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OSCILLATION RESULTS FOR SYSTEM OF DIFFERENTIAL EQUATIONS WITH SEVERAL DELAY TERMS

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Abstract. In this paper, we provide sufficient conditions for the oscillation of every solution of the system of differential equations with several delay terms

$$x'(t) + \sum_{i=1}^{m} P_i x(t - \tau_i) = 0,$$

where $P_i \in \mathbb{R}^{r\times r}$ and $\tau_i \in \mathbb{R}^+$ for $i = 1, 2, ..., m$. Furthermore, we provide sufficient conditions for the oscillation of all solutions of the system of difference equations with continuous arguments

$$x(t) - x(t - \tau) + \sum_{i=1}^{m} P_i x(t - \sigma_i) = 0,$$

where $P_i \in \mathbb{R}^{r\times r}$ and $\tau, \sigma_i \in \mathbb{R}^+$ for $i = 1, 2, ..., m$. The conditions are given in terms of the eigenvalues of the P_i matrix for $i = 1, 2, ..., m$.

1. Introduction

Recently, there has been a lot of studies concerning the oscillatory behaviour of differential and difference equations, see [1-14] and the references cited therein. In [9], Ladas and Sficas obtained some results about oscillatory behaviour of all solutions of the following differential equations;

$$x'(t) + px(t - \tau) = 0, \tag{1.1}$$

where $p, \tau \in \mathbb{R}$ and

$$x'(t) + \sum_{i=1}^{m} p_i x(t - \tau_i) = 0, \tag{1.2}$$

where $p_i \in \mathbb{R}$ and $\tau_i \in \mathbb{R}^+$ for $i = 1, 2, ..., m$. (Also see [6])
In [4], Ferreira and Györi obtained the necessary and sufficient conditions for the oscillation of all solutions of linear autonomous system of differential equations

$$x'(t) + Px(t - \tau) = 0, \tag{1.3}$$

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where $P \in \mathbb{R}^{r\times r}$and $\tau\geq 0$. Furthermore, they obtained the sufficient conditions for the oscillation of all solutions of the system of differential equations

$$x'(t) + \sum_{i=1}^{m} P_i x(t - \tau_i) = 0, \tag{1.4}$$

where $P_i \in \mathbb{R}^{r\times r}$ and $\tau_i \in \mathbb{R}^+$ for $i = 1, 2, ..., m$.
In [12], Öğünmez and Öcalan obtained the necessary and sufficient conditions for the oscillation of all solutions of linear autonomous system of differential equations

$$x^{(m)}(t) + Px(t - \tau) = 0, \tag{1.5}$$

where $P \in \mathbb{R}^{s\times s}$and $\tau \in \mathbb{R}^+$. Also, the authors obtained the sufficient conditions for the oscillation of all solutions of the system of differential equations

$$x^{(m)}(t) + \sum_{i=1}^{n} P_i x(t - \tau_i) = 0,$$

where $P_i \in \mathbb{R}^{s\times s}$ and $\tau_i \in \mathbb{R}^+$ for $i = 1, 2, ..., n$.
In [11], Meng et al. obtained the sufficient conditions for the oscillation of all solutions of the system of difference equations with continuous arguments

$$x(t) - x(t - \tau) + \sum_{i=1}^{m} P_i x(t - \sigma_i) = 0, \tag{1.6}$$

where $P_i \in \mathbb{R}^{r\times r}$ and $\tau, \sigma_i \in \mathbb{R}^+$ for $i = 1, 2, ..., m$.
In [4], [11] and [12] the authors, to obtain results, used to logarithmic norm of P which is denoted $\mu(P)$ and defined by

$$\mu(P) = \max_{\|\xi\|=1} (P\xi, \xi), \tag{1.7}$$

where $(,)$ is an inner product in $\mathbb{R}^r$ and $\| \xi \| = (\xi, \xi)^{\frac{1}{2}}$.
In the present paper we obtain sufficient conditions for the oscillation of all solutions of equations (1.4) and (1.6) without using logarithmic norm defined by (1.7). According to us, the results of this paper more useful than known results in [4, 6, 11].
By a solution of the equation (1.4), we mean a function $x \in C\left[(t_1 - \tau, \infty), \mathbb{R}^r\right]$ for some $t_1 \geq t_0$ such that x is continuously differentiable on $[t_1, \infty)$ and x satisfies equation (1.4) for $t \geq t_1$. A solution of the equation (1.4) with $x(t) = [x_1(t), x_2(t), \ldots, x_s(t)]^T$ is said to oscillate if every component $x_i(t)$ of the solution has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory.

2. Sufficient condition for oscillation of (1.4)

In this section we obtain sufficient condition for the oscillation of all solutions of the differential equation with the matrix coefficients of $P_1, P_2, ..., P_m$ and several delay terms

$$x'(t) + \sum_{i=1}^{m} P_i x(t - \tau_i) = 0.$$

The condition will be given in terms of the τ_i and eigenvalues of the matrices P_i for each $i = 1, 2, ..., m$.
We need the following lemma, which proved in [3]. (Also see [6]).

Lemma 2.1. *Assume that* $P_i \in \mathbb{R}^{rxr}$ *and* $\tau_i \in \mathbb{R}^+$ *for* $i = 1, 2, ..., m$. *Then the following statements are equivalent.*
(a) *Every solution of equation* (1.4) *oscillates componentwise,*
(b) *The characteristic equation of* (1.4)

$$\det\left[\gamma I + \sum_{i=1}^{m} P_i e^{-\gamma\tau_i}\right] = 0 \tag{2.1}$$

has no real roots.

Theorem 2.2. *Let* $P_i \in \mathbb{R}^{rxr}$ *and* $\tau_i \geq 0$ *for* $i = 1, 2, ..., m$. *Then every solution of equation* (1.4) *oscillates (componentwise) provided that*

$$\lambda\left(\sum_{i=1}^{m} P_i\tau_i\right) > \frac{1}{e}, \tag{2.2}$$

where $\lambda(P)$ *denotes any real eigenvalues of* P.

Proof. If $\tau_i = \tau(\neq 0)$ for all $i = 1, 2, ..., m$, then every solution of equation (1.4) oscillates if and only if

$$\lambda\left(\sum_{i=1}^{m} P_i\right) > \frac{1}{e\tau},$$

which is given in [4]. Furthermore, $\tau_i = 0$ for all $i = 1, 2, ..., m$, then every solution of equation (1.4) oscillates if and only if $\sum\limits_{i=1}^{m} P_i$ has no eigenvalues in the interval $(-\infty, \infty)$. Thus, we assume that at least two τ_i for $i = 1, 2, ..., m$ are different from each other. Assume, for the sake of contradiction, that equation (2.1) has a γ_0 real root. If $\gamma_0 \in (0, \infty)$, then equation (2.1) becomes

$$\det\left[I + \sum_{i=1}^{m} P_i \frac{e^{-\gamma_0\tau_i}}{\gamma_0}\right] = 0$$

and

$$\det\left[-I - \sum_{i=1}^{m} P_i \frac{e^{-\gamma_0\tau_i}}{\gamma_0}\right] = 0.$$

Hence, we have

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{e^{-\gamma_0\tau_i}}{\gamma_0}\right) = -1.$$

But, by the condition (2.2), this is impossible. Indeed, we observe that for $i = 1, 2, ..., m$,

$$\lim_{\gamma_0\to 0^+} \frac{e^{-\gamma_0\tau_i}}{\gamma_0} = \infty \quad \text{and} \quad \lim_{\gamma_0\to\infty} \frac{e^{-\gamma_0\tau_i}}{\gamma_0} = 0.$$

Thus, this is a contradiction to (2.2).
If $\gamma_0 = 0$, then equation (2.1) becomes

$$\det\left[\sum_{i=1}^{m} P_i\right] = 0$$

and also we get

$$\det\left[\sum_{i=1}^{m} P_i\tau_i\right] = 0$$

which means that at least one eigenvalue of $\sum_{i=1}^{m} P_i\tau_i$ is zero. So, we arrive a contradiction to (2.2).
Next, assume that $\gamma_0 < 0$, hence we have

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{e^{-\gamma_0\tau_i}}{\gamma_0}\right) = -1.$$

But, by the condition (2.2) this is impossible. Indeed, we observe that for $i = 1, 2, ..., m$,

$$\max_{\gamma_0<0} \frac{e^{-\gamma_0\tau_i}}{\gamma_0} = -e\tau_i.$$

Then, we have

$$\lambda\left(\sum_{i=1}^{m} P_i(-e\tau_i)\right) \geq -1$$

and

$$\lambda\left(\sum_{i=1}^{m} P_i e\tau_i\right) \leq 1$$

which this is a contradiction to (2.2). Thus, the proof is complete. □

3. Sufficient condition for oscillation of (1.6)

In this section, we obtain sufficient condition for the oscillation of all solutions of the equation (1.6). The condition will be given in terms of the eigenvalues of the matrices P_i for $i = 1, 2, ..., m$.
We need the following lemma which is proved in [11].

Lemma 3.1. *Assume that* $P_i \in \mathbb{R}^{rxr}$ *and* $\tau, \sigma_i \in \mathbb{R}^+$ *for* $i = 1, 2, ..., m$. *Then the following statements are equivalent.*
(*a*) *Every solution of equation* (1.6) *oscillates componentwise,*
(*b*) *The characteristic equation of* (1.6)

$$\det\left[(1 - e^{-\gamma\tau})I + \sum_{i=1}^{m} P_i e^{-\gamma\sigma_i}\right] = 0 \tag{3.1}$$

has no real roots.

Theorem 3.2. *Let* $P_i \in \mathbb{R}^{rxr}$ *and* $\sigma_i > \tau > 0$ *for* $i = 1, 2, ..., m$. *Then every solution of equation* (1.6) *oscillates (componentwise) provided that*

$$\lambda\left[\sum_{i=1}^{m} P_i \left(\frac{\sigma_i^{\sigma_i}}{(\sigma_i - \tau)^{\sigma_i - \tau}}\right)^{\frac{1}{\tau}}\right] > \tau, \tag{3.2}$$

where $\lambda(P)$ *denotes the real eigenvalues of* P.

Proof. If $\sigma_i = \sigma(\neq 0)$ for all $i = 1, 2, ..., m$, then every solution of equation (1.6) oscillates if and only if

$$\lambda\left(\sum_{i=1}^{m} P_i\right) > \tau\left(\frac{(\sigma - \tau)^{\sigma-\tau}}{\sigma^{\sigma}}\right)^{\frac{1}{\tau}},$$

which is given in [11]. Furthermore, $\sigma_i = 0$ for all $i = 1, 2, ..., m$, then every solution of equation (1.6) oscillates if and only if $\sum_{i=1}^{m} P_i$ has no eigenvalues in the interval $(-\infty, \infty)$. Thus, we assume that at least two σ_i for $i = 1, 2, ..., m$ are different from each other. Assume, for the sake of contradiction, that equation (3.1) has a γ_0 real root. If $\gamma_0 \in (0, \infty)$, then equation (3.1) becomes

$$\det\left[I + \sum_{i=1}^{m} P_i \frac{e^{-\gamma_0\sigma_i}}{(1 - e^{-\gamma_0\tau})}\right] = 0$$

and

$$\det\left[-I - \sum_{i=1}^{m} P_i \frac{e^{-\gamma_0\sigma_i}}{(1 - e^{-\gamma_0\tau})}\right] = 0.$$

Hence we have

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{e^{-\gamma_0\sigma_i}}{(1 - e^{-\gamma_0\tau})}\right) = -1.$$

But, by the condition (3.2) this is impossible. Indeed, we observe that for $i = 1, 2, ..., m$,

$$\lim_{\gamma_0 \to 0^+} \frac{e^{-\gamma_0\sigma_i}}{(1 - e^{-\gamma_0\tau})} = \infty \quad \text{and} \quad \lim_{\gamma_0 \to \infty} \frac{e^{-\gamma_0\sigma_i}}{(1 - e^{-\gamma_0\tau})} = 0.$$

So, this is a contradiction to (3.2).
If $\gamma_0 = 0$, then equation (3.1) becomes

$$\det\left[\sum_{i=1}^{m} P_i\right] = 0$$

and also we get

$$\det\left[\sum_{i=1}^{m} P_i \left(\frac{\sigma_i^{\sigma_i}}{\tau^\tau(\sigma_i - \tau)^{\sigma_i - \tau}}\right)^{\frac{1}{\tau}}\right] = 0$$

which this is a contradiction to (3.2).
Next, assume that $\gamma_0 < 0$, then we have

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{e^{-\gamma_0\sigma_i}}{(1 - e^{-\gamma_0\tau})}\right) = -1.$$

But, by the condition (3.2) this is impossible. Indeed, we observe that for $i = 1, 2, ..., m$,

$$\max_{\gamma_0 < 0} \frac{e^{-\gamma_0\sigma_i}}{(1 - e^{-\gamma_0\tau})} = -\left(\frac{\sigma_i^{\sigma_i}}{\tau^\tau(\sigma_i - \tau)^{\sigma_i - \tau}}\right)^{\frac{1}{\tau}}.$$

Then, we have

$$\lambda\left[\sum_{i=1}^{m} -P_i \left(\frac{\sigma_i^{\sigma_i}}{\tau^\tau(\sigma_i - \tau)^{\sigma_i - \tau}}\right)^{\frac{1}{\tau}}\right] \geq -1$$

and

$$\lambda\left[P_i \left(\frac{\sigma_i^{\sigma_i}}{\tau^\tau(\sigma_i - \tau)^{\sigma_i - \tau}}\right)^{\frac{1}{\tau}}\right] \leq 1,$$

which this is a contradiction to (3.2). Thus, the proof is complete. □

Example 3.3. *We consider the following system of differential equations*

$$x'(t) + P_1 x(t-1) + P_2 x(t-2) = 0, \tag{3.3}$$

where

$$P_1 = \begin{bmatrix} \frac{1}{3} & -3 \\ \frac{1}{4} & 2 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{3}{8} & 1 \end{bmatrix}.$$

We observe that for $i = 1, 2$

$$\begin{aligned} \lambda\left(\sum_{i=1}^{2} P_i \tau_i\right) &= \lambda\left(\begin{bmatrix} \frac{1}{3} & -3 \\ \frac{1}{4} & 2 \end{bmatrix} + \begin{bmatrix} \frac{2}{3} & 1 \\ \frac{3}{4} & 2 \end{bmatrix}\right) \\ &= \lambda\left(\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\right). \end{aligned}$$

So, we obtain that the eigenvalues of the matrix $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ *are $\lambda_1 = 2$ and $\lambda_2 = 3$. Consequently, the condition* (2.2) *is provided and every solution of equation* (3.3) *oscillates. On the other hand, to say the same result from* [4] *and [6, Theorem 5.2.1] we must find the eigenvector that corresponds to each eigenvalue so that we calculate the logarithmic norm $\mu(P)$.*

Clearly, this example shows that our results are sharper than the results in the literature.

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