K-BANHATTI INDICES, K-HYPER BANHATTI INDICES, FORGOTTEN INDEX, FIRST HYPER ZAGREB INDEX OF GENERALIZED TRANSFORMATION GRAPHS

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Abstract. Let $G$ be a simple connected graph. The K-Banhatti indices and K-Hyper Banhatti indices are introduced by Kulli in 2016. The Zagreb indices were introduced in 1972. In this paper, we established the expressions for the K-Banhatti indices, K-Hyper Banhatti indices, Zagreb indices, first Hyper Zagreb indices and Forgotten index of the generalized transformation graphs $G^{xy}$ and their complement graphs are obtained in terms of some parameters of a graph.

1. Introduction

The graphs considered here are finite, undirected without loops and multiple edges. Let $G$ be a connected graph with $n$ vertices and $m$ edges. The degree $d_G(u)$ of a vertex $u$. The edge connecting the vertices $u$ and $v$ will be denoted by $uv$. Let $d_G(e)$ denote the degree of an edge $e = uv$ in $G$. Which is denoted by $d_G(e) = d_G(u) + d_G(v) - 2$.

Topological indices are useful tool for modeling physical and chemical properties of molecules for design of pharmacoalogically active compounds for recognizing environmentally hazardons materials. A number of chemical applications especially to multiple quantum NMR Spectroscopy. Chemical graph theory is the topology branch of Mathematical Chemistry which applies graph theory to mathematical modelling of chemical phenomena.

A chemical graph is a graph in which the vertices correspond to atoms and edges to the bonds of a chemical structure. A single number that can be finding from the chemical graph and used to characterize some property of the underlying chemical is said to be a topological index or molecular structure descriptor. Lot of such descriptors have been considered in theoretical chemistry and have some applications especially in QSPR/QSAR fields of research see (5, 7).

The first and second Banhatti indices are first introduced by Kulli and are
denoted and defined as below

\[ B_1(G) = \sum_{ue} [d_G(u) + d_G(e)] \quad \text{and} \quad B_2(G) = \sum_{ue} d_G(u)d_G(e). \]

Where \(ue\) means that the vertex \(u\) and edge \(e\) are incident in \(G\).

In [7, 8] Kulli introduced the first and second K-Hyper Banhatti Indices and are defined as

\[ HB_1(G) = \sum_{ue} [d_G(u) + d_G(e)]^2 \quad \text{and} \quad HB_2(G) = \sum_{ue} [d_G(u)d_G(e)]^2. \]

The first and second Zagreb indices of a graph \(G\) are defined as,

\[ M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} [d_G(u)d_G(v)]. \]

In [2], Furtula et al., introduced the forgotten topological index \(F\), defined as

\[ F(G) = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2]. \]

In [13], Shirdel et al., introduced the first Hyper Zagreb index of \(G\) and defined as

\[ HM_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2. \]

The generalized transformation graph \(G^{xy}\) introduced recently by Basavanagoud et al [14], is a graph whose vertex set is \(V(G) \cup E(G)\) and \(\alpha, \beta \in V(G^{xy})\) the vertices \(\alpha, \beta\) are adjacent in \(G^{xy}\) if and only if (i) and (ii) satisfied.

(i) \(\alpha, \beta \in V(G)\), \(\alpha, \beta\) are adjacent in \(G\) if \(x = +\), and \(\alpha, \beta\) are not adjacent in \(G\) if \(x = -\).

(ii) \(\alpha \in V(G)\) and \(\beta \in E(G)\), \(\alpha, \beta\) are incident in \(G\) if \(y = +\), and \(\alpha, \beta\) are not incident in \(G\) if \(y = -\).

One can obtain the four graphical transformations of graphs as \(G^{++}, G^{+-}, G^{-+}, G^{--}\). An example of generalized transformation graphs and their complements are depicted in the Fig.1. Note that \(G^{++}\) is just the semitotal point graph of \(G\), which was introduced by Sampathkumar and Chikkodimath [12]. The vertex \(u\) of \(G^{xy}\) corresponding to a vertex \(u\) of \(G\) is referred to as a point vertex. The vertex \(e\) of \(G^{xy}\) corresponding to an edge of \(G\) is referred to as a line vertex.

**Lemma 1.1.** [1] Let \(G\) be a graph with \(n\) vertices and \(m\) edges, Let \(u \in V(G)\) and \(e \in E(G)\) then the degree of point and line vertices in \(G^{xy}\) are,

(i) \(d_{G^{xy}}^+(u) = 2d_G(u)\) and \(d_{G^{xy}}^+(e) = 2\)

(ii) \(d_{G^{xy}}^-(u) = m\) and \(d_{G^{xy}}^-(e) = n - 2\)

(iii) \(d_{G^{xy}}^+(u) = n - 1\) and \(d_{G^{xy}}^-(e) = 2\)

(iv) \(d_{G^{xy}}^-(u) = n + m - 1 - 2d_G(u)\) and \(d_{G^{xy}}^-(e) = n - 2\).

The complement of \(G\) will be denoted by \(\overline{G}\). If \(G\) has \(n\) vertices and \(m\) edges then the number of vertices of \(G^{xy}\) is \(n + m\). By Lemma 1.1 and taking into account that \(d_{\overline{G}}(u) = n - 1 - d_G(u)\). We have following Lemma.

**Lemma 1.2.** [1] Let \(G\) be a graph with \(n\) vertices and \(m\) edges, Let \(u \in V(G)\) and \(e \in E(G)\) then the degree of point and line vertices in \(G^{xy}\) are,

(i) \(d_{G^{xy}}^+(u) = 2d_G(u)\) and \(d_{G^{xy}}^-(e) = 2\)

(ii) \(d_{G^{xy}}^-(u) = m\) and \(d_{G^{xy}}^+(e) = n - 2\)

(iii) \(d_{G^{xy}}^+(u) = n - 1\) and \(d_{G^{xy}}^-(e) = 2\)

(iv) \(d_{G^{xy}}^-(u) = n + m - 1 - 2d_G(u)\) and \(d_{G^{xy}}^-(e) = n - 2\).
Lemma 1.3. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges, then

$$B_2(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2 - 2[d_G(u) + d_G(v)].$$

In this paper, we obtain the expressions for the K-Banhatti indices, K-Hyper Banhatti indices, Zagreb indices, Forgotten index and first Hyper Zagreb index of generalized transformation graphs $G^{xg}$ and of their complements $\overline{G^{xg}}$ in terms of some parameters of a graph.

![Graphs and Complements]

Fig.1: the Graph $G$, its generalized transformations $G^{xg}$ and their complements $\overline{G^{xg}}$.

2. The First Banhatti Index of $G^{xg}$

Theorem 2.1. Let $G$ be a graph with $n$ vertices and $m$ edges, then

$$B_1(G^{++}) = 6M_1(G) - 4m + \sum_{u \in V(G)} 2d_G(u)[3d_G(u) + 1].$$

Proof. Partition the edge set $E(G^{++})$ into subsets $E_1$ and $E_2$. Where $E_1 = \{uv/uv \in E(G)\}$ and $E_2 = \{ue/ the vertex u is incident to the edge e in G\}$. It is easy to check that $|E_1| = m$ and $|E_2| = 2m$.

By Lemma 1.1, if $u \in V(G)$ then $d_G^{++}(u) = 2d_G(u)$ and if $e \in E(G)$ then
\( d_G^+(e) = 2 \). Therefore,

\[
B_1(G^{++}) = \sum_{ue} [d_G^+(u) + d_G^+(e)]
\]

\[
= \sum_{uv \in E(G^{++})} [3d_G^+(u) + 3d_G^+(v) - 4]
\]

\[
= 6M_1(G) - 4m + \sum_{u \in V(G)} [3d_G(u) + 2].
\]

In the second part of above equation, the quantity \([6d_G(u) + 2]\) appears \(d_G(u)\) times, hence above expression can be written as,

\[
B_1(G^{++}) = 6M_1(G) - 4m + \sum_{u \in V(G)} 2d_G(u)[3d_G(u) + 1].
\]

\[\square\]

**Theorem 2.2.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges, then

\[
B_1(G^{-+}) = 2m(3m - 2) + m(n - 2)(3m + 3n - 10).
\]

**Proof.** Partition the edge set \( E(G^{-+}) \) into subsets \( E_1 \) and \( E_2 \).

Where \( E_1 = \{uv/uv \in E(G)\} \) and \( E_2= \{ue / the vertex u is not incident to the edge e in G \} \). It is easy to check that \(|E_1| = m\) and \(|E_2| = m(n - 2)\). By Lemma 1.1, if \( u \in V(G) \) then \( d_G^+(u) = n - 1 \) and if \( e \in E(G) \) then \( d_G^+(e) = 2 \). Therefore,

\[
B_1(G^{-+}) = \sum_{ue} [d_G^-(u) + d_G^-(e)]
\]

\[
= \sum_{uv \in E(G^{-+})} [3d_G^-(u) + 3d_G^-(v) - 4]
\]

\[
= m(6m - 4) + \sum_{u \in E_2} (3m + 3n - 10)
\]

\[
= 2m(3m - 2) + m(n - 2)(3m + 3n - 10).
\]

\[\square\]

**Theorem 2.3.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges, then

\[
B_1(G^{-+}) = n(n - 1)(3n - 5) + 8m.
\]

**Proof.** Partition the edge set \( E(G^{-+}) \) into subsets \( E_1 \) and \( E_2 \).

Where \( E_1 = \{uv/uv \notin E(G)\} \) and \( E_2= \{ue / the vertex u is incident to the edge e in G \} \). It is easy to check that \(|E_1| = \left( \frac{n}{2} \right) - m\) and \(|E_2| = 2m\). By Lemma 1.1, if \( u \in V(G) \) then \( d_G^+(u) = n - 1 \) and if \( e \in E(G) \) then \( d_G^+(e) = 2 \). Therefore,
Theorem 3.1. Partition the edge set

\[ B_1(G^{-+}) = \sum_{u \in E} [d_G^+(u) + d_G^-(e)] \]

\[ = \sum_{u \in E(G^{-+})} [3d_G^+(u) + 3d_G^-(v) - 4] \]

\[ = \left( \frac{n}{2} \right) - m(6m - 4) + 2m(3n - 1) \]

\[ = n(n - 1)(3n - 5) + 8m. \]

\[ \square \]

Theorem 2.4. Let G be a graph with n vertices and m edges, then

\[ B_1(G^{-+}) = \sum_{uv \notin E(G)} (6m + 6n - 10) - 2M_1(G) + \sum_{uv \in E_2} [2n + m - 7 - 2d_G(u)]. \]

Proof. Partition the edge set \( E(G^{-+}) \) into subsets \( E_1 \) and \( E_2 \).

Where \( E_1 = \{ uv / uv \notin E(G) \} \) and \( E_2 = \{ uv / \text{the vertex u is not incident to the edge e in G} \} \). It is easy to check that \( |E_1| = \left( \frac{n}{2} \right) - m \) and \( |E_2| = m(n - 2) \). By Lemma 1.1, if \( u \in V(G) \) then \( d_G^-(u) = n + m - 1 - 2d_G(u) \) and if \( e \in E(G) \) then \( d_G^-(e) = n - 2 \). Therefore,

\[ B_1(G^{-+}) = \sum_{uv \notin E(G)} [d_G^-(u) + d_G^-(e)] \]

\[ = \sum_{uv \notin E(G)} [3d_G^-(u) + 3d_G^-(v) - 4] \]

\[ = \sum_{uv \notin E(G)} (6n + 6m - 10) - 2[d_G(u) + d_G(v)] + \sum_{uv \in E_2} [2n + m - 7 - 2d_G(u)] \]

\[ = \sum_{uv \notin E(G)} (6n + 6m - 10) - 2M_1(G) + \sum_{uv \in E_2} [2n + m - 7 - 2d_G(u)]. \]

\[ \square \]

Note 1: In Theorems 2.2 and 2.3, graphs \( G \) having same number of vertices and edges, then \( B_1(G^{-+}) \) and \( B_1(G^{-+}) \) are same.

3. The First Banhatti Index of \( G^{xy} \)

Theorem 3.1. Let \( G \) be a graph with \( n \) vertices and \( m \) edges, then

\[ B_1(G^{++}) = \sum_{uv \notin E(G)} 2(3n + 3m - 5) - 2M_1(G) + \sum_{uv \in V(G)} [2m - d_G(u)] \left[ 3n + 3m - 7 - d_G(u) \right] + \sum_{uv \in E(G)} [2n + m - 7 - 2d_G(u)] \]

\[ + m(m - 1)(3n + 3m - 11). \]

Proof. Partition the edge set \( E(G^{++}) \) into subsets \( E_1, E_2 \) and \( E_3 \). Where \( E_1 = \{ uv / uv \notin E(G) \} \), \( E_2 = \{ uv / \text{the vertex u is not incident to the edge e in G} \} \) and \( E_3 = \{ ef / e, f \in E(G) \} \). It is easy to check that \( |E_1| = \left( \frac{n}{2} \right) - m \), \( |E_2| = m(n - 2) \), and \( |E_3| = \left( \frac{m}{2} \right) \). By Lemma 1.2, If \( u \in V(G) \) then \( d_G^{++}(u) = n + m - 1 - 2d_G(u) \)
and If $e \in E(G)$ then $d_{G^{-+}}(e) = n + m - 3$. Therefore,

$$B_1(G^{-+}) = \sum_{u \in E(G^{-+})} [d_{G^{-+}}(u) + d_{G^{-+}}(e)]$$

$$= \sum_{u \in E(G^{-+})} [3d_{G^{-+}}(u) + 3d_{G^{-+}}(v) - 4]$$

$$= \sum_{uv \in E(G)} (6n + 6m - 10) - 2M_1(G) + \sum_{ue \in E_2} [6n + 6m - 14 - 2d_G(u)]$$

$$+ \sum_{ef \in E_3} [6n + 6m - 22]$$

$$= \sum_{uv \in E(G)} 3(3n + 3m - 5) - 2M_1(G) + \sum_{u \in V(G)} 2|m - d_G(u)| |3n + 3m - 7 - d_G(u)|$$

$$+ m(m - 1)(3n + 3m - 11).$$

□

**Theorem 3.2.** Let $G$ be a graph with $n$ vertices and $m$ edges, then

$$B_1(G^{-+}) = n(n-1)(3n-5) - 2m(3n-5) + 2m(3n+3m-4) + m(m-1)(3m+1).$$

**Proof.** Partition the edge set $E(G^{-+})$ into subsets $E_1$, $E_2$ and $E_3$ Where $E_1 = \{uv /uv \notin E(G)\}$, $E_2 = \{ue / the vertex $u$ is incident to the edge $e$ in $G$\}$ and $E_3 = \{ef / ef \in E(G)\}$.

It is easy to check that $|E_1| = \left(\frac{n}{2}\right) - m$, $|E_2| = 2m$ and $|E_3| = \left(\frac{m}{2}\right)$. By Lemma 1.2, if $u \in V(G)$ then $d_{G^{-+}}(u) = n - 1$ and if $e \in E(G)$ then $d_{G^{-+}}(e) = m + 1$. Therefore,

$$B_1(G^{-+}) = \sum_{u \in E(G^{-+})} [d_{G^{-+}}(u) + d_{G^{-+}}(e)]$$

$$= \sum_{u \in E(G^{-+})} [3d_{G^{-+}}(u) + 3d_{G^{-+}}(v) - 4]$$

$$= \left[\left(\frac{n}{2}\right) - m\right] + 2m(3n + 3m - 4) + \left(\frac{m}{2}\right)(6m + 2)$$

$$= n(n - 1)(3n - 5) - 2m(3n - 5) + 2m(3n + 3m - 4) + m(m - 1)(3m + 1).$$

□

**Theorem 3.3.** Let $G$ be a graph with $n$ vertices and $m$ edges, then

$$B_1(G^{-+}) = m(6m - 4) + m(n - 2)(n + 2m - 7) + m(m - 1)(3n + 3m - 11).$$

**Proof.** Partition the edge set $E(G^{-+})$ into subsets $E_1$, $E_2$ and $E_3$. Where $E_1 = \{uv /uv \in E(G)\}$, $E_2 = \{ue / the vertex $u$ is not incident to the edge $e$ in $G$\}$ and $E_3 = \{ef / ef \in E(G)\}$.

It is easy to check that $|E_1| = m$, $|E_2| = (n - 2)m$ and $|E_3| = \left(\frac{m}{2}\right)$. By Lemma 1.2, if $u \in V(G)$ then $d_{G^{-+}}(u) = m$ and if $e \in E(G)$ then $d_{G^{-+}}(e) = n + m - 3$. 

□
Therefore,

\[ B_1(G^{-\rightarrow}) = \sum_{ue} [d_{G^{-\rightarrow}}(u) + d_{G^{-\rightarrow}}(e)] \]

\[ = \sum_{uv \in E(G^{-\rightarrow})} [3d_{G^{-\rightarrow}}(u) + d_{G^{-\rightarrow}}(v) - 4] \]

\[ = \sum_{uv \in E(G^-)} (6m - 4) + \sum_{ue \in E_2} (n + 2m - 7) + \sum_{ef \in E_3} (6n + 6m - 22) \]

\[ = m(6m - 4) + m(n - 2)(n + 2m - 7) + m(m - 1)(3n + 3m - 11). \]

\[ \square \]

**Theorem 3.4.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges, then

\[ B_1(G^{-\rightarrow}) = 6M_1(G) - 4m + \sum_{u \in V(G)} [6d_G(u) + 3m - 1] + m(m - 1)(3m + 1). \]

**Proof.** Partition the edge set \( E(G^{-\rightarrow}) \) into subsets \( E_1, E_2 \) and \( E_3 \). Where \( E_1 = \{uv / uv \in E(G)\}, E_2 = \{ue / \text{the vertex } u \text{ is incident to the edge } e \in G\} \) and \( E_3 = \{ef / ef \in E(G)\} \). It is easy to check that \(|E_1| = m, |E_2| = 2m \) and \(|E_3| = \binom{m}{2} \). By Lemma 1.2, If \( u \in V(G) \) then \( d_{G^{-\rightarrow}}(u) = 2d_G(u) \) and If \( e \in E(G) \) then \( d_{G^{-\rightarrow}}(e) = m + 1. \) Therefore,

\[ B_1(G^{-\rightarrow}) = \sum_{ue} [d_{G^{-\rightarrow}}(u) + d_{G^{-\rightarrow}}(e)] \]

\[ = \sum_{uv \in E(G^{-\rightarrow})} [3d_{G^{-\rightarrow}}(u) + d_{G^{-\rightarrow}}(v) - 4] \]

\[ = 6 \sum_{uv \in E(G)} (d_G(u) + d_G(v)) - 4 \sum_{ue \in E_2} [6d_G(u) + 3m - 1] + m(m - 1)(3m + 1) \]

\[ = 6M_1(G) - 4m + \sum_{u \in V(G)} [6d_G(u) + 3m - 1] + m(m - 1)(3m + 1). \]

\[ \square \]

**Note 2:** In Theorems 3.2 and 3.3, graphs \( G \) having same number of vertices and edges, then \( B_1(G^{+\rightarrow}) \) and \( B_1(G^{-\rightarrow}) \) are same.

### 4. The Second Banhatti Index, K Hyper Banhatti Indices, Forgotten Index and First Hyper-Zagreb Index of \( G_{xy} \) and \( \overline{G}_{xy} \)

In this section, we present some results without proof on \( G_{xy} \) and its complement related to some indices. Because the proof technique adopted here similar to the previous theorems.
**Theorem 4.1.** Let $G$ be a graph with $n$ vertices and $m$ edges, then

\[ i) B_2(G^{++}) = 4HM_1(G) - 4M_1(G) + 4 \sum_{u \in V(G)} d_G(u)^2[d_G(u) + 1]. \]

\[ ii) B_2(G^{+-}) = 4m^2(m - 1) + m(m - 2)(m + n - 2)(m + n - 4). \]

\[ iii) B_2(G^{-+}) = 2m(n - 1)^2(n - 2) - 4m(n - 1)(n - 2) + 2m(n - 1)(n + 1). \]

\[ iv) B_2(G^{- -}) = \sum_{uv \in E(G)} 4[n + m - 1 - d_G(u) - d_G(v)][n + m - 3 - d_G(u) - d_G(v)] \]

\[ + \sum_{u \in V(G)} [m - d_G(u)][2n + m - 3 - 2d_G(u)][2n + m - 5 - 2d_G(u)]. \]

\[ v) B_2(G^{++}) = \sum_{uv \in E(G)} 4[n + m - 1 - d(u) - d_G(v)][n + m - 2 - d_G(u) - d_G(v)] \]

\[ + \sum_{u \in V(G)} 4[m - d_G(u)][n + m - 2 - d_G(u)][n + m - 3 - d_G(u)] \]

\[ + 2m(m - 1)(n + m - 4). \]

\[ vi) B_2(G^{+-}) = 2m(n - 1)^2(n - 2) - 4m(n - 1)(n - 2) + 2m(n + m)(n + m - 2) \]

\[ + 2m^2(m - 1)(m + 1). \]

\[ vii) B_2(G^{-+}) = 4m^2(m - 1) + m(n - 2)(n + 2m - 3)(n + 2m - 5) \]

\[ + 2m(m - 1)(n + m - 3)(n + m - 4). \]

\[ viii) B_2(G^{- -}) = 4HM_1(G) - 4M_1(G) + \sum_{u \in V(G)} d_G(u)[2d_G(u) + m + 1][2d_G(u) + m - 1] \]

\[ + 2m^2(m - 1)(m + 1). \]

**Theorem 4.2.** Let $G$ be a graph with $n$ vertices and $m$ edges, then

\[ i) HB_1(G^{++}) = m[6d_G(u) + 6d_G(v) - 4]^2 + 8m[3d_G(u) + 1]^2. \]

\[ ii) HB_1(G^{+-}) = 4m(3m - 2)^2 + m(n - 2)(3m + 3n - 10)^2. \]

\[ iii) HB_1(G^{-+}) = 2n(n - 1)(3n - 5)^2 - 4m(3n - 5)^2 + 2m(3n - 1)^2. \]

\[ iv) HB_1(G^{- -}) = \sum_{uv \in E(G)} [6(n + m - d_G(u) - d_G(v)) - 10]^2 \]

\[ + [m - d_G(u)][6(n + m - d_G(u) - d_G(v)) - 10]^2. \]

\[ v) HB_2(G^{++}) = \sum_{uv \in E(G)} 64d_G(u)^2[2d_G(u) - 1]^2 + \sum_{u \in E_2} 4d_G(u)^2[2d_G(u) + 2]^2. \]

\[ vi) HB_2(G^{+-}) = 16m^3(m - 1)^2 + m(n - 2)(m + n - 2)^2(m + n - 4)^2. \]

\[ vii) HB_2(G^{-+}) = 8n(n - 1)^3(n - 2)^2 - 16m(n - 1)^2(n - 2)^2 + 2m(n - 1)^2(n + 1)^2. \]

\[ viii) HB_2(G^{- -}) = \sum_{uv \in E(G)} 16[n + m - 1 - d_G(u) - d_G(v)]^2[n + m - 2 - d_G(u) - d_G(v)]^2 \]

\[ + m[n - 2][2n + m - 3 - 2d_G(u)]^2[2n + m - 5 - 2d_G(u)]^2. \]
Theorem 4.3. Let $G$ be a graph with $n$ vertices and $m$ edges, then

\[ i) HB_1(G^{++}) = \sum_{uv \in E(G)} \{6[n + m - 1 - d_G(u) - d_G(v)] - 4\}^2 \]
\[ + \sum_{uv \in V(G)} [m - d_G(u)][6(n + m - 1 - d_G(u)) - 4]^2 + m(m - 1)(3n + 3m - 11)^2. \]

\[ ii) HB_1(G^{+-}) = 2n(2n - 1)(3n - 5)^2 - 4m(3n - 5)^2 + 2m(3n + 3m - 4)^2 \]
\[ + 2m(m - 1)(3m + 1)^2. \]

\[ iii) HB_1(G^{-+}) = m(6m - 4)^2 + m(m - 2)(3n + 6m - 13) + m(m - 1)(3n + 3m - 11)^2. \]

\[ iv) HB_1(G^{--}) = \sum_{uv \in E(G)} [6d_G(u) + 6d_G(v) - 4]^2 \]
\[ + 2m(m - 1)(3m + 1)^2. \]

\[ v) HB_2(G^{++}) = \sum_{uv \in E(G)} [6(n + m - 1 - d_G(u) - d_G(v))]^2 \]
\[ + \sum_{uv \in V(G)} [m - d_G(u)][2n + 2m - 4 - 2d_G(u) - d_G(v)]^2 \]
\[ + 8m(m - 1)(n + m - 3)^2(n + m - 7). \]

\[ vi) HB_2(G^{+-}) = \sum_{uv \in E(G)} [16(n - 1)^2(n - 2)^2 + 2m(n + m)^2(n + m - 2)^2 \]
\[ + 8m(m - 1)(m + 1)^2(m - 3)^2. \]

\[ vii) HB_2(G^{-+}) = 16m^3(m - 1)^2 + m(n - 2)(n + 2m - 3)^2(n + 2m - 5)^2 \]
\[ + 8m(m - 1)(n + m - 3)^2(n + m - 7)^2. \]

\[ viii) HB_2(G^{--}) = \sum_{uv \in E(G)} [16d_G(u) + d_G(v)]^2 \]
\[ + \sum_{uv \in V(G)} [2d_G(u) + m + 1]^2[2d_G(u) + m - 1]^2 + 8m^3(m - 1)(m + 1)^2. \]

Theorem 4.4. Let $G$ be a graph with $n$ vertices and $m$ edges, then

\[ i) F(G^{++}) = 4F(G) + 4 \sum_{uv \in E_2} [d_G(u)^2 + 1]. \]

\[ ii) F(G^{+-}) = 2m^3 + m(n - 2)[m^2 + (n - 2)^2] \]

\[ iii) F(G^{-+}) = n(n - 1)^3 + 8m. \]

\[ iv) F(G^{--}) = \sum_{uv \in E(G)} [2n + m - 1 - 2d_G(u)]^2 \]
\[ + \sum_{uv \in E_2} \{n + m - 1 - 2d_G(u)]^2 + (n - 2)^2\}. \]
Let Theorem 4.5.

\begin{enumerate}
\item \(vHM_1(G^{++}) = 4HM_1(G) + 4 \sum_{u \in E_2} [d_G(u) + 1]^2 \). \\
\item \(viHM_1(G^{+-}) = 4m^3 + mn(n-2)(m+n-2)^2 \). \\
\item \(viiHM_1(G^{-+}) = 2n(n-1)^3 - 4m(n-1)^2 + 2m(n+1)^2 \). \\
\item \(viiiHM_1(G^{--}) = \sum_{uv \notin E} 4[n + m - 1 - 2d_G(u)]^2 + \sum_{u \in E_2} [2n + m - 3 - 2d_G(u)]^2 \).
\end{enumerate}

**Theorem 4.5.** Let \(G\) be a graph with \(n\) vertices and \(m\) edges, then

\[ i) F(G^{++}) = \sum_{uv \notin E(G)} \{[n + m - 1 - 2d_G(u)]^2 + [n + m - 1 - 2d_G(v)]^2 \} \]

\[ + \sum_{u \in V(G)} \{m - d_G(u)\{[n + m - 1 - 2d_G(u)]^2 + [n + m - 3]^2 \} \]

\[ + m(m-1)(m+n-3)^2 \].

\[ ii) F(G^{+-}) = n(n-1)^3 - 2m(n-1)^2 + 2m[(n-1)^2 + (m+1)^2] + m(m-1)(m+1)^2 \].

\[ iii) F(G^{-+}) = 2m^3 + m(n-2)[m^2 + (n+m-3)^2] + m(m-1)(n+m-3)^2 \].

\[ iv) F(G^{--}) = 4F(G) + \sum_{u \in V(G)} 2d_G(u)[4d_G(u)^2 + (m+1)^2] + m(m-1)(m+1)^2 \].

\[ v)HM_1(G^{++}) = 4 \sum_{uv \notin E(G)} [n + m - 1 - d_G(u) - d_G(v)]^2 + \\
\]

\[ 4 \sum_{u \in V(G)} [m - d_G(u)][n + m - 2 - d_G(u)]^2 + 2m(m-1)(n+m-3)^2 \].

\[ vi)HM_1(G^{+-}) = 2n(n-1)^3 - 4m(n-1)^2 + 2m(n+m)^2 + 2m(m-1)(m+1)^2 \].

\[ vii)HM_1(G^{-+}) = 4m^3 + m(n-2)(n+2m-3)^2 + 2m(m-1)(n+m-3)^2 \].

\[ viii)HM_1(G^{--}) = 4HM_1(G) + \sum_{u \in V(G)} 2d_G(u)[2d_G(u) + m + 1]^2 + 2m(m-1)(m+1)^2 \].

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