\right\} \\
\leq\ & (1+\beta)\left|\frac{zD_q(D^n_{\lambda,q}f(z))}{(1-\gamma)D^n_{\lambda,q}f(z)+\gamma zD_q(D^n_{\lambda,q}f(z))}-1\right| \\
\leq\ & \frac{(1+\beta)\sum\limits_{k=2}^{\infty}([k]_q-1)(1-\gamma)\left[1+\lambda([k]_q-1)\right]^n a_k}{1-\sum\limits_{k=2}^{\infty}[1+\gamma([k]_q-1)]\left[1+\lambda([k]_q-1)\right]^n a_k}.
\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ since (12) holds.

Conversely if $f(z) \in TY^{\lambda}_q(n,\alpha,\beta,\gamma)$ and z is real, then

$$\frac{1-\sum\limits_{k=2}^{\infty}[k]_q\left[1+\lambda([k]_q-1)\right]^n a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}[1+\gamma([k]_q-1)]\left[1+\lambda([k]_q-1)\right]^n a_k z^{k-1}}-\alpha \geq \beta\left|\frac{\sum\limits_{k=2}^{\infty}([k]_q-1)(1-\gamma)\left[1+\lambda([k]_q-1)\right]^n a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}[1+\gamma([k]_q-1)]\left[1+\lambda([k]_q-1)\right]^n a_k z^{k-1}}\right|.$$

Letting $z \to 1^-$ along the real axis, we obtain (12). Hence the proof is completed. □

Corollary 1. For $f(z) \in TY^{\lambda}_q(n,\alpha,\beta,\gamma)$, we have

$$a_k \leq \frac{1-\alpha}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}\ (k\geq 2) \tag{13}$$

and

$$f(z) = z-\frac{1-\alpha}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}z^k\ (k\geq 2), \tag{14}$$

gives the sharpness.

3. GROWTH AND DISTORTION THEOREMS

Theorem 2. Let $f(z) \in TY^{\lambda}_q(n,\alpha,\beta,\gamma)$. Then for $0 \leq i \leq n$,

$$\left|D^i_{\lambda,q}f(z)\right| \geq |z| - \frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n-i}}|z|^2 \tag{15}$$

and

$$\left|D^i_{\lambda,q}f(z)\right| \leq |z| + \frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n-i}}|z|^2. \tag{16}$$

The equalities in (15) and (16) are attained

$$f(z) = z-\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n-i}}z^2. \tag{17}$$

Proof. Note that $f(z) \in TY^{\lambda}_q(n,\alpha,\beta,\gamma)$ if and only if $D^i_{\lambda,q}f(z) \in TY^{\lambda}_q(n-i,\alpha,\beta,\gamma)$, where

$$D^i_{\lambda,q}f(z) = z-\sum_{k=2}^{\infty}\left[1+\lambda([k]_q-1)\right]^i a_k z^k. \tag{18}$$

Using Theorem 1, we get

$$\begin{aligned}&\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n-i}\sum_{k=2}^{\infty}\left[1+\lambda([k]_q-1)\right]^i a_k\\ \leq\quad & 1-\alpha, \qquad (19)\end{aligned}$$

that is, that

$$\sum_{k=2}^{\infty}\left[1+\lambda([k]_q-1)\right]^i a_k \leq \frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n-i}}. \qquad (20)$$

It follows from (18) and (20) that

$$\begin{aligned}\left|D^i_{\lambda,q}f(z)\right| &\geq |z|-|z|^2\sum_{k=2}^{\infty}\left[1+\lambda([k]_q-1)\right]^i a_k\\ &\geq |z|-\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n-i}}|z|^2, \qquad (21)\end{aligned}$$

and

$$\begin{aligned}\left|D^i_{\lambda,q}f(z)\right| &\leq |z|+|z|^2\sum_{k=2}^{\infty}\left[1+\lambda([k]_q-1)\right]^i a_k\\ &\leq |z|+\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n-i}}|z|^2. \qquad (22)\end{aligned}$$

Finally, we note that the bounds in (15) and (16) are attained for $f(z)$ defined by

$$D^i_{\lambda,q}f(z)=z-\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n-i}}z^2 \quad (z\in\mathbb{U}). \qquad (23)$$

This completes the proof. □

Corollary 2. Let $f(z)\in TY^{\lambda}_q(n,\alpha,\beta,\gamma)$. Then

$$|f(z)|\geq|z|-\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n}}|z|^2, \qquad (24)$$

and

$$|f(z)|\leq|z|+\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^{n}}|z|^2. \qquad (25)$$

The sharpenss are attained for $f(z)$ given by (17).

Proof. Taking $i=0$ in Theorem 2, we can easily obtain (24) and (25). □

4. CLOSURE THEOREMS

Let $f_i(z)$ be defined, for $i=1,2,...,m$, by

$$f_i(z)=z-\sum_{k=2}^{\infty}a_{k,i}z^k \quad (a_{k,i}\geq 0,\ z\in\mathbb{U}). \qquad (26)$$

Theorem 3. Let $f_i(z)\in TY^{\lambda}_q(n,\alpha,\beta,\gamma)$ for $i=1,2,...,m$. Then

$$g(z)=\sum_{i=1}^{m}c_if_i(z), \qquad (27)$$

is also in the same class, where $c_i\geq 0$, $\sum\limits_{i=1}^{m}c_i=1$.

Proof. According to (27), we can write

$$g(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{m} c_i a_{k,i} \right) z^k. \tag{28}$$

Further, since $f_i(z) \in TY_q^{\lambda}(n, \alpha, \beta, \gamma)$, we get

$$\sum_{k=2}^{\infty} \{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q - 1)]\} \left[1 + \lambda([k]_q - 1)\right]^n a_{k,i} \leq 1 - \alpha. \tag{29}$$

Hence

$$\begin{aligned} & \sum_{k=2}^{\infty} \{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q - 1)]\} \left[1 + \lambda([k]_q - 1)\right]^n \left(\sum_{i=1}^{m} c_i a_{k,i} \right) \\ = & \sum_{i=1}^{m} c_i \left[\sum_{k=2}^{\infty} \{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q - 1)]\} \left[1 + \lambda([k]_q - 1)\right]^n a_{k,i} \right] \\ \leq & \left(\sum_{i=1}^{m} c_i \right) (1-\alpha) = 1 - \alpha, \end{aligned} \tag{30}$$

which implies that $g(z) \in TY_q^{\lambda}(n, \alpha, \beta, \gamma)$. Thus we have the theorem. □

Corollary 3. The class $TY_q^{\lambda}(n, \alpha, \beta, \gamma)$ is closed under convex linear combination.

Proof. Let $f_i(z) \in TY_q^{\lambda}(n, \alpha, \beta, \gamma)$ $(i = 1, 2)$ and

$$g(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \leq \mu \leq 1), \tag{31}$$

Then by, taking $m = 2$, $c_1 = \mu$ and $c_2 = 1 - \mu$ in Theorem 3, we have $g(z) \in TY_q^{\lambda}(n, \alpha, \beta, \gamma)$. □

Theorem 4. Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{1-\alpha}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n} z^k \quad (k \geq 2). \tag{32}$$

Then $f(z) \in TY_q^{\lambda}(n, \alpha, \beta, \gamma)$ if and only if

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{33}$$

where $\mu_k \geq 0$ $(k \geq 1)$ and $\sum\limits_{k=1}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1-\alpha}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n} \mu_k z^k. \tag{34}$$

Then it follows that

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{1-\alpha} \cdot \frac{1-\alpha}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n} \mu_k \\ = & \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1. \end{aligned} \tag{35}$$

So by Theorem 1, $f(z) \in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$.

Conversely, assume that $f(z) \in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$. Then

$$a_k \leq \frac{1-\alpha}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n} \quad (k \geq 2). \tag{36}$$

Setting

$$\mu_k = \frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{1-\alpha} a_k \quad (k \geq 2), \tag{37}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \tag{38}$$

we see that $f(z)$ can be expressed in the form (33). This completes the proof. □

Corollary 4. The extreme points of $TY_q^{\lambda}(n,\alpha,\beta,\gamma)$ are $f_k(z)$ $(k \geq 1)$ given by Theorem 4.

5. SOME RADII OF THE CLASS $TY_q^{\lambda}(n,\alpha,\beta,\gamma)$

Theorem 5. Let $f(z) \in TY_q^{\lambda}(n,\alpha,\beta)$. Then for $0 \leq \rho < 1, k \geq 2, f(z)$ is

(i) close -to- convex of order ρ in $|z| < r_1$, where

$$r_1 = r_1(q,\alpha,\beta,\lambda,\gamma,\rho) := \inf_k \left[\frac{(1-\rho)\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{k(1-\alpha)} \right]^{\frac{1}{(k-1)}}, \tag{39}$$

(ii) starlike of order ρ in $|z| < r_2$, where

$$r_2 = r_2(q,\alpha,\beta,\lambda,\gamma,\rho) := \inf_k \left[\frac{(1-\rho)\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(k-\rho)(1-\alpha)} \right]^{\frac{1}{(k-1)}}, \tag{40}$$

(iii) convex of order ρ in $|z| < r_3$, where

$$r_3 = r_3(q,\alpha,\beta,\lambda,\gamma,\rho) := \inf_k \left[\frac{(1-\rho)\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{k(k-\rho)(1-\alpha)} \right]^{\frac{1}{(k-1)}}. \tag{41}$$

The result is sharp, for $f(z)$ given by (14).

Proof. To prove (i) we must show that

$$\left| f'(z) - 1 \right| \leq 1 - \rho \quad for \ \ |z| < r_1(q,\alpha,\beta,\gamma,\rho).$$

From (10), we have

$$\left| f'(z) - 1 \right| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$\left| f'(z) - 1 \right| \leq 1 - \rho,$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho} \right) a_k |z|^{k-1} \leq 1. \tag{42}$$

But, by Theorem 1, (42) will be true if

$$\left(\frac{k}{1-\rho} \right) |z|^{k-1} \leq \frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{k(1-\alpha)}\right]^{\frac{1}{(k-1)}} \quad (k \geq 2), \tag{43}$$

which gives (39).

To prove (ii) and (iii) it may to show

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho \ \text{ for } \ |z| < r_2, \tag{44}$$

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\rho \ \text{ for } \ |z| < r_3, \tag{45}$$

respectively, by using arguments as in proving (i). □

6. A FAMILY OF INTEGRAL OPERATORS

Theorem 6. Let the function $f(z)$ defined by (10) be in the class $TY_q^\lambda(n,\alpha,\beta,\gamma)$ and let $c > -1$ be real number. Then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (c > -1), \tag{46}$$

also belongs to the class $TY_q^\lambda(n,\alpha,\beta,\gamma)$.

Proof. It follows from (46) that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{47}$$

where

$$b_k = \frac{c+1}{c+k}a_k \leq a_k. \tag{48}$$

Therefore,

$$\begin{aligned}
&\sum_{k=2}^{\infty}\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)\}\left[1+\lambda([k]_q-1)\right]^n b_k \\
=\ &\sum_{k=2}^{\infty}\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n\left(\tfrac{c+1}{c+k}\right)a_k \\
\leq\ &\sum_{k=2}^{\infty}\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n a_k \leq 1-\alpha,
\end{aligned} \tag{49}$$

since $f(z) \in TY_q^\lambda(n,\alpha,\beta,\gamma)$. Hence, by Theorem 1, $F(z) \in TY_q^\lambda(n,\alpha,\beta,\gamma)$. □

Theorem 7. Let $F(z) = z - \sum_{k=2}^{\infty} a_k z^k \in TY_q^\lambda(n,\alpha,\beta)$, $(a_k \geq 0)$ and $c > -1$ be real number. Then $f(z)$ given by (46) is univalent in $|z| < R^*$, where

$$R^* = \inf_k \left[\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n(c+1)}{k(1-\alpha)(c+k)}\right]^{\frac{1}{k-1}} (k \geq 2). \tag{50}$$

The result is sharp.

Proof. From (46), we have

$$f(z)=\frac{z^{1-c}(z^cF(z))'}{c+1}=z-\sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right)a_kz^k. \tag{51}$$

In order to obtain the required result, it suffices to show that $\left|f'(z)-1\right|<1$ whenever $|z|<R^*$. Now

$$\left|f'(z)-1\right|\leq\sum_{k=2}^{\infty}k\left(\frac{c+k}{c+1}\right)a_k\left|z\right|^{k-1}.$$

Thus $\left|f'(z)-1\right|<1$, if

$$\sum_{k=2}^{\infty}k\left(\frac{c+k}{c+1}\right)a_k\left|z\right|^{k-1}<1. \tag{52}$$

But Theorem 1 confirms that

$$\sum_{k=2}^{\infty}\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^na_k}{1-\alpha}\leq 1. \tag{53}$$

Hence (52) will be satisfied if

$$k\left(\frac{c+k}{c+1}\right)\left|z\right|^{k-1}<\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{1-\alpha},$$

that is, if

$$|z|<\left[\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n(c+1)}{k(1-\alpha)(c+k)}\right]^{\frac{1}{k-1}}\ (k\geq 2). \tag{54}$$

Therefore the function $f(z)$ given by (46) is univalent in $|z|<R^*$. The sharpness follows if

$$f(z)=z-\frac{(1-\alpha)(c+k)}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n(c+1)}z^k\quad(k\geq 2;\ c>-1). \tag{55}$$

This completes the proof. □

7. MODIFIED HADAMARD PRODUCTS

Let $f_j(z)$ be defined, for $j=1,2,...,m$, by (26). Then for $m=2$, the modified Hadamard product is defined by

$$\left(f_1*f_2\right)(z)=z-\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k=\left(f_2*f_1\right)(z). \tag{56}$$

Theorem 8. If $f_j(z)\in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$, $(j=1,2)$ then

$$\left(f_1*f_2\right)(z)\in TY_q^{\lambda}(n,\tau,\beta,\gamma),$$

where

$$\tau=1-\frac{q(1+\beta)(1-\alpha)^2}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}^2(1+\lambda q)^n-(1-\alpha)^2}. \tag{57}$$

The result is sharp.

Proof. Employing the techniques used by Schild and Silverman [33], we need to find the largest τ such that

$$\sum_{k=2}^{\infty}\frac{\{[k]_q(1+\beta)-(\tau+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(1-\tau)}a_{k,1}a_{k,2}\leq 1. \tag{58}$$

Since

$$\sum_{k=2}^{\infty}\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(1-\alpha)}a_{k,j}\leq 1\ (j=1,2), \tag{59}$$

then Cauchy-Schwarz inequality yields

$$\sum_{k=2}^{\infty}\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(1-\alpha)}\sqrt{a_{k,1}a_{k,2}}\leq 1. \tag{60}$$

Thus it is sufficient to show that

$$\begin{aligned}&\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(1-\tau)}a_{k,1}a_{k,2}\\ \leq\ &\frac{\{[k]_q(1+\beta)-(\tau+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(1-\alpha)}\sqrt{a_{k,1}a_{k,2}},\end{aligned} \tag{61}$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}(1-\tau)}{\{[k]_q(1+\beta)-(\tau+\beta)[1+\gamma([k]_q-1)]\}(1-\alpha)}\ (k\geq 2). \tag{62}$$

Note that

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{(1-\alpha)}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}\quad(k\geq 2). \tag{63}$$

Consequently, we need only to prove that

$$\frac{(1-\alpha)}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}\leq\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}(1-\tau)}{\{[k]_q(1+\beta)-(\tau+\beta)[1+\gamma([k]_q-1)]\}(1-\alpha)}(k\geq 2), \tag{64}$$

or, equivalently, that

$$\tau\leq 1-\frac{([k]_q-1)(1+\beta)(1-\alpha)^2}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}^2\left[1+\lambda([k]_q-1)\right]^n-(1-\alpha)^2}\quad(k\geq 2). \tag{65}$$

Since

$$\Phi_q(k)=1-\frac{([k]_q-1)(1+\beta)(1-\alpha)^2}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}^2\left[1+\lambda([k]_q-1)\right]^n-(1-\alpha)^2}, \tag{66}$$

is an increasing function of k $(k\geq 2)$, letting $k=2$ in (66), we obtain

$$\tau\leq\Phi_q(2)=1-\frac{q(1+\beta)(1-\alpha)^2}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}^2(1+\lambda q)^n-(1-\alpha)^2}, \tag{67}$$

which is (57).

Finally, taking $f_j(z)$ $(j=1,2)$ of the form

$$f_j(z)=z-\frac{(1-\alpha)}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^n}z^2\quad(j=1,2), \tag{68}$$

we can see that the result is sharp. □

Theorem 9. Let $f_1(z)\in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$ and $f_2(z)\in TY_q^{\lambda}(n,\rho,\beta,\gamma)$. Then

$$\left(f_1*f_2\right)(z)\in TY_q^{\lambda}(n,\xi,\beta,\gamma),$$

where

$$\xi = 1-\frac{q(1+\beta)(1-\alpha)(1-\rho)}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}\{[2]_q(1+\beta)-(\rho+\beta)(1+\gamma q)\}(1+\lambda q)^n-(1-\alpha)(1-\rho)}. \tag{69}$$

The result is the best possible for the functions

$$f_1(z)=z-\frac{(1-\alpha)}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^n}z^2, \tag{70}$$

and

$$f_2(z)=z-\frac{(1-\rho)}{\{[2]_q(1+\beta)-(\rho+\beta)(1+\gamma q)\}(1+\lambda q)^n}z^2.$$

Proof. Proceeding as in the proof of Theorem 8, we get

$$\begin{aligned}\xi &\le B_q(k)\\ &= 1-\frac{([k]_q-1)(1+\beta)(1-\alpha)(1-\rho)}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\{[k]_q(1+\beta)-(\rho+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n-(1-\alpha)(1-\rho)} (k\ge 2).\end{aligned} \tag{71}$$

Since the function $B_q(k)$ is an increasing function of k $(k\ge 2)$, setting $k=2$ in (71), we get

$$\begin{aligned}\xi &\le B_q(2)\\ &= 1-\frac{q(1+\beta)(1-\alpha)(1-\rho)}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}\{[2]_q(1+\beta)-(\rho+\beta)(1+\gamma q)\}(1+\lambda q)^n-(1-\alpha)(1-\rho)}.\end{aligned} \tag{72}$$

This completes the proof. □

Corollary 5. Let $f_j(z)$ $(j=1,2,3)\in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$. Then

$$\left(f_1*f_2*f_3\right)(z)\in TY_q^{\lambda}(n,\delta,\beta,\gamma),$$

where

$$\delta=1-\frac{q(1+\beta)(1-\alpha)^3}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}^3(1+\lambda q)^{2n}-(1-\alpha)^3}. \tag{73}$$

The result is the best possible for $f_j(z)$ given by (69); $j=1,2,3$.

Proof. From Theorem 8, we have $\left(f_1*f_2\right)(z)\in TY_q^{\lambda}(n,\tau,\beta,\gamma)$, where τ is given by (57). By using Theorem 9, we get $\left(f_1*f_2*f_3\right)(z)\in TY_q^{\lambda}(n,\delta,\beta,\gamma)$, where

$$\begin{aligned}\delta &= 1-\frac{q(1+\beta)(1-\alpha)(1-\tau)}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}\{[2]_q(1+\beta)-(\tau+\beta)(1+\gamma q)\}(1+\lambda q)^n-(1-\alpha)(1-\tau)}\\ &= 1-\frac{q(1+\beta)(1-\alpha)^3}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}^3(1+\lambda q)^{2n}-(1-\alpha)^3}.\end{aligned}$$

This completes the proof. □

Theorem 10. Let $f_j(z)$ $(j=1,2)\in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$. Then

$$h(z)=z-\sum_{k=2}^{\infty}(a_{k,1}^2+a_{k,2}^2)z^k, \tag{74}$$

belongs to the class $TY_q^{\lambda}(n,\upsilon,\beta,\gamma)$, where

$$\upsilon=1-\frac{2q(1+\beta)(1-\alpha)^2}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}^2(1+\lambda q)^n-2(1-\alpha)^2}. \tag{75}$$

The result is sharp for $f_j(z)$ $(j=1,2)$ defined by (68).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(1-\alpha)}\right]^2 a_{k,j}^2$$

$$\leq \left[\sum_{k=2}^{\infty}\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(1-\alpha)}a_{k,j}\right]^2 \leq 1 \quad (j=1,2), \tag{76}$$

It follows that

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(1-\alpha)}\right]^2\left(a_{k,1}^2+a_{k,2}^2\right)\leq 1. \tag{77}$$

Therefore, we need to find the largest υ such that

$$\frac{\left[\{[k]_q(1+\beta)-(\upsilon+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n\right]}{(1-\tau)} \tag{78}$$

$$\leq \frac{1}{2}\left[\frac{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n}{(1-\alpha)}\right]^2 \quad (k\geq 2), \tag{79}$$

that is,

$$\upsilon \leq 1-\frac{2([k]_q-1)(1+\beta)(1-\alpha)^2}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}^2\left[1+\lambda([k]_q-1)\right]^n-2(1-\alpha)^2} \quad (k\geq 2). \tag{80}$$

Since

$$Q_q(k)=1-\frac{2([k]_q-1)(1+\beta)(1-\alpha)^2}{\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}^2\left[1+\lambda([k]_q-1)\right]^n-2(1-\alpha)^2}, \tag{81}$$

is an increasing function of k $(k\geq 2)$, we readily have

$$\upsilon \leq Q_q(2)=1-\frac{2q(1+\beta)(1-\alpha)^2}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}^2(1+\lambda q)^n-2(1-\alpha)^2}, \tag{82}$$

and Theorem 10 follows at once. □

Remarks 1:
(i) Letting $q\to 1^-$ and putting $\gamma=0$ in the above results, we obtain the results of Aouf and Mostafa [8] and Aouf et al. [13];
(ii) Letting $q\to 1^-$ and putting $\lambda=1$ and $\gamma=0$ in the above results, we obtain the results of Rosy and Murugusundaramoorthy [23].

8. Integral means

Applying Lemma 1, Theorem 1 and Corollary 1, we can prove the following results.

Theorem 11. Let $f(z)\in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$,$\zeta>0$ and $f_2(z)$ is defined by

$$f_2(z)=z-\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^n}z^2.$$

Then for $z=re^{i\theta}, 0<r<1$, we have

$$\int_0^{2\pi}|f(z)|^{\zeta}d\theta \leq \int_0^{2\pi}|f_2(z)|^{\zeta}d\theta. \tag{83}$$

Proof. From (10) and (83) is equivalent to proving that

$$\int_0^{2\pi}\left|1-\sum_{k=2}^{\infty}a_kz^{k-1}\right|^{\zeta}d\theta\le\int_0^{2\pi}\left|1-\tfrac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^n}z\right|^{\zeta}d\theta.$$

By Lemma 1, it sufficient to show that

$$1-\sum_{k=2}^{\infty}a_kz^{k-1}\prec 1-\tfrac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^n}z.$$

Setting

$$1-\sum_{k=2}^{\infty}a_kz^{k-1}=1-\tfrac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^n}\omega(z),$$

and using (13), we get

$$\begin{aligned}|\omega(z)| &= \left|\sum_{k=2}^{\infty}\tfrac{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^n}{1-\alpha}a_kz^{k-1}\right|\\ &\le |z|\sum_{k=2}^{\infty}\tfrac{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}(1+\lambda q)^n}{1-\alpha}a_k\le|z|.\end{aligned}$$

This completes the proof. □

9. PARTIAL SUMS

For $f(z)$ of the form (1), its sequence of partial sums is denoted by

$$f_m(z)=z+\sum_{k=2}^{m}a_kz^k(m\in\mathbb{N}\backslash\{1\}).$$

Silverman [32] determined sharp lower bounds for

$$\Re\left\{\frac{f(z)}{f_m(z)}\right\},\Re\left\{\frac{f_m(z)}{f(z)}\right\},\Re\left\{\frac{D_qf(z)}{D_qf_m(z)}\right\},\Re\left\{\frac{D_qf_m(z)}{D_qf(z)}\right\}.$$

We will follow the work of [32] and also the work cited in ([10], [11], [15], [16], [24], [28] and [21]) on partial sums of analytic functions. Also, we will make use of the well-known results that $\Re\left\{\frac{1+g(z)}{1-g(z)}\right\}>0$ if and only if $g(z)=\sum_{k=1}^{\infty}c_kz^k$ satisfies the inequality $|g(z)|\le|z|$.

We let

$$\begin{aligned}\Phi_{q,k}^n &= \Phi_q^n(k,\alpha,\beta,\gamma,\lambda)\\ &= \{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[1+\lambda([k]_q-1)\right]^n. \quad (84)\end{aligned}$$

Theorem 12. Let $f(z)\in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$ and $\frac{f(z)}{z}\ne0(0<|z|<1)$, then

$$\Re\left\{\frac{f(z)}{f_m(z)}\right\}\ge\frac{\Phi_{q,m+1}^n-(1-\alpha)}{\Phi_{q,m+1}^n}, \quad (85)$$

where

$$\Phi_{q,k}^n\ge\begin{cases}1-\alpha, & if\ k=2,3,...,m\\ \Phi_{q,m+1}^n & if\ k=m+1,m+2,...\end{cases}.$$

The result is sharp for every m, for the function

$$f(z)=z+\tfrac{1-\alpha}{\Phi^n_{q,m+1}}z^{m+1}. \tag{86}$$

Proof. Define $g(z)$ by

$$\begin{aligned}\frac{1+g(z)}{1-g(z)} &= \frac{\Phi^n_{q,m+1}}{1-\alpha}\left\{\frac{f(z)}{f_m(z)}-\frac{\Phi^n_{q,m+1}-(1-\alpha)}{\Phi^n_{q,m+1}}\right\}\\ &= \frac{1+\sum_{k=2}^m a_kz^{k-1}+\frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^\infty a_kz^{k-1}}{1+\sum_{k=2}^m a_kz^{k-1}}. \end{aligned}\tag{87}$$

It sufficient to show that $|g(z)|\le 1$. Now from (87), we have

$$g(z)=\frac{\frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^\infty a_kz^{k-1}}{2+2\sum_{k=2}^m a_kz^{k-1}+\frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^\infty a_kz^{k-1}}$$

and

$$|g(z)|\le\frac{\frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^\infty |a_k|}{2-2\sum_{k=2}^m |a_k|-\frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^\infty |a_k|}.$$

Now $|g(z)|\le 1$ iff

$$2\tfrac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^\infty|a_k|\le 2-2\sum_{k=2}^m|a_k|,$$

or

$$\sum_{k=2}^m|a_k|+\tfrac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^\infty|a_k|\le 1. \tag{88}$$

It sufficient to show that the left hand side of (88) is bounded above by

$$\sum_{k=2}^\infty\frac{\Phi^n_{q,k}}{1-\alpha}|a_k|,$$

which is equivalent to

$$\sum_{k=2}^m\frac{\Phi^n_{q,k}-(1-\alpha)}{1-\alpha}|a_k|+\sum_{k=m+1}^\infty\tfrac{\Phi^n_{q,k}-\Phi^n_{q,m+1}}{1-\alpha}|a_k|\ge 0.$$

To see that f given by (86) gives the sharp result, we observe for $z=re^{\frac{i\pi}{m}}$ that

$$\begin{aligned}\frac{f(z)}{f_m(z)} &= 1+\frac{1-\alpha}{\Phi^n_{q,m+1}}z^k\to 1-\frac{1-\alpha}{\Phi^n_{q,m+1}}\\ &= \frac{\Phi^n_{q,m+1}-(1-\alpha)}{\Phi^n_{q,m+1}}\end{aligned}$$

when $r\to 1^-$. Therefore we complete the proof. □

Theorem 13. Let $f(z)\in TY^\lambda_q(n,\alpha,\beta,\gamma)$ and $\frac{f(z)}{z}\neq 0(0<|z|<1)$, then

$$\Re\left\{\frac{f_m(z)}{f(z)}\right\}\ge\frac{\Phi^n_{q,m+1}}{\Phi^n_{q,m+1}+1-\alpha}. \tag{89}$$

The result is sharp for every m, for the function given by (86).

Proof. The proof follows by defining

$$\begin{aligned}\frac{1+g(z)}{1-g(z)} &= \tfrac{\Phi^n_{q,m+1}+1-\alpha}{1-\alpha}\left\{\tfrac{f_m(z)}{f(z)} - \tfrac{\Phi^n_{q,m+1}}{\Phi^n_{q,m+1}+1-\alpha}\right\}\\ &= \frac{1+\sum_{k=2}^{m} a_k z^{k-1} - \frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^{\infty} a_k z^{k-1}}{1+\sum_{k=2}^{\infty} a_k z^{k-1}},\end{aligned}$$

and much akin are to similiar arguments in Theorem 12. So, we omit it. □

Theorem 14. Let $f(z) \in TY_q^{\lambda}(n,\alpha,\beta,\gamma)$, then

$$\Re\left\{\frac{D_q f(z)}{D_q f_m(z)}\right\} \geq \tfrac{\Phi^n_{q,m+1}-[m+1]_q(1-\alpha)}{\Phi^n_{q,m+1}}, \tag{90}$$

and

$$\Re\left\{\frac{D_q f_m(z)}{D_q f(z)}\right\} \geq \tfrac{\Phi^n_{q,m+1}}{\Phi^n_{q,m+1}+[m+1]_q(1-\alpha)} \tag{91}$$

where $\Phi^n_{q,m+1} \geq [m+1]_q(1-\alpha)$ and

$$\Phi^n_{q,k} \geq \begin{cases} [k]_q(1-\alpha), & if\ k=2,3,...,m \\ [k]_q\left(\frac{\Phi^n_{q,m+1}}{[m+1]_q}\right) & if\ k=m+1,m+2,...\ . \end{cases}$$

The result is sharp for every m, for the function given by (86).

Proof. We prove only (90), which is similar in spirit of the proof of Theorem 12. The proof of (91) follows the pattern of that in Theorem 13. We write

$$\frac{1+g(z)}{1-g(z)} = \tfrac{\Phi^n_{q,m+1}}{1-\alpha}\left\{\frac{D_q f(z)}{D_q f_m(z)} - \tfrac{\Phi^n_{q,m+1}-[m+1]_q(1-\alpha)}{\Phi^n_{q,m+1}}\right\},$$

where

$$g(z) = \frac{\frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^{\infty}[k]_q a_k z^{k-1}}{2+2\sum_{k=2}^{m}[k]_q a_k z^{k-1} + \frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^{\infty}[k]_q a_k z^{k-1}}$$

and

$$|g(z)| \leq \frac{\frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^{\infty}[k]_q|a_k|}{2-2\sum_{k=2}^{m}[k]_q|a_k| - \frac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^{\infty}[k]_q|a_k|}$$

$|g(z)| \leq 1$ iff

$$2\tfrac{\Phi^n_{q,m+1}}{1-\alpha}\sum[k]_q|a_k| \leq 2-2\sum[k]_q|a_k|,$$

or

$$\sum_{k=2}^{m}[k]_q|a_k| + \tfrac{\Phi^n_{q,m+1}}{1-\alpha}\sum_{k=m+1}^{\infty}[k]_q|a_k| \leq 1. \tag{92}$$

The left hand side of (92) is bounded above by

$$\sum_{k=2}^{\infty}\frac{\Phi^n_{q,k}}{1-\alpha}|a_k|,$$

this complete the proof. □

Remarks 2:

(i) Letting $q \to 1^-$ and putting $\lambda = 1$ and $\gamma = 0$ in Theorems 12,13 and 14, we have results obtained by Mostafa [21];

(ii) Putting $\lambda = 1$ and $\gamma = 0$ in our results, we obtain corresponding results for the class $TY_q(n, \alpha, \beta)$;

(iii) Letting $q \to 1^-$ in our results, we obtain corresponding results for the class $TY^\lambda(n, \alpha, \beta, \gamma)$;

(iv) Putting $\gamma = 0$ in our results, we obtain corresponding results for the class $TY_q^\lambda(n, \alpha, \beta)$;

(v) Putting $n = 0$ in our results, we obtain corresponding results for the class $TY_q^\lambda(\alpha, \beta, \gamma)$.

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