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INTERVAL OSCILLATION CRITERIA FOR SECOND-ORDER IMPULSIVE DELAY DIFFERENTIAL EQUATIONS WITH MIXED NONLINEARITIES

V. MUTHULAKSHMI AND R. MANJURAM

Abstract. In this paper, we study the oscillatory behavior of second-order forced impulsive delay differential equations with mixed nonlinearities. By using Riccati transformation technique, integral averaging method and some inequalities, we obtain sufficient conditions for oscillation of all solutions. Finally, two examples are presented to illustrate the theoretical results.

1. Introduction

In this paper, we investigate the oscillation of the following second-order impulsive delay differential equation with mixed nonlinearities:

$$\begin{aligned}\big(r(t)\Phi_\alpha(x'(t))\big)' + p(t)\Phi_\alpha(x(t-\tau(t))) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t-\tau(t))) = e(t), t \ge t_0, t \ne t_k,\\ x({t_k}^{+}) = a_k x(t_k), \quad x'({t_k}^{+}) = b_k x'(t_k),\ k = 1,2,3,\ldots.\end{aligned} \tag{1}$$

Here,

$$x(t_k^-) := \lim_{t\to t_k^-} x(t), \quad x(t_k^+) := \lim_{t\to t_k^+} x(t),$$

$$x'(t_k^-) := \lim_{h\to 0^-} \frac{x(t_k+h)-x(t_k)}{h}, \quad x'(t_k^+) := \lim_{h\to 0^+} \frac{x(t_k+h)-x(t_k)}{h},$$

where $\Phi_*(s) := |s|^{*-1}s$, $\{t_k\}$ denotes the impulsive moment sequence with $0 \le t_0 < t_1 < \ldots < t_k < \ldots$, $\lim_{k\to\infty} t_k = \infty$.
Let $J \subset \mathbb{R}$ be an interval, we define

$$\begin{aligned} PLC(J,\mathbb{R}) := \big\{h : J \to \mathbb{R} \mid\ & h \text{ is continuous on each interval } (t_k, t_{k+1}),\\ & h(t_k^{\pm}) \text{ exists and } h(t_k) = h(t_k^-) \text{ for all } k \in \mathbb{N}\big\}.\end{aligned}$$

For given t_0 and $\phi \in PLC(E_{t_0},\mathbb{R})$, we say $x \in PLC\big(E_{t_0},\mathbb{R}\big)$ is a solution of equation (1) with the initial value ϕ if $x(t)$ satisfies equation (1) for $t \ge t_0$ and $x(t) = \phi(t)$

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for $t \in E_{t_0}$, where $E_{t_0} = t_0 \cup \{t - \tau(t) : t - \tau(t) < t_0,\, t \geq t_0\}$. As usual, a nontrivial solution $x(t)$ of equation (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is termed nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, population dynamics, pharmacokinetics, information science, electronics, automatic control systems, computer networking, artificial intelligence, robotics, and telecommunications, do exhibit impulsive effects. Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems. Therefore, it is important and necessary to study impulsive dynamical systems. For the general theory and applications of impulsive differential equations, we refer the reader to [4, 5, 11, 18].

Concerning delay differential equations, they are in the form of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at earlier times. Their systems are widely used to describe many scientific phenomena such as population dynamics, communication network model, economical systems, propagation and transport [10, 13, 14].

It is well known that most of the differential equations cannot be solvable in terms of elementary functions. Though there are various analytical methods for solving nonlinear oscillation systems, qualitative properties of solutions, in particular, the oscillatory behavior of solutions, of such equations assume importance in the absence of closed form solutions.

In the absence of impulses and delays, equation (1) reduces to

$$\big(r(t)|y'(t)|^{\alpha-1}y'(t)\big)' + p(t)|y(t)|^{\alpha-1}y(t) + \sum_{j=1}^{m} q_j(t)|y(t)|^{\beta_j-1}y(t) = e(t),\ t \geq t_0, \quad (2)$$

which can be considered as a generalization of the nonhomogeneous equation

$$\big(r(t)y'(t)\big)' + f(t, y(t)) = e(t),\ t \geq t_0. \quad (3)$$

The oscillation of equations (2) and (3) has been received great attention during the last 50 years, see, for example, [2, 19, 23], and the references cited therein.

In recent years, a great deal of effort has been spent in getting sufficient conditions for the oscillation of solutions of second order nonlinear impulsive delay differential equations. Recently, interval oscillation of impulsive delay differential equations was attracting the interest of many researchers, see [6, 7, 12, 15, 16, 20, 21, 22]. However, for the impulsive equations, almost all of interval oscillation results in the existing literature were established only for the case of "constant delay ". Recently, Zhou and Wang [24] considered the second order nonlinear impulsive differential equations with variable delay of the form

$$\begin{gathered} x''(t) + p(t)f(x(t - \tau(t))) = f(t),\ t \geq t_0,\ t \neq \theta_k, \\ x(t^+) = a_k x(t), \quad x'(t^+) = b_k x'(t),\ t = t_k,\ k = 1, 2, ..., \end{gathered}$$

and obtained some interval oscillation criteria. To the best of our knowledge, this is the first research work in this area.

Our purpose here is to obtain some interval oscillation criteria for equation (1). The results of this study generalize and improve some known results in [7, 12, 15, 20]. Examples are given to illustrate the effectiveness of our main results.

2. Main results

Throughout this paper, assume that the following conditions hold without further mention:

(A1) $r(t) \in C([t_0,\infty),(0,\infty))$ is non-decreasing, $p(t), q_i(t), e(t) \in PLC([t_0,\infty),\mathbb{R})$, $i = 1, 2 \dots, n$;
(A2) $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$ are constants;
(A3) α is a quotient of odd positive integers, $b_k \ge a_k > 0$, $k \in \mathbb{N}$ are constants;
(A4) $\tau(t) \in C([t_0,\infty))$ and there exists a non-negative constant τ such that $0 \le \tau(t) \le \tau$ for all $t \ge t_0$ and $t_{k+1} - t_k > \tau$ for all $k = 1, 2, \dots$.

Let $k(s) := \max\{i : t_0 < t_i < s\}$, for $c_j < d_j$, let $M_j := \max\{r(t) : t \in [c_j, d_j]\}$ and $\Omega_j := \{\omega \in C^1[c_j, d_j] : \omega(t) \not\equiv 0, \omega(c_j) = \omega(d_j) = 0\}$, $j = 1, 2$. For two constants $c, d \notin \{t_k\}$ with $c < d$ and a function $\phi \in C([c,d],\mathbb{R})$, we define an operator $\Psi : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$\Psi_c^d[\phi] = \begin{cases} 0, & \text{for } k(c) = k(d), \\ \phi(t_{k(c)+1})\theta(c) + \sum\limits_{i=k(c)+2}^{k(d)} \phi(t_i)\varepsilon(t_i), & \text{for } k(c) < k(d), \end{cases}$$

where

$$\theta(c) = \frac{b^{\alpha}_{k(c)+1} - a^{\alpha}_{k(c)+1}}{(a^{\alpha}_{k(c)+1}(t_{k(c)+1} - c)^{\alpha})}\ , \quad \varepsilon(t_i) = \frac{b_i^{\alpha} - a_i^{\alpha}}{(a_i^{\alpha}(t_i - t_{i-1})^{\alpha})},$$

and $\sum\limits_s^t = 0$ if $s > t$.

In the discussion of the impulse moments of $x(t)$ and $x(t-\tau(t))$, we need to consider the following four cases for $k(c_j) < k(d_j)$,

(s_1) $t_{k(c_j)} + \tau < c_j$ and $t_{k(d_j)} + \tau > d_j$; (s_2) $t_{k(c_j)} + \tau < c_j$ and $t_{k(d_j)} + \tau < d_j$;
(s_3) $t_{k(c_j)} + \tau > c_j$ and $t_{k(d_j)} + \tau > d_j$; (s_4) $t_{k(c_j)} + \tau > c_j$ and $t_{k(d_j)} + \tau < d_j$,

and the three cases for $k(c_j) = k(d_j)$,

$(\tilde{s_1})$ $t_{k(c_j)} + \tau < c_j$; $(\tilde{s_2})$ $c_j < t_{k(c_j)} + \tau < d_j$; $(\tilde{s_3})$ $t_{k(c_j)} + \tau > d_j$, $j = 1, 2$.

Combining (s_*) with $(\tilde{s_*})$, we can get 12 cases. Throughout the paper, we study equation (1) under the case of combination of (s_1) with $(\tilde{s_1})$ only. The discussions for other cases are similar and so omitted. We define a function

$$D_k(t) = t - t_k - \tau(t),\ t \in [t_k, t_{k+1}], k = 1, 2, \dots,$$

as in [24], and assume that the following condition holds throughout.

(A5) There is one zero point $z_k \in (t_k, t_{k+1}]$ such that $D_k(z_k) = 0, D_k(t) < 0$ for $t \in (t_k, z_k)$ and $D_k(t) > 0$ for $t \in (z_k, t_{k+1}]$.

First, let us see some lemmas which will be useful to prove our main results.

Lemma 1. [1] *For any given n-tuple* $\{\beta_1, \beta_2, \dots, \beta_n\}$ *satisfying* $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$, *there corresponds an n-tuple* $(\eta_1, \eta_2, \dots, \eta_n)$ *such that*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \qquad \sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1. \tag{4}$$

Lemma 2. [1] *Let* γ *and* δ *be positive real numbers with* $\gamma > \delta$. *Then*

(i) $Ax^{\gamma} + B \geq \gamma \delta^{-\delta/\gamma}(\gamma-\delta)^{(\delta/\gamma)-1} A^{\delta/\gamma} B^{1-\delta/\gamma} x^{\delta}$ *for all* $A, B, x \geq 0$, (5)

(ii) $Cx^{\delta} - D \leq \delta^{-\gamma/\delta}\delta(\gamma-\delta)^{(\gamma/\delta)-1} C^{\gamma/\delta} D^{1-\gamma/\delta} x^{\gamma}$ *for all* $C, x \geq 0$ *and* $D > 0$ (6)

Lemma 3. [8] *If* X *and* Y *are non-negative real numbers, then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda-1) Y^{\lambda}, \lambda > 1, \tag{7}$$

where equality holds if and only if $X = Y$.

Lemma 4. *Assume that for any* $T \geq t_0$, *there exist* $c_j, d_j \notin \{t_k\}$, $j = 1, 2$, *such that* $T < c_1 - \tau < c_1 < d_1 \leq c_2 - \tau < c_2 < d_2$ *and*

$$\begin{aligned} p(t), q_i(t) \geq 0, &\quad t \in [c_1-\tau, d_1] \cup [c_2-\tau, d_2] \backslash \{t_k\},\ i = 1, 2, 3, ..., n, \\ e(t) \leq 0, &\quad t \in [c_1-\tau, d_1] \backslash \{t_k\}, \\ e(t) \geq 0, &\quad t \in [c_2-\tau, d_2] \backslash \{t_k\}. \end{aligned} \tag{8}$$

If $x(t)$ *is a non-oscillatory solution of equation* (1), *then there exist the following estimations for* $\frac{x(t-\tau(t))}{x(t)}$:

(a) $\dfrac{x(t-\tau(t))}{x(t)} > \left(\dfrac{t-t_i-\tau(t)}{t-t_i}\right)$ *for* $t \in (z_i, t_{i+1}]$,

(b) $\dfrac{x(t-\tau(t))}{x(t)} > \left(\dfrac{t-t_i}{b_i(t+\tau(t)-t_i)}\right)$ *for* $t \in (t_i, z_i)$,

(c) $\dfrac{x(t-\tau(t))}{x(t)} > \left(\dfrac{t-t_{k(c_j)}-\tau(t)}{t-t_{k(c_j)}}\right)$ *for* $t \in [c_j, t_{k(c_j)+1}]$,

(d) $\dfrac{x(t-\tau(t))}{x(t)} > \left(\dfrac{t-t_{k(d_j)}-\tau(t)}{t-t_{k(d_j)}}\right)$ *for* $t \in [z_{k(d_j)}, d_j]$,

(e) $\dfrac{x(t-\tau(t))}{x(t)} > \left(\dfrac{t-t_{k(d_j)}}{b_{k(d_j)}(t+\tau(t)-t_{k(d_j)})}\right)$ *for* $t \in (t_{k(d_j)}, z_{k(d_j)}]$,

where $z_i \in (t_i, t_{i+1}]$ *for* $i = k(c_j)+1, ..., k(d_j)-1$, $j = 1, 2$.

Proof. Without loss of generality, we assume that $x(t) > 0$ and $x(t-\tau(t)) > 0$ for $t \geq t_0$. In this case, the selected interval of t is $[c_1, d_1]$. From equations (1) and (8), we obtain

$$\left[r(t)\Phi_{\alpha}(x'(t))\right]' = e(t) - p(t)\Phi_{\alpha}(x(t-\tau(t))) - \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t-\tau(t))) \leq 0. \tag{9}$$

Hence $r(t)\Phi_{\alpha}(x'(t))$ is non-increasing on the interval $[c_1, d_1]\backslash\{t_k\}$.

Case(a): If $z_i < t \leq t_{i+1}$, then $(t-\tau(t), t) \subset (t_i, t_{i+1}]$, and hence there is no impulsive moment in $(t-\tau(t), t)$. By mean value theorem, for any $s \in (t-\tau(t), t)$, we have

$$x(s) - x(t_i^+) = x'(\xi_1)(s-t_i) \text{ for some } \xi_1 \in (t_i, s).$$

Since $x(t_i^+) > 0$, we have

$$x(s) > x'(\xi_1)(s-t_i),\ \xi_1 \in (t_i, s).$$

Since $\Phi_\alpha(*)$ is a strictly increasing function and since $r(s)\Phi_\alpha(x'(s))$ is non-increasing on (t_i, t_{i+1}), we have

$$\Phi_\alpha(x(s)) > \Phi_\alpha(x'(\xi_1)(s-t_i)) = \frac{r(\xi_1)\Phi_\alpha(x'(\xi_1))(s-t_i)^\alpha}{r(\xi_1)} \geq \frac{r(s)\Phi_\alpha(x'(s))}{r(\xi_1)}(s-t_i)^\alpha.$$

Further, since $r(s)$ is positive and non-decreasing as mentioned in (A1), the above inequality becomes

$$\Phi_\alpha(x(s)) > \Phi_\alpha(x'(s)(s-t_i)) \text{ for some } \xi_1 \in (t_i, s).$$

Then, by the definition of $\Phi_\alpha(*)$, we have

$$\frac{x'(s)}{x(s)} < \frac{1}{(s-t_i)}.$$

Integrating both sides from $t-\tau(t)$ to t, we obtain

$$\frac{x(t-\tau(t))}{x(t)} > \left(\frac{t-t_i-\tau(t)}{t-t_i}\right),\ t \in (z_i, t_{i+1}]. \tag{10}$$

Case(b): If $t_i < t < z_i$, then $t-\tau(t) \in (t_i-\tau(t), t_i)$. ie, $t_i - \tau < t-\tau(t) < t_i < t < t_i+\tau$. Then there is an impulsive moment t_i in $(t-\tau(t), t)$. For any $t \in (t_i, z_i)$, we have

$$x(t) - x(t_i^+) = x'(\xi_2)(t-t_i) \text{ for some } \xi_2 \in (t_i, t).$$

Using the impulsive condition of equation (1), we get

$$x(t) - a_i x(t_i) = x'(\xi_2)(t-t_i),\ \xi_2 \in (t_i, t).$$

Then by using the monotone property of $r(t)\Phi_\alpha(x'(t))$, we get

$$\Phi_\alpha(x(t)-a_i x(t_i)) \leq \frac{r(t_i^+)\Phi_\alpha(x'(t_i^+))}{r(\xi_2)}(t-t_i)^\alpha.$$

Again, by using the impulsive condition of equation (1) and the monotone property of $r(t)$, the above inequality becomes

$$\Phi_\alpha(x(t)-a_i x(t_i)) \leq \Phi_\alpha(b_i x'(t_i)(t-t_i)).$$

Since $x(t_i) > 0$, we have

$$\Phi_\alpha\left(\frac{x(t)}{x(t_i)} - a_i\right) \leq \Phi_\alpha\left(b_i \frac{x'(t_i)}{x(t_i)}(t-t_i)\right). \tag{11}$$

In addition, by mean value theorem on $[t_i - \tau(t), t_i]$, we have

$$x(t_i) - x(t_i - \tau(t)) = x'(\xi_3)\tau(t) \text{ for some } \xi_3 \in (t_i-\tau(t), t_i).$$

Then as in Case (a), by using the monotone properties of $r(t)$, $\Phi_\alpha(t)$ and $r(t)\Phi_\alpha(x'(t))$, we get

$$\frac{x'(t_i)}{x(t_i)} < \frac{1}{\tau(t)}. \tag{12}$$

Thus, from (11) and (12), we have

$$\Phi_\alpha\left(\frac{x(t)}{x(t_i)} - a_i\right) < \Phi_\alpha\left(\frac{b_i(t-t_i)}{\tau(t)}\right)$$

and hence

$$\Phi_\alpha\left(\frac{x(t)}{x(t_i)}\right) < \Phi_\alpha\left(\frac{b_i(t-t_i)}{\tau(t)} + a_i\right).$$

Therefore, by (A3), we get

$$\frac{x(t)}{x(t_i)} < \frac{b_i(t-t_i+\tau(t))}{\tau(t)}. \tag{13}$$

Now, for some $s \in (t_i - \tau(t), t_i)$, we have

$$x(s) - x(t_i - \tau(t)) = x'(\xi_4)(s - t_i + \tau(t)) \text{ for some } \xi_4 \in (t_i - \tau(t), s).$$

Again, by using the monotone properties of $r(t)$, $\Phi_\alpha(*)$ and $r(t)\Phi_\alpha(x'(t))$ as in Case(a), we get

$$\frac{x'(s)}{x(s)} < \frac{1}{(s-t_i+\tau(t))}.$$

Integrating both sides from $t-\tau(t)$ to t_i, we have

$$\frac{x(t-\tau(t))}{x(t_i)} > \frac{t-t_i}{\tau(t)} \text{ for } t \in (t_i, z_i). \tag{14}$$

Thus, from (13) and (14), we have

$$\frac{x(t-\tau(t))}{x(t)} > \left(\frac{t-t_i}{b_i(t-t_i+\tau(t))}\right) \text{ for } t \in (t_i, z_i). \tag{15}$$

For other cases, the proof is similar and hence omitted. This concludes the lemma. □

Theorem 1. *Assume that for any $T \geq t_0$, there exist $c_j, d_j \notin \{t_k\}$, $j = 1, 2$, such that $T < c_1 - \tau < c_1 < d_1 \leq c_2 - \tau < c_2 < d_2$ and (8) holds. If there exist $\omega_j(t) \in \Omega_j(c_j, d_j)$, $j = 1, 2$, such that, for $k(c_j) < k(d_j)$,*

$$\begin{aligned}
&\Bigg\{\int_{c_j}^{t_{k(c_j)+1}} W_j(t)\left(\frac{t-t_{k(c_j)}-\tau(t)}{t-t_{k(c_j)}}\right)^\alpha dt \\
&+\sum_{i=k(c_j)+1}^{k(d_j)-1}\left[\int_{t_i}^{z_i} W_j(t)\left(\frac{t-t_i}{b_i(t-t_i+\tau(t))}\right)^\alpha dt + \int_{z_i}^{t_i+1} W_j(t)\left(\frac{t-t_i-\tau(t)}{t-t_i}\right)^\alpha dt\right] \\
&+\int_{t_{k(d_j)}}^{z_{k(d_j)}} W_j(t)\left(\frac{t-t_{k(d_j)}}{b_{k(d_j)}(t-t_{k(d_j)}+\tau(t))}\right)^\alpha dt + \int_{z_{k(d_j)}}^{d_j} W_j(t)\left(\frac{t-t_{k(d_j)}-\tau(t)}{t-t_{k(d_j)}}\right)^\alpha dt \\
&-\int_{c_j}^{d_j}(r(t)\left|\omega_j'(t)\right|^{\alpha+1})dt\Bigg\} > M_j \Psi_{c_j}^{d_j}[\omega_j^{\alpha+1}],
\end{aligned} \tag{16}$$

and for $k(c_j) = k(d_j)$,

$$\int_{c_j}^{d_j}\left(W_j(t)\left(\frac{t-c_j}{t-c_j+\tau(t)}\right)^\alpha - r(t)\left|\omega_j'(t)\right|^{\alpha+1}\right)dt > 0, \tag{17}$$

where $W_j(t) = Q(t)\omega_j^{\alpha+1}(t)$ and

$$Q(t) = \left(p(t) + \eta_0^{-\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i} q_i^{\eta_i}(t)|e(t)|^{\eta_0}\right), \tag{18}$$

and $\eta_i > 0$ are chosen according to given $\beta_1, \beta_2, \ldots, \beta_n$ as in Lemma 1 satisfying (4). Then equation (1) is oscillatory.

Proof. To arrive at a contradiction, let us suppose that $x(t)$ is a non-oscillatory solution of equation (1). Without loss of generality, we assume that $x(t) > 0$ and $x(t-\tau(t)) > 0$ for $t \geq t_0$. In this case, the interval of t selected for the following discussion is $[c_1, d_1]$. We define

$$u(t) = r(t)\frac{\Phi_\alpha(x'(t))}{x^\alpha(t)} \text{ for } t \in [c_1, d_1]. \tag{19}$$

Then

$$u'(t) = -\left(p(t)\frac{x^\alpha(t-\tau(t))}{x^\alpha(t)} + \frac{\sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t-\tau(t)))}{x^\alpha(t)} + \frac{|e(t)|}{x^\alpha(t)}\right) - \alpha u(t)\frac{x'(t)}{x(t)}, \tag{20}$$

for all $t \neq t_k, t \geq t_0$, and $u(t_k^+) = \frac{b_k}{a_k}u(t_k)$ for all $k \in \mathbb{N}$.
From our assumption, we can choose $c_1, d_1 \geq t_0$ such that $p(t) \geq 0$ and $q_i(t) \geq 0$ for $t \in [c_1-\tau, d_1]$, $i = 1, 2, \ldots, n$, and $e(t) \leq 0$ for $t \in [c_1-\tau, d_1]$. Also, by Lemma 1, there exist $\eta_i > 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \beta_i\eta_i = \alpha$ and $\sum_{i=1}^{n} \eta_i < 1$.
Now, define $\eta_0 := 1 - \sum_{i=1}^{n} \eta_i$ and let

$$\begin{aligned} u_0 &:= \eta_0^{-1}\left|\frac{e(t)x(t-\tau(t))}{x^\alpha(t)}\right| x^{-1}(t-\tau(t)), \\ u_i &:= \eta_i^{-1}q_i(t)\frac{x(t-\tau(t))}{x^\alpha(t)}\, x^{\beta_i-1}(t-\tau(t)) \text{ for } \ i = 1, 2, \ldots, n\,. \end{aligned}$$

Then, by the arithmetic-geometric mean inequality (see Beckenbach and Bellman [3])

$$\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i},\ u_i \geq 0,$$

we have

$$\frac{|e(t)|}{x^\alpha(t)} + \frac{\sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t-\tau(t)))}{x^\alpha(t)} \geq \eta_0^{-\eta_0}\frac{|e(t)|^{\eta_0}}{x^{\alpha\eta_0}(t)}\prod_{i=1}^{n}\left(\eta_i^{-\eta_i}q_i^{\eta_i}(t)\frac{x^{\beta_i\eta_i}(t-\tau(t))}{x^{\alpha\eta_i}(t)}\right). \tag{21}$$

Using (21) in (20), we get

$$\begin{aligned} u'(t) \leq -p(t)\frac{x^\alpha(t-\tau(t))}{x^\alpha(t)} - \eta_0^{-\eta_0}\frac{|e(t)|^{\eta_0}}{x^{\alpha\eta_0}(t)}\prod_{i=1}^{n}\left(\eta_i^{-\eta_i}q_i^{\eta_i}(t)\frac{x^{\beta_i\eta_i}(t-\tau(t))}{x^{\alpha\eta_i}(t)}\right) \\ -\alpha u(t)\frac{x'(t)}{x(t)},\ t \neq t_k. \end{aligned} \tag{22}$$

Since

$$\begin{aligned}\frac{1}{x^{\alpha\eta_0}(t)}\prod_{i=1}^{n}\left(\frac{x^{\beta_i\eta_i}(t-\tau(t))}{x^{\alpha\eta_i}(t)}\right) &= \frac{1}{x^{\alpha\eta_0}(t)}\left(\frac{x^{\sum_{i=1}^{n}\beta_i\eta_i}(t-\tau(t))}{x^{\alpha\sum_{i=1}^{n}\eta_i}(t)}\right)\\ &= \frac{1}{x^{\alpha\eta_0}(t)}\left(\frac{x^{\alpha}(t-\tau(t))}{x^{\alpha(1-\eta_0)}(t)}\right)\\ &= \frac{x^{\alpha}(t-\tau(t))}{x^{\alpha}(t)},\end{aligned}$$

equation (22) becomes

$$\begin{aligned}u'(t) \le -p(t)\frac{x^{\alpha}(t-\tau(t))}{x^{\alpha}(t)} - \eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}q_i^{\eta_i}(t)\left(\frac{x^{\alpha}(t-\tau(t))}{x^{\alpha}(t)}\right)\\ -\alpha u(t)\frac{x'(t)}{x(t)},\ t\neq t_k. \qquad (23)\end{aligned}$$

Now, from (19), we have

$$\begin{aligned}\alpha u(t)\frac{x'(t)}{x(t)} &= \alpha u(t)\frac{x'(t)}{x(t)}\left(\frac{r(t)\Phi_\alpha(x'(t))}{r(t)\Phi_\alpha(x'(t))}\right)^{1/\alpha}\\ &= \frac{\alpha}{r^{1/\alpha}(t)}u(t)\left(\frac{r(t)\Phi_\alpha(x'(t))}{x^{\alpha}(t)}\right)^{1/\alpha}\left(\frac{x'(t)}{(r(t)\Phi_\alpha(x'(t)))^{1/\alpha}}\right)\\ &= \frac{\alpha}{r^{1/\alpha}(t)}u(t)^{\frac{\alpha+1}{\alpha}}.\end{aligned}$$

Hence, (23) becomes

$$\begin{aligned}u'(t) &\le -\left[p(t)+\eta_0^{-\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}q_i^{\eta_i}(t)|e(t)|^{\eta_0}\right]\left(\frac{x^{\alpha}(t-\tau(t))}{x^{\alpha}(t)}\right) - \frac{\alpha}{r^{1/\alpha}(t)}u(t)^{\frac{\alpha+1}{\alpha}}\\ &= -Q(t)\left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} - \frac{\alpha}{r^{1/\alpha}(t)}u(t)^{\frac{\alpha+1}{\alpha}},\ t\neq t_k, \qquad (24)\end{aligned}$$

where $Q(t)$ is as in (18).
To estimate $\frac{x(t-\tau(t))}{x(t)}$, we first consider the case $k(c_1) < k(d_1)$. In this case, the impulsive moments in $[c_1, d_1]$ are $t_{k(c_1)+1},\ t_{k(c_1)+2}, \dots,\ t_{k(d_1)}$, and the zero points of $D_i(t)$ in intervals (t_i, t_{i+1}) are $z_i, i = k(c_1)+1, \dots, k(d_1)-1$. Multiplying both

sides of (24) by $\omega_1^{\alpha+1}(t)$ and integrating it from c_1 to d_1, we get

$$\begin{aligned}
&\int_{c_1}^{t_{k(c_1)+1}} u'(t)\omega_1^{\alpha+1}(t)dt + \int_{t_{k(c_1)+1}}^{t_{k(c_1)+2}} u'(t)\omega_1^{\alpha+1}(t)dt + \cdots + \int_{t_{k(d_1)}}^{d_1} u'(t)\omega_1^{\alpha+1}(t)dt \\
&\leq \Bigg\{ -\int_{c_1}^{t_{k(c_1)+1}} \frac{\alpha}{r^{1/\alpha}(t)} u(t)^{\frac{\alpha+1}{\alpha}} \omega_1^{\alpha+1}(t)dt - \int_{t_{k(c_1)+1}}^{t_{k(c_1)+2}} \frac{\alpha}{r^{1/\alpha}(t)} u(t)^{\frac{\alpha+1}{\alpha}} \omega_1^{\alpha+1}(t)dt \\
&\quad - \cdots - \int_{t_{k(d_1)}}^{d_1} \frac{\alpha}{r^{1/\alpha}(t)} u(t)^{\frac{\alpha+1}{\alpha}} \omega_1^{\alpha+1}(t)dt - \int_{c_1}^{t_{k(c_1)+1}} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt \\
&\quad - \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{t_i}^{z_i} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt + \int_{z_i}^{t_{i+1}} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt\right] \\
&\quad - \int_{t_{k(d_1)}}^{z_{k(d_1)}} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt - \int_{z_{k(d_1)}}^{d_1} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt \Bigg\}. \quad (25)
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} \omega_1^{\alpha+1}(t_i)[u(t_i) - u(t_i^+)] \\
&\leq \Bigg\{ \int_{c_1}^{d_1} \left[(\alpha+1)\left|\omega_1^{\alpha}(t)\omega_1'(t)\right| |u(t)| - \frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(\alpha+1)/\alpha} \omega_1^{\alpha+1}(t)\right] dt \\
&\quad - \int_{c_1}^{t_{k(c_1)+1}} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt \\
&\quad - \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{t_i}^{z_i} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt + \int_{z_i}^{t_{i+1}} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt\right] \\
&\quad - \int_{t_{k(d_1)}}^{z_{k(d_1)}} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt - \int_{z_{k(d_1)}}^{d_1} \left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt \Bigg\}. \quad (26)
\end{aligned}$$

Letting

$$\lambda = 1 + \frac{1}{\alpha},\ X = \left(\frac{\alpha}{r^{1/\alpha}(t)}\right)^{\alpha/\alpha+1} |\omega_1^{\alpha}(t)|\,|u(t)| \ \text{ and } Y = [\alpha r(t)]^{\alpha/\alpha+1} |\omega_1'(t)|^{\alpha},$$

and using Lemma 3, we get

$$(\alpha+1)\left|\omega_1^{\alpha}(t)\omega_1'(t)\right| |u(t)| - \frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(\alpha+1)/\alpha} \omega_1^{\alpha+1}(t) \leq r(t) \left|\omega_1'(t)\right|^{\alpha+1}. \quad (27)$$

Meanwhile, for $t = t_k$, $k = 1, 2, \ldots,$

$$u(t_k^+) = \left(\frac{b_k}{a_k}\right)^{\alpha} u(t_k). \quad (28)$$

Then the left hand side of the inequality (26) becomes

$$\sum_{i=k(c_1)+1}^{k(d_1)} \omega_1^{\alpha+1}(t_i)[u(t_i) - u(t_i^+)] = \sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i^{\alpha} - b_i^{\alpha}}{a_i^{\alpha}} \omega_1^{\alpha+1}(t_i) u(t_i). \quad (29)$$

Substituting (27) and (29) in (26), we get

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i^\alpha - b_i^\alpha}{a_i^\alpha}\omega_1^{\alpha+1}(t_i)u(t_i) \\
&\leq \left\{ \int_{c_1}^{d_1} r(t)\left|\omega_1'(t)\right|^{\alpha+1} dt - \int_{c_1}^{t_{k(c_1)+1}} \left(\frac{x(t-\tau(t))}{x(t)}\right)^\alpha W_1(t)dt \right. \\
&\quad - \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{t_i}^{z_i} \left(\frac{x(t-\tau(t))}{x(t)}\right)^\alpha W_1(t)dt + \int_{z_i}^{t_{i+1}} \left(\frac{x(t-\tau(t))}{x(t)}\right)^\alpha W_1(t)dt\right] \\
&\quad \left. - \int_{t_{k(d_1)}}^{z_{k(d_1)}} \left(\frac{x(t-\tau(t))}{x(t)}\right)^\alpha W_1(t)dt - \int_{z_{k(d_1)}}^{d_1} \left(\frac{x(t-\tau(t))}{x(t)}\right)^\alpha W_1(t)dt \right\}. \quad (30)
\end{aligned}$$

On the other hand, for $t \in (t_{i-1}, t_i] \subset [c_1, d_1]$, $i = k(c_1) + 2, \ldots, k(d_1)$, we have

$$x(t) - x(t_{i-1}) = x'(\xi)(t - t_{i-1}),\ \xi \in (t_{i-1}, t).$$

In view of $x(t_{i-1}) > 0$ and the monotone properties of $\Phi_\alpha(t)$, $r(t)\Phi_\alpha(x'(t))$ and $r(t)$ we obtain

$$\frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))} \leq \frac{r(\xi)}{(t - t_{i-1})^\alpha}.$$

Letting $t \to t_i^-$, we get

$$u(t_i) = \frac{r(t_i)\Phi_\alpha(x'(t_i))}{\Phi_\alpha(x(t_i))} \leq \frac{M_1}{(t_i - t_{i-1})^\alpha},\ i = k(c_1) + 2, \ldots, k(d_1). \quad (31)$$

Making a similar analysis on $(c_1, t_{k(c_1)+1}]$, we get

$$u(t_{k(c_1)+1}) = \frac{r(t_{k(c_1)+1})\Phi_\alpha(x'(t_{k(c_1)+1}))}{\Phi_\alpha(x(t_{k(c_1)+1}))} \leq \frac{M_1}{(t_{k(c_1)+1} - c_1)^\alpha}. \quad (32)$$

Thus from (31), (32) and (A3), we obtain

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha}\omega_1^{\alpha+1}(t_i)u(t_i) \\
&\leq M_1\left[\omega_1^{\alpha+1}(t_{k(c_1)+1})\theta(c_1) + \sum_{i=k(c_1)+2}^{k(d_1)} \omega_1^{\alpha+1}(t_i)\varepsilon(t_i)\right] = M_1\Psi_{c_1}^{d_1}\left[\omega_1^{\alpha+1}\right]. \quad (33)
\end{aligned}$$

Hence, from (30), (33) and Lemma 4, we obtain

$$\begin{aligned}&\Bigg\{\int_{c_1}^{t_{k(c_1)+1}} W_1(t)\left(\frac{t-t_{k(c_1)}-\tau(t)}{t-t_{k(c_1)}}\right)^{\alpha} dt\\&+\sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{t_i}^{z_i} W_1(t)\left(\frac{t-t_i}{b_i(t-t_i+\tau(t))}\right)^{\alpha} dt+\int_{z_i}^{t_i+1} W_1(t)\left(\frac{t-t_i-\tau(t)}{t-t_i}\right)^{\alpha} dt\right]\\&+\int_{t_{k(d_1)}}^{z_{k(d_1)}} W_1(t)\left(\frac{t-t_{k(d_1)}}{b_{k(d_1)}(t-t_{k(d_1)}+\tau(t))}\right)^{\alpha} dt\\&+\int_{z_{k(d_1)}}^{d_1} W_1(t)\left(\frac{t-t_{k(d_1)}-\tau(t)}{t-t_{k(d_1)}}\right)^{\alpha} dt-\int_{c_1}^{d_1} r(t)\left|\omega_1'(t)\right|^{\alpha+1} dt\Bigg\}\leq M_1\Psi_{c_1}^{d_1}[\omega_1^{\alpha+1}].\end{aligned} \tag{34}$$

This contradicts (16).

If $k(c_1)=k(d_1)$, then there are no impulse moments in $[c_1,d_1]$. Multiplying both sides of (24) by $\omega_1^{\alpha+1}(t)$ and integrating it from c_1 to d_1, we obtain

$$\begin{aligned}\int_{c_1}^{d_1} u'(t)\omega_1^{\alpha+1}(t)dt\leq&-\int_{c_1}^{d_1}\frac{\alpha}{r^{1/\alpha}(t)}\left|u(t)\right|^{(\alpha+1)/\alpha}\omega_1^{\alpha+1}(t)dt\\&-\int_{c_1}^{d_1}\left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt.\end{aligned}$$

Using integration by parts on the left hand side and noting the condition $\omega_1(c_1)=\omega_1(d_1)=0$, we obtain

$$\begin{aligned}&\int_{c_1}^{d_1}\left[(\alpha+1)\omega_1^{\alpha}(t)\omega_1'(t)u(t)-\frac{\alpha}{r^{1/\alpha}(t)}\left|u(t)\right|^{(\alpha+1)/\alpha}\omega_1^{\alpha+1}(t)\right]dt\\&-\int_{c_1}^{d_1}\left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt\geq 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{c_1}^{d_1}\left[(\alpha+1)\left|\omega_1^{\alpha}(t)\omega_1'(t)\right|\left|u(t)\right|-\frac{\alpha}{r^{1/\alpha}(t)}\omega_1^{\alpha+1}(t)\left|u(t)\right|^{(\alpha+1)/\alpha}\right]dt\\&-\int_{c_1}^{d_1}\left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)dt\geq 0.\end{aligned} \tag{35}$$

Letting

$$\lambda=1+\frac{1}{\alpha},\ X=\left(\frac{\alpha}{r^{1/\alpha}(t)}\right)^{\alpha/\alpha+1}\left|\omega_1^{\alpha}(t)\right|\left|u(t)\right|\ \text{ and } Y=\left[\alpha r(t)\right]^{\alpha/\alpha+1}\left|\omega_1'(t)\right|^{\alpha},$$

and using Lemma 3, we get

$$(\alpha+1)\left|\omega_1^{\alpha}(t)\omega_1'(t)\right|\left|u(t)\right|-\frac{\alpha}{r^{1/\alpha}(t)}\left|u(t)\right|^{(\alpha+1)/\alpha}\omega_1^{\alpha+1}(t)\leq r(t)\left|\omega_1'(t)\right|^{\alpha+1}. \tag{36}$$

Thus from (35) and (36), we have

$$\int_{c_1}^{d_1}\left[r(t)\left|\omega_1'(t)\right|^{\alpha+1}-\left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} W_1(t)\right]dt\geq 0. \tag{37}$$

Now let us estimate $\frac{x(t-\tau(t))}{x(t)}$ on $[c_1, d_1]$. If $t \in [c_1, d_1]$ then $t-\tau(t) \in [c_1-\tau(t), d_1-\tau(t)]$, and then there are no impulsive moment in $(t-\tau(t), t)$. For any $t \in (t-\tau(t), t)$, we have

$$x(t) - x(c_1 - \tau(t)) = x'(\xi)(t - c_1 + \tau(t)) \text{ for some } \xi \in (c_1 - \tau(t), t).$$

By using the monotone properties of $r(t), \Phi_\alpha(*)$ and $r(t)\Phi_\alpha(x'(t))$, we get

$$\Phi_\alpha(x(t)) \geq \frac{r(t)\Phi_\alpha(x'(t))}{r(t)}(t - c_1 + \tau(t))^\alpha = \Phi_\alpha(x'(t)(t - c_1 + \tau(t))).$$

Then, by the definition of $\Phi_\alpha(*)$, we have

$$\frac{x'(t)}{x(t)} < \frac{1}{(t - c_1 + \tau(t))}.$$

Integrating both sides of the above inequality from $t - \tau(t)$ to t, we obtain

$$\frac{x(t-\tau(t))}{x(t)} > \left(\frac{t - c_1}{t - c_1 + \tau(t)}\right) \text{ for } t \in [c_1, d_1]. \tag{38}$$

From (37) and (38), we obtain

$$\int_{c_1}^{d_1} \left[W_1(t)\left(\frac{t - c_1}{t - c_1 + \tau(t)}\right)^\alpha - r(t)\left|\omega_1'(t)\right|^{\alpha+1}\right] dt \leq 0.$$

This again contradicts our assumption.

When $x(t)$ is eventually negative, we can consider the interval $[c_2, d_2]$, and reach a similar contradiction. Thus the proof is complete. □

Following Kong [9] and Philos [17], we consider the following class of functions: Let $D = \{(t, s) : t_0 \leq s \leq t\}$, $H_1, H_2 \in C^1(D, \mathbb{R})$. A pair of functions (H_1, H_2) is said to belong to a function class $\mathcal{H}$, if $H_1(t, t) = H_2(t, t) = 0$, $H_1(t, s) > 0$, $H_2(t, s) > 0$ for $t > s$ and there exist $h_1, h_2 \in L_{\text{loc}}(D, \mathbb{R})$ such that

$$\frac{\partial H_1(t, s)}{\partial t} = h_1(t, s)H_1(t, s), \quad \frac{\partial H_2(t, s)}{\partial s} = -h_2(t, s)H_2(t, s).$$

We assume that there exist c_j, d_j, $\delta_j \notin \{t_k\}$, $k = 1, 2, \ldots, (j = 1, 2)$ such that $T < c_1 - \tau < c_1 < \delta_1 < d_1 \leq c_2 - \tau < c_2 < \delta_2 < d_2$ for any $T \geq t_0$. Notice that whether there are or not impulse moments of $x(t)$ in $[c_j, \delta_j]$ and $[\delta_j, d_j]$, we should consider the following four cases,

$$(S_1)\ k(c_j) < k(\delta_j) < k(d_j); \quad (S_2)\ k(c_j) = k(\delta_j) < k(d_j);$$
$$(S_3)\ k(c_j) < k(\delta_j) = k(d_j); \quad (S_4)\ k(c_j) = k(\delta_j) = k(d_j),\ j = 1, 2.$$

Moreover in the discussion of impulse moments of $x(t - \tau(t))$, it is necessary to consider the following two cases,

$$(\bar{S}_1)\ t_{k(\delta_j)} + \tau > \delta_j; \quad (\bar{S}_2)\ t_{k(\delta_j)} + \tau \leq \delta_j,\ j = 1, 2.$$

In the following theorem, we only consider the case of combination of (S_1) with $(\bar{S}_1)$. For the other cases, similar conclusions can be given, and hence their proofs

are omitted.
For convenience, we define

$$\begin{aligned}\Pi_{1,j} =: \frac{1}{H_1(\delta_j,c_j)}\Big\{ &\int_{c_j}^{t_{k(c_j)+1}} \tilde{H}_1(t,c_j)\left(\frac{t-t_{k(c_j)}-\tau(t)}{t-t_{k(c_j)}}\right)^{\alpha} dt \\ &+\sum_{i=k(c_j)+1}^{k(\delta_1)-1}\Big[\int_{t_i}^{z_i}\tilde{H}_1(t,c_j)\left(\frac{t-t_i}{b_i(t-t_i+\tau(t))}\right)^{\alpha} dt \\ &\qquad +\int_{z_i}^{t_{i+1}}\tilde{H}_1(t,c_j)\left(\frac{t-t_i-\tau(t)}{t-t_i}\right)^{\alpha} dt\Big] \\ &+\int_{t_{k(\delta_j)}}^{z_{k(\delta_j)}}\tilde{H}_1(t,c_j)\left(\frac{t-t_{k(\delta_j)}}{b_{k(\delta_j)}(t-t_{k(\delta_j)}+\tau(t))}\right)^{\alpha} dt \\ &+\int_{z_{k(\delta_j)}}^{\delta_j}\tilde{H}_1(t,c_j)\left(\frac{t-t_{k(\delta_j)}-\tau(t)}{(t-t_{k(\delta_j)})}\right)^{\alpha} dt \\ &-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{c_j}^{\delta_j} r(t)H_1(t,c_j)\left|h_1(t,c_j)\right|^{\alpha+1} dt\Big\}, \quad (39)\end{aligned}$$

and

$$\begin{aligned}\Pi_{2,j} =: \frac{1}{H_2(d_j,\delta_j)}\Big\{ &\int_{\delta_j}^{t_{k(\delta_j)+1}} \tilde{H}_2(d_j,t)\left(\frac{t-t_{k(\delta_j)}-\tau(t)}{t-t_{k(\delta_j)}}\right)^{\alpha} dt \\ &+\sum_{i=k(\delta_j)+1}^{k(d_j)-1}\Big[\int_{t_i}^{z_i}\tilde{H}_2(d_j,t)\left(\frac{t-t_i}{b_i(t-t_i+\tau(t))}\right)^{\alpha} dt \\ &\qquad +\int_{z_i}^{t_{i+1}}\tilde{H}_2(d_j,t)\left(\frac{t-t_i-\tau(t)}{t-t_i}\right)^{\alpha} dt\Big] \\ &+\int_{t_{k(d_j)}}^{z_{k(d_j)}}\tilde{H}_2(d_j,t)\left(\frac{t-t_{k(d_j)}}{b_{k(d_j)}(t-t_{k(d_j)}+\tau(t))}\right)^{\alpha} dt \\ &+\int_{z_{k(d_j)}}^{d_j}\tilde{H}_2(d_j,t)\left(\frac{t-t_{k(d_j)}-\tau(t)}{(t-t_{k(d_j)})}\right)^{\alpha} dt \\ &-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{\delta_j}^{d_j} r(t)H_2(d_j,t)\left|h_2(d_j,t)\right|^{\alpha+1} dt\Big\}, \quad (40)\end{aligned}$$

where $\tilde{H}_1(t,c_j) = H_1(t,c_j)Q(t)$, $\tilde{H}_2(d_j,t) = H_2(d_j,t)Q(t)$, $(j = 1,2)$ and $Q(t)$ is defined as in (18).

Theorem 2. *Assume that for any $T \geq t_0$, there exist c_j, d_j, $\delta_j \notin \{t_k\}$, $j = 1,2$, such that $T < c_1 - \tau < c_1 < \delta_1 < d_1 \leq c_2 - \tau < c_2 < \delta_2 < d_2$ and (8) holds. If there exists $(H_1, H_2) \in \mathcal{H}$ such that*

$$\Pi_{1,j} + \Pi_{2,j} > \frac{M_j}{H_1(\delta_j,c_j)}\Psi_{c_j}^{\delta_j}[H_1(.,c_j)] + \frac{M_j}{H_2(d_j,\delta_j)}\Psi_{\delta_j}^{d_j}[H_2(d_j,.)], \; j = 1,2, \quad (41)$$

then equation (1) is oscillatory.

Proof. To arrive at a contradiction, let us suppose that $x(t)$ is a non-oscillatory solution of equation (1). Without loss of generality, we assume that $x(t) > 0$ and $x(t-\tau(t)) > 0$ for $t \geq t_0$. In this case, the interval of t selected for the following discussion is $[c_1, d_1]$. Continuing as in the proof of Theorem 1, we can get (24). Multiplying both sides of (24) by $H_1(t, c_1)$ and integrating it from c_1 to δ_1, we have

$$\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt \le -\int_{c_1}^{\delta_1} H_1(t,c_1)\frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(\alpha+1)/\alpha} dt - \int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} dt. \tag{42}$$

Noticing the impulsive moments $t_{k(c_1)+1}$, $t_{k(c_1)+2}$, $\ldots$, $t_{k(\delta_1)}$ in $[c_1, \delta_1]$ and using integration by parts on the left-hand side of the above inequality, we obtain

$$\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt = \sum_{i=k(c_1)+1}^{k(\delta_1)} [u(t_i) - u(t_i^+)]H_1(t_i,c_1) + u(\delta_1)H(\delta_1,c_1) - \left(\int_{c_1}^{t_{k(c_1)+1}} + \int_{t_{k(c_1)+1}}^{t_{k(c_1)+2}} + ... + \int_{t_{k(\delta_1)}}^{\delta_1}\right) u(t)h_1(t,c_1)H_1(t,c_1)dt. \tag{43}$$

From (28) and (43), we have

$$\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt = \sum_{i=k(c_1)+1}^{k(\delta_1)} \left(\frac{a_i^\alpha - b_i^\alpha}{a_i^\alpha}\right) H_1(t_i,c_1)u(t_i) + H_1(\delta_1,c_1)u(\delta_1) - \int_{c_1}^{\delta_1} u(t)h_1(t,c_1)H_1(t,c_1)dt. \tag{44}$$

Substituting (44) in (42), we have

$$\int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} dt \le \sum_{i=k(c_1)+1}^{k(\delta_1)} \left(\frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha}\right) H_1(t_i,c_1)u(t_i) - H_1(\delta_1,c_1)u(\delta_1) + \int_{c_1}^{\delta_1} H_1(t,c_1)\left[|h_1(t,c_1)|\,|u(t)| - \frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(1+\alpha)/\alpha}\right] dt. \tag{45}$$

Letting

$$\lambda = 1 + \frac{1}{\alpha},\ X = \frac{\alpha^{\alpha/\alpha+1}|u(t)|}{[r(t)]^{1/\alpha+1}} \text{ and } Y = \left[\alpha(\alpha+1)^{-(\alpha+1)}r(t)\right]^{\alpha/\alpha+1} |h_1(t,c_1)|^{\alpha},$$

and using by Lemma 3, (45) becomes

$$\int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} dt \le \sum_{i=k(c_1)+1}^{k(\delta_1)} \left(\frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha}\right) H_1(t_i,c_1)u(t_i) - H_1(\delta_1,c_1)u(\delta_1) + \frac{1}{(\alpha+1)^{\alpha+1}} \int_{c_1}^{\delta_1} r(t)H_1(t,c_1)\,|h_1(t,c_1)|^{\alpha+1} dt. \tag{46}$$

Similar to the proof of Theorem 1, we need to divide the interval $[c_1, \delta_1]$ into several sub intervals for estimating the function $\frac{x(t-\tau(t))}{x(t)}$. Using Lemma 4, we get the estimation for the left hand side of the above inequality as follows:

$$\begin{aligned}
&\int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau(t))}{x(t)}\right)^{\alpha} dt \\
&> \Bigg\{\int_{c_1}^{t_{k(c_1)+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(c_1)}-\tau(t)}{t-t_{k(c_1)}}\right)^{\alpha} dt \\
&\quad + \sum_{i=k(c_1)+1}^{k(\delta_1)-1}\Bigg[\int_{t_i}^{z_i} \tilde{H}_1(t,c_1)\left(\frac{t-t_i}{b_i(t-t_i+\tau(t))}\right)^{\alpha} dt \\
&\qquad\qquad + \int_{z_i}^{t_{i+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_i-\tau(t)}{t-t_i}\right)^{\alpha} dt\Bigg] \\
&\quad + \int_{t_{k(\delta_1)}}^{z_{k(\delta_1)}} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(\delta_1)}}{b_{k(\delta_1)}(t-t_{k(\delta_1)}+\tau(t))}\right)^{\alpha} dt \\
&\quad + \int_{z_{k(\delta_1)}}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(\delta_1)}-\tau(t)}{(t-t_{k(\delta_1)})}\right)^{\alpha} dt\Bigg\}. \qquad (47)
\end{aligned}$$

Comparing (46) and (47), we have

$$\begin{aligned}
&\Bigg\{\int_{c_1}^{t_{k(c_1)+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(c_1)}-\tau(t)}{t-t_{k(c_1)}}\right)^{\alpha} dt \\
&+ \sum_{i=k(c_1)+1}^{k(\delta_1)-1}\Bigg[\int_{t_i}^{z_i} \tilde{H}_1(t,c_1)\left(\frac{t-t_i}{b_i(t-t_i+\tau(t))}\right)^{\alpha} dt \\
&\qquad\qquad + \int_{z_i}^{t_{i+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_i-\tau(t)}{t-t_i}\right)^{\alpha} dt\Bigg] \\
&+ \int_{t_{k(\delta_1)}}^{z_{k(\delta_1)}} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(\delta_1)}}{b_{k(\delta_1)}(t-t_{k(\delta_1)}+\tau(t))}\right)^{\alpha} dt \\
&+ \int_{z_{k(\delta_1)}}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(\delta_1)}-\tau(t)}{(t-t_{k(\delta_1)})}\right)^{\alpha} dt \\
&- \frac{1}{(\alpha+1)^{\alpha+1}}\int_{c_1}^{\delta_1} r(t)H_1(t,c_1)\left|h_1(t,c_1)\right|^{\alpha+1} dt\Bigg\} \\
&\quad < \sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}\right) H_1(t_i,c_1)u(t_i) - H_1(\delta_1,c_1)u(\delta_1). \qquad (48)
\end{aligned}$$

Multiplying both sides of (24) by $H_2(d_1,t)$, and using similar analysis as above, we can obtain

$$\begin{aligned}&\Bigg\{\int_{\delta_1}^{t_{k(\delta_1)+1}} \tilde{H}_2(d_1,t)\left(\frac{t-t_{k(\delta_1)}-\tau(t)}{t-t_{k(\delta_1)}}\right)^{\alpha} dt\\&+\sum_{i=k(\delta_1)+1}^{k(d_1)-1}\Bigg[\int_{t_i}^{z_i} \tilde{H}_2(d_1,t)\left(\frac{t-t_i}{b_i(t-t_i+\tau(t))}\right)^{\alpha} dt\\&\qquad+\int_{z_i}^{t_{i+1}} \tilde{H}_2(d_1,t)\left(\frac{t-t_i-\tau(t)}{t-t_i}\right)^{\alpha} dt\Bigg]\\&+\int_{t_{k(d_1)}}^{z_{k(d_1)}} \tilde{H}_2(d_1,t)\left(\frac{t-t_{k(d_1)}}{b_{k(d_1)}(t-t_{k(d_1)}+\tau(t))}\right)^{\alpha} dt\\&+\int_{z_{k(d_1)}}^{d_1} \tilde{H}_2(d_1,t)\left(\frac{t-t_{k(d_1)}-\tau(t)}{(t-t_{k(d_1)})}\right)^{\alpha} dt\\&-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{\delta_1}^{d_1} r(t)H_2(d_1,t)\left|h_2(d_1,t)\right|^{\alpha+1} dt\Bigg\}\\&<\sum_{i=k(\delta_1)+1}^{k(d_1)}\left(\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}\right)H_2(d_1,t_i)u(t_i)+H_2(d_1,\delta_1)u(\delta_1).\end{aligned} \tag{49}$$

Dividing (48) and (49) by $H_1(\delta_1, c_1)$ and $H_2(d_1, \delta_1)$ respectively, and adding them, we get

$$\begin{aligned}\Pi_{1,1}+\Pi_{2,1}<\Bigg\{&\frac{1}{H_1(\delta_1,c_1)}\sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}\right)H_1(t_i,c_1)u(t_i)\\&+\frac{1}{H_2(d_1,\delta_1)}\sum_{i=k(\delta_1)+1}^{k(d_1)}\left(\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}\right)H_2(d_1,t_i)u(t_i)\Bigg\}.\end{aligned} \tag{50}$$

On the other hand, similar to (33), we have

$$\sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}\right)H_1(t_i,c_1)u(t_i)\le M_1\Psi_{c_1}^{\delta_1}[H_1(.,c_1)], \tag{51}$$

and

$$\sum_{i=k(\delta_1)+1}^{k(d_1)}\left(\frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}}\right)H_2(d_1,t_i)u(t_i)\le M_1\Psi_{\delta_1}^{d_1}[H_2(d_1,.)]. \tag{52}$$

Substituting (51) and (52) in (50), we obtain a contradiction to the condition (41).

When $x(t)$ is eventually negative, we can consider the interval $[c_2, d_2]$ and reach a similar contradiction. Hence the proof is complete. □

As shown in [20] for the sub-linear terms case, we can also remove the sign condition imposed on the coefficients of the sub-half-linear terms to obtain interval oscillation criteria. More precisely, we consider

$$\begin{aligned}&\big(r(t)\Phi_\alpha(x'(t))\big)'+p(t)\Phi_\alpha(x(t-\tau(t)))+\mathcal{R}(x,t)=e(t),\ t\ge t_0,\ t\ne t_k,\\&x(t_k{}^+)=a_kx(t_k),\quad x'(t_k{}^+)=b_kx'(t_k),\ k=1,2,3,...,\end{aligned} \tag{53}$$

where

$$\mathcal{R}(x,t)=\sum_{i=1}^{m}q_i(t)\Phi_{\beta_i}(x-\tau(t))+\sum_{k=m+1}^{n}q_k(t)\Phi_{\beta_k}(x-\tau(t)).$$

When some or all of the functions $q_i(t), i=m+1,\dots,n$, are nonpositive, we can get the following results.

Theorem 3. *Assume that for any $T\geq t_0$, there exist $c_j,d_j\notin\{t_k\}$, $j=1,2$, such that $T<c_1-\tau<c_1<d_1\leq c_2-\tau<c_2<d_2$ and*

$$\begin{aligned} p(t),q_i(t)\geq 0, &\quad t\in[c_1-\tau,d_1]\cup[c_2-\tau,d_2]\backslash\{t_k\},\ i=1,2,3,...,m,\\ e(t)\leq 0, &\quad t\in[c_1-\tau,d_1]\backslash\{t_k\},\\ e(t)\geq 0, &\quad t\in[c_2-\tau,d_2]\backslash\{t_k\}.\end{aligned}\tag{54}$$

If there exist $\omega_j(t)\in\Omega_j(c_j,d_j)$, $j=1,2$, and positive numbers π_i and ψ_k with

$$\sum_{i=1}^{m}\pi_i+\sum_{k=m+1}^{n}\psi_k=1,\tag{55}$$

such that, for $k(c_j)<k(d_j)$,

$$\begin{aligned}&\left\{\int_{c_j}^{t_{k(c_j)+1}}\tilde{W_j}(t)\left(\frac{t-t_{k(c_j)}-\tau(t)}{t-t_{k(c_j)}}\right)^{\alpha}dt\right.\\ &+\sum_{i=k(c_j)+1}^{k(d_j)-1}\left[\int_{t_i}^{z_i}\tilde{W_j}(t)\left(\frac{t-t_i}{b_i(t-t_i+\tau(t))}\right)^{\alpha}dt+\int_{z_i}^{t_i+1}\tilde{W_j}(t)\left(\frac{t-t_i-\tau(t)}{t-t_i}\right)^{\alpha}\right]\\ &+\int_{t_{k(d_j)}}^{z_{k(d_j)}}\tilde{W_j}(t)\left(\frac{t-t_{k(d_j)}}{b_{k(d_j)}(t-t_{k(d_j)}+\tau(t))}\right)^{\alpha}dt\\ &+\int_{z_{k(d_j)}}^{d_j}\tilde{W_j}(t)\left(\frac{t-t_{k(d_j)}-\tau(t)}{t-t_{k(d_j)}}\right)^{\alpha}dt\\ &\left.-\int_{c_j}^{d_j}(r(t)\left|\omega_j'(t)\right|^{\alpha+1})dt\right\}>M_j\Psi_{c_j}^{d_j}[\omega_j^{\alpha+1}],\end{aligned}\tag{56}$$

and for $k(c_j)=k(d_j)$,

$$\int_{c_j}^{d_j}\left(\tilde{W_j}(t)\left(\frac{t-c_j}{t-c_j+\tau(t)}\right)^{\alpha}-r(t)\left|\omega_j'(t)\right|^{\alpha+1}\right)dt>0,\tag{57}$$

where $\tilde{W_j}(t)=\Theta(t)\omega_j^{\alpha+1}(t)$, $j=1,2$, and

$$\Theta(t)=\left(p(t)+\sum_{i=1}^{m}\Gamma_i(t)-\sum_{k=m+1}^{n}\Upsilon_k(t)\right),\tag{58}$$

$$\Gamma_i(t)=\beta_i(\beta_i-\alpha)^{(\alpha/\beta_i)-1}\alpha^{-\alpha/\beta_i}\pi_i^{1-(\alpha/\beta_i)}[q_i(t)]^{\alpha/\beta_i}|e(t)|^{1-(\alpha/\beta_i)},\ i=1,2,\dots,m,$$
$$\Upsilon_k(t)=\beta_k(\alpha-\beta_k)^{(\alpha/\beta_k)-1}\alpha^{-\alpha/\beta_k}\psi_k^{1-(\alpha/\beta_k)}[\tilde{q}_k(t)]^{\alpha/\beta_k}|e(t)|^{1-(\alpha/\beta_k)},$$

$k=m+1,\dots,n$, with $\tilde{q}_k(t)=\max\{-q_k(t),0\}$, then equation (53) is oscillatory.

Proof. Suppose that equation (53) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t)$ is eventually positive on $[c_1,d_1]$. If $x(t)$ is eventually negative, then one can repeat the proof on the interval $[c_2, d_2]$. Clearly, from the definition of $\mathcal{R}(x,t)$ in (53) , we have

$$\mathcal{R}(x,t) - e(t) = \sum_{i=1}^{m} [q_i(t)\Phi_{\beta_i}(x(t-\tau(t))) + \pi_i|e(t)|] - \sum_{k=m+1}^{n} [-q_k(t)\Phi_{\beta_k}(x(t-\tau(t))) - \psi_k|e(t)|],$$

this gives

$$\mathcal{R}(x,t) - e(t) \geq \sum_{i=1}^{m} [q_i(t)\Phi_{\beta_i}(x(t-\tau(t))) + \pi_i|e(t)|] - \sum_{k=m+1}^{n} [\tilde{q}_k(t)\Phi_{\beta_k}(x(t-\tau(t))) - \psi_k|e(t)|]. \quad (59)$$

Applying Lemma 2 to each summation on the right side of (59) with

$$A = q_i(t),\ \ B = \pi_i|e(t)|,\ \ \gamma = \beta_i,\ \ \delta = \alpha,\ \text{for } \beta_i > \alpha \text{ and } i = 1, \dots, m,$$

and

$$C = \tilde{q}_k(t),\ \ D = \psi_i|e(t)|,\ \ \delta = \beta_k,\ \ \gamma = \alpha,\ \text{for } \beta_k < \alpha \text{ and } k = m+1, \dots, n,$$

we get

$$\mathcal{R}(x,t) - e(t) \geq \sum_{i=1}^{m} \Gamma_i(t)\Phi_\alpha(x(t-\tau(t))) - \sum_{k=m+1}^{n} \Upsilon_i(t)\Phi_\alpha(x(t-\tau(t))). \quad (60)$$

From equation (53) and inequality (60), we have

$$(r(t)\phi_\alpha(x'(t)))' + p(t)\phi_\alpha x(t-\tau(t)) + \sum_{i=1}^{m} \Gamma_i(t)\Phi_\alpha(x(t-\tau(t))) - \sum_{k=m+1}^{n} \Upsilon_i(t)\Phi_\alpha(x(t-\tau(t))) \leq 0. (61)$$

Defining $u(t)$ as in (19), and continuing as in the proof of Theorem 1, we get

$$u'(t) \leq -\Theta(t)\left(\frac{x(t-\tau(t))}{x(t)}\right)^\alpha - \frac{1}{r^{1/\alpha}(t)} u^{\frac{\alpha+1}{\alpha}}(t) \text{ for } t \in [c_1, d_1], \quad (62)$$

where $\Theta(t)$ is as in (58). The remainder of the proof is the same as that of Theorem 1, hence omitted. □

Theorem 4. *Assume that for any $T \geq t_0$, there exist c_j, d_j, $\delta_j \notin \{t_k\}$, $j = 1, 2$, such that $T < c_1 - \tau < c_1 < \delta_1 < d_1 \leq c_2 - \tau < c_2 < \delta_2 < d_2$ and (54) holds. If there exist $(H_1, H_2) \in \mathcal{H}$ and positive numbers π_i and ψ_k with (55), such that*

$$\Pi_{1,j} + \Pi_{2,j} > \frac{M_j}{H_1(\delta_j, c_j)} \Psi_{c_j}^{\delta_j}[H_1(., c_j)] + \frac{M_j}{H_2(d_j, \delta_j)} \Psi_{\delta_j}^{d_j}[H_2(d_j, .)],\ j = 1, 2, \quad (63)$$

where $\tilde{H}_1(t, c_j) = H_1(t, c_j)\Theta(t)$, $\tilde{H}_2(d_j, t) = H_2(d_j, t)\Theta(t)$, $j = 1, 2$, and $\Theta(t)$ is defined as in (58) with $\tilde{q}_k(t) = \max\{-q_k(t), 0\}$, then equation (53) is oscillatory.

Proof. Continuing as in the proof of Theorem 3, we can get (62). The remainder of the proof is the same as that of Theorem 2, and hence is omitted. □

3. Examples

In this section, we give two examples to illustrate our main results.

Example 1. Consider the following impulsive differential equation with variable delay

$$\begin{aligned}
&(\Phi_\alpha(x'(t)))' + \gamma_0(1+\sin t)\Phi_\alpha\left(x\left(t-\frac{\pi}{12}\sin^3 t\right)\right)\\
&\qquad + \gamma_1\exp(-t/2)\Phi_{\beta_1}\left(x\left(t-\frac{\pi}{12}\sin^3 t\right)\right)\\
&\qquad + \gamma_2 t^2\Phi_{\beta_2}\left(x\left(t-\frac{\pi}{12}\sin^3 t\right)\right) = e(t),\ t\geq t_0,\ t\neq t_{k,i}, \qquad (64)\\
&\qquad x(t_{k,i}^+) = a_{k,i}x(t_{k,i}),\quad x'(t_{k,i}^+) = b_{k,i}x'(t_{k,i}),\\
&\text{where}\quad t_{k,i} = 2k\pi + \frac{3\pi}{8} + (-1)^{i-2}\left(\frac{\pi}{4}\right),\quad i=1,2,\ \text{and}\ k=1,2,\ldots.
\end{aligned}$$

Here,

$$r(t)=1, p(t)=\gamma_0(1+\sin t), q_1(t)=\gamma_1\exp(-t/2), q_2(t)=\gamma_2 t^2,$$

and

$$e(t) = \begin{cases} -\sin 2t, & t\in\left[2k\pi,\ 2k\pi+\frac{\pi}{6}\right],\\ 1+\sin t, & t\in\left[2k\pi+\frac{\pi}{6},\ 2k\pi+\frac{4\pi}{3}\right],\end{cases}$$

where γ_0,γ_1 and γ_2 are positive constants.

If we choose $\eta_0 = 1/2$, $\beta_1 = 19/2$, $\beta_2 = 5/2$ and $\alpha = 3$, then by Lemma 1, we can easily find $\eta_1=\eta_2=1/4$. For any $T>0$, if we choose k large enough such that $T<c_1 = 2k\pi+\frac{\pi}{12} < d_1 = 2k\pi+\frac{\pi}{6}$ and $c_2 = 2k\pi+\frac{\pi}{4} < d_2 = 2k\pi+\frac{4\pi}{3}$, then there are impulsive moments $t_{k,1} = 2k\pi+\frac{\pi}{8}$ in $[c_1,d_1]$ and $t_{k,2} = 2k\pi+\frac{5\pi}{8}$ in $[c_2,d_2]$. From $t_{k,2}-t_{k,1} = \frac{\pi}{2} > \frac{\pi}{12}$ and $t_{k+1,1}-t_{k,2} = \frac{3\pi}{2} > \frac{\pi}{12}$ for all $k\in\mathbb{N}$, we know that the condition $t_{k+1}-t_k > \tau$ is satisfied. Also the variable delay $\tau(t) = \frac{\pi}{12}\sin^3 t$ satisfies $0\leq\tau(t)\leq\tau=\frac{\pi}{12}$. If we take $D_{k,i}(t) = t-t_{k,i}-\tau(t)$, then $D'_{k,i}(t) = 1-\frac{\pi}{4}\sin^2 t\cos t > 0$ for all t and there exist zero points $z_{k,i}$ of $D_{k,i}(t)$ in $(t_{k,i},d_i)$. Moreover, we also see that the condition (8) is satisfied. Let

$$\omega_j(t) = \sin 12t \in \Omega_j(c_j,d_j) \text{ for } j=1,2.$$

For $t\in[c_1,d_1]$, by simple calculation, we get $z_{k,1}\in(t_{k,1},d_1)$ and $z_{k,1}\approx 0.408$. In view of $\sum_{i=k(c_j)+1}^{k(d_j)-1} = 0$ as $k(c_j)+1 > k(d_j)-1$, $j=1,2$, the left hand side of (16) is the following

$$\begin{aligned}
&\int_{\pi/12}^{\pi/8} W_1(t)\left(\frac{t+11\pi/8-\pi/12\sin^3(t)}{t-11\pi/8)}\right)^3 dt\\
&+\int_{\pi/8}^{0.408} W_1(t)\left(\frac{t-\pi/8}{b_{k,1}(t-\pi/8+\pi/12\sin^3(t))}\right)^3 dt\\
&+\int_{0.408}^{\pi/6} W_1(t)\left(\frac{t-\pi/8-\pi/12\sin^3(t)}{t-11\pi/8)}\right)^3 dt - 12^4\int_{\pi/12}^{\pi/6}\left(\cos^4 12t\right)dt\\
&\approx\left[0.0722\gamma_0+0.0714\gamma_1^{1/4}\gamma_2^{1/4}\right] + b_{k,1}^{-3}\left[0.0009\gamma_0+0.0014\gamma_1^{1/4}\gamma_2^{1/4}\right] - 648\pi. \quad (65)
\end{aligned}$$

On the other hand, the right hand side of (16) is

$$\begin{aligned}\Psi_{c_1}^{d_1}[\omega_1^{\alpha+1}] &= \omega_1^{\alpha+1}(t_{k(c_1)+1})\frac{b_{k(c_1)+1}^{\alpha}-a_{k(c_1)+1}^{\alpha}}{(a_{k(c_1)+1}^{\alpha}(t_{k(c_1)+1}-c_1)^{\alpha})} \\ &\quad+\sum_{i=k(c_1)+2}^{k(d_1)}\omega_1^{\alpha+1}(t_i)\frac{b_i^{\alpha}-a_i^{\alpha}}{(a_i^{\alpha}(t_i-t_{i-1})^{\alpha})} \\ &= \sin^4 12(2k\pi+\pi/8)\left(\frac{b_{k,1}-a_{k,1}}{a_{k,1}(2k\pi+\pi/8-(2k\pi+\pi/12))}\right)^3 \\ &= \left(\frac{24}{\pi}\right)^3\left(\frac{b_{k,1}-a_{k,1}}{a_{k,1}}\right)^3. \end{aligned} \tag{66}$$

Thus for $t\in[c_1,d_1]$, if we choose γ_0, γ_1 and γ_2 large enough so that

$$\begin{aligned}&0.0722\gamma_0+0.0714\gamma_1^{1/4}\gamma_2^{1/4}+b_{k,1}^{-3}\left[0.0009\gamma_0+0.0014\gamma_1^{1/4}\gamma_2^{1/4}\right]-648\pi \\ &>\left(\frac{24}{\pi}\right)^3\left(\frac{b_{k,1}-a_{k,1}}{a_{k,1}}\right)^3,\end{aligned} \tag{67}$$

then (16) will be satisfied.

Similarly for $t\in[c_2,d_2]$, we can get the following condition

$$\begin{aligned}&2.4899\gamma_0+2.8481\gamma_1^{1/4}\gamma_2^{1/4}+b_{k,2}^{-3}\left(0.0074\gamma_0+0.0011\gamma_1^{1/4}\gamma_2^{1/4}\right)-8424\pi \\ &>\left(\frac{8}{3\pi}\right)^3\left(\frac{b_{k,2}-a_{k,2}}{a_{k,2}}\right)^3.\end{aligned} \tag{68}$$

Thus it is clear that the condition (16) is satisfied by properly choosing γ_0, γ_1, γ_2, and so by Theorem 1, equation (64) is oscillatory.

Example 2. Consider the following impulsive differential equation with variable delay

$$\begin{aligned}(\Phi_\alpha(x'(t)))'+\kappa_0 p(t)\Phi_\alpha\left(x\left(t-\frac{\pi}{16}\cos^2(\pi t)\right)\right)+\kappa_1 q_1(t)\Phi_{\beta_1}\left(x\left(t-\frac{\pi}{16}\cos^2(\pi t)\right)\right)\\ +\kappa_2 q_2(t)\Phi_{\beta_2}\left(x\left(t-\frac{\pi}{16}\cos^2(\pi t)\right)\right)=e(t),\ t\ge t_0,\ t\ne t_{n,i},\\ x(t_{n,i}^+)=a_{n,i}x(t_{n,i}),\quad x'(t_{n,i}^+)=b_{n,i}x'(t_{n,i}),\end{aligned} \tag{69}$$

where κ_0,κ_1 and κ_2 are positive constants, and

$$t_{n,1}=2n\pi+\frac{\pi}{8},\ t_{n,2}=2n\pi+\frac{3\pi}{8},\ t_{n,3}=2n\pi+\frac{13\pi}{8}\text{ and }t_{n,4}=2n\pi+\frac{17\pi}{8}.$$

In addition let,

$$p(t)=\begin{cases} e^{12t}, & t\in\left[2n\pi+\frac{\pi}{12},\,2n\pi+\frac{\pi}{6}\right],\\ e^{4t}, & t\in\left[2n\pi+\frac{\pi}{6},\,2n\pi+\frac{\pi}{2}\right],\\ \sin^2 t, & t\in\left[2n\pi+\frac{3\pi}{2},\,2n\pi+\frac{5\pi}{2}\right],\end{cases}$$

$$q_1(t)=3+\cos(t/2),$$

$$q_2(t) = \begin{cases} \cos(t/4), & t \in \left[2n\pi + \frac{\pi}{12},\, 2n\pi + \frac{\pi}{2}\right], \\ \cos t, & t \in \left[2n\pi + \frac{3\pi}{2},\, 2n\pi + \frac{5\pi}{2}\right], \end{cases}$$

and

$$e(t) = \begin{cases} -\sin 2t, & t \in \left[2n\pi + \frac{\pi}{12},\, 2n\pi + \frac{\pi}{2}\right], \\ \sin^2 t, & t \in \left[2n\pi + \frac{3\pi}{2},\, 2n\pi + \frac{5\pi}{2}\right]. \end{cases}$$

For any $t_0 > 0$, we choose n large enough such that $t_0 < 2n\pi + \frac{\pi}{12}$ and let $[c_1, d_1] = \left[2n\pi + \frac{\pi}{12}, 2n\pi + \frac{\pi}{2}\right]$, $[c_2, d_2] = \left[2n\pi + \frac{3\pi}{2}, 2n\pi + \frac{5\pi}{2}\right]$, $\delta_1 = 2n\pi + \frac{\pi}{6}$ and $\delta_2 = 2n\pi + \frac{5\pi}{3}$. It is easy to see that condition (8) is satisfied. Moreover, we easily see that there exist zero point $z_{n,i}\,(i = 1, 2, 3, 4)$ of $D_{n,i}(t) = t - t_{n,i} - \frac{\pi}{12}\cos^2(\pi t)$ in $(t_{n,1}, \delta_1), (t_{n,2}, d_1), (t_{n,3}, \delta_2)$ and $(t_{n,4}, d_2)$ respectively. Let $H_1(t, s) = H_2(t, s) = (t - s)^2$. Then $h_1(t, s) = -h_2(t, s) = \frac{2}{(t-s)}$. Next, if we choose $\eta_0 = 1/2$, $\beta_1 = 5/2$, $\beta_2 = 1/2$ and $\alpha = 1$, then one can easily find $\eta_1 = 3/8$ and $\eta_2 = 1/8$.
Also by a simple calculation, we get

$$\begin{aligned} \Pi_{1,1} = \frac{1}{H_1(\frac{\pi}{6}, \frac{\pi}{12})} \Bigg\{ & \int_{\pi/12}^{\pi/8} H_1(t, \pi/12) Q(t) \left(\frac{t + 13\pi/8 - \pi/16\cos^2(\pi t)}{t + 13\pi/8} \right) dt \\ & + \int_{\pi/8}^{0.408} H_1(t, \pi/12) Q(t) \left(\frac{t - \pi/8}{b_{n,1}(t - \pi/8) + \pi/16\cos^2(\pi t)} \right) dt \\ & + \int_{0.408}^{\pi/6} H_1(t, \pi/12) Q(t) \left(\frac{t - \pi/8 - \pi/16\cos^2(\pi t)}{t - \pi/8} \right) dt \\ & - \frac{1}{2^2} \int_{\pi/12}^{\pi/6} H_1(t, \pi/12) \left| h_1(t, \pi/12) \right|^2 dt \Bigg\} \\ \approx \kappa_0 \left(23.2539 + \frac{0.1532}{b_{n,1}} \right) & + {\kappa_1}^{3/8} {\kappa_2}^{1/8} \left(0.3035 + \frac{0.0044}{b_{n,1}} \right) - 3.8197, \end{aligned} \tag{70}$$

and

$$\begin{aligned} \Pi_{2,1} = \frac{1}{H_2(\frac{\pi}{2}, \frac{\pi}{6})} \Bigg\{ & \int_{\pi/6}^{3\pi/8} H_2(\pi/2, t) Q(t) \left(\frac{t - \pi/8 - \pi/16\cos^2(\pi t)}{t - \pi/8} \right) dt \\ & + \int_{3\pi/8}^{1.265} H_2(\pi/2, t) Q(t) \left(\frac{t - 3\pi/8}{b_{n,2}(t - 3\pi/8 + \pi/16\cos^2(\pi t))} \right) dt \\ & + \int_{1.265}^{\pi/2} H_2(\pi/2, t) Q(t) \left(\frac{t - 3\pi/8 - \pi/16\cos^2(\pi t)}{t - 3\pi/8} \right) dt \\ & - \frac{1}{(2)^2} \int_{\pi/6}^{\pi/2} H_2(\pi/2, t) \left| h_2(\pi/2, t) \right|^2 dt \Bigg\} \\ \approx \kappa_0 \left(7.8212 + \frac{0.3295}{b_{n,2}} \right) & + {\kappa_1}^{3/8} {\kappa_2}^{1/8} \left(1.1289 + \frac{0.0082}{b_{n,2}} \right) - 0.9549. \end{aligned} \tag{71}$$

From (70) and (71), we get

$$\begin{aligned} \Pi_{1,1} + \Pi_{2,1} \approx \kappa_0 & \left(31.0751 + \frac{0.1532}{b_{n,1}} + \frac{0.3295}{b_{n,2}} \right) \\ & + {\kappa_1}^{3/8} {\kappa_2}^{1/8} \left(1.4324 + \frac{0.0044}{b_{n,1}} + \frac{0.0082}{b_{n,2}} \right) - 4.7746, \end{aligned} \tag{72}$$

which gives the left hand side of (41).
On the other hand,

$$\begin{aligned}\frac{M_1}{H_1(\delta_1, c_1)}\Psi_{c_1}^{\delta_1}[H_1(., c_1)] &= \frac{1}{H_1(\pi/6, \pi/12)} H_1(\pi/8, \pi/12) \times \left(\frac{b_{n,1} - a_{n,1}}{a_{n,1}(\pi/8 - \pi/12)}\right) \\ &\approx 1.9098 \left(\frac{b_{n,1} - a_{n,1}}{a_{n,1}}\right), \qquad (73)\end{aligned}$$

and

$$\begin{aligned}\frac{M_1}{H_2(d_1, \delta_1)}\Psi_{\delta_1}^{d_1}[H_2(d_1, .)] &= \frac{1}{(\pi/2 - \pi/6)^2} (\pi/2 - 3\pi/8)^2 \times \left(\frac{b_{n,2} - a_{n,2}}{a_{n,2}(3\pi/8 - \pi/6))}\right) \\ &\approx 0.2148 \left(\frac{b_{n,2} - a_{n,2}}{a_{n,2}}\right). \qquad (74)\end{aligned}$$

From (73) and (74), we have the right hand side of (41) as

$$\begin{aligned}&\frac{M_1}{H_1(\delta_1, c_1)}\Psi_{c_1}^{\delta_1}[H_1(., c_1)] + \frac{M_1}{H_2(d_1, \delta_1)}\Psi_{\delta_1}^{d_1}[H_2(d_1, .)] \\ &\approx 1.9098 \left(\frac{b_{n,1} - a_{n,1}}{a_{n,1}}\right) + 0.2148 \left(\frac{b_{n,2} - a_{n,2}}{a_{n,2}}\right). \qquad (75)\end{aligned}$$

Thus (41) is satisfied for $j = 1$, if

$$\begin{aligned}&\kappa_0 \left(31.0751 + \frac{0.1532}{b_{n,1}} + \frac{0.3295}{b_{n,2}}\right) + {\kappa_1}^{3/8}{\kappa_2}^{1/8} \left(1.4324 + \frac{0.0044}{b_{n,1}} + \frac{0.0082}{b_{n,2}}\right) \\ &> 4.7716 + 1.9098 \left(\frac{b_{n,1} - a_{n,1}}{a_{n,1}}\right) + 0.2148 \left(\frac{b_{n,2} - a_{n,2}}{a_{n,2}}\right). \qquad (76)\end{aligned}$$

Similarly for $t \in [c_2, d_2]$, we have

$$\begin{aligned}\Pi_{1,2} + \Pi_{2,2} &\approx \kappa_0 \left(0.2871 + \frac{0.0012}{b_{n,3}} + \frac{0.0016}{b_{n,4}}\right) \\ &+ {\kappa_1}^{3/8}{\kappa_2}^{1/8} \left(-0.6978 + \frac{0.0042}{b_{n,3}} + \frac{0.0124}{b_{n,4}}\right) - 2.2917 \qquad (77)\end{aligned}$$

and

$$\begin{aligned}&\frac{M_2}{H_1(\delta_2, c_2)}\Psi_{c_2}^{\delta_2}[H_1(., c_2)] + \frac{M_2}{H_2(d_2, \delta_2)}\Psi_{\delta_2}^{d_2}[H_2(d_2, .)] \\ &\approx 1.4324 \left(\frac{b_{n,3} - a_{n,3}}{a_{n,3}}\right) + 0.1406 \left(\frac{b_{n,4} - a_{n,4}}{a_{n,4}}\right). \qquad (78)\end{aligned}$$

Thus (41) is satisfied for $j = 2$, if

$$\begin{aligned}&\kappa_0 \left(0.2871 + \frac{0.0012}{b_{n,3}} + \frac{0.0016}{b_{n,4}}\right) + {\kappa_1}^{3/8}{\kappa_2}^{1/8} \left(-0.6978 + \frac{0.0042}{b_{n,3}} + \frac{0.0124}{b_{n,4}}\right) \\ &> 2.2917 + 1.4324 \left(\frac{b_{n,3} - a_{n,3}}{a_{n,3}}\right) + 0.1406 \left(\frac{b_{n,4} - a_{n,4}}{a_{n,4}}\right). \qquad (79)\end{aligned}$$

Note that all the conditions of Theorem 2 are thus verified and it is possible to choose the constants so that (76) and (79) are also satisfied. Hence, by Theorem 2, equation (69) is oscillatory.

4. Conclusion

In this paper, we have established interval oscillation results for equation (1) using Riccati transformation technique. Our results generalize and improve the results in the existing literature as follows:

- When $\alpha = 1$ and $\tau(t) = \tau$, Theorem 1 reduces to Theorem 2.2 and Theorem 2 reduces Theorem 2.4 of [12].
- When $\tau(t) = 0$, our results reduces to the results of [7] for the case $\rho(t) = 1$.
- When $\tau(t) = 0$ and $\alpha = 1$, Theorem 1 reduces to Theorem 2.1 of [15].
- When $a_k = b_k = 1$ for all $k = 1, 2, 3, ...,$ $\tau(t) = 0$ and $\alpha = 1$, our results reduces to the results of [20] for the case $\rho(t) = 1$.

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V. Muthulakshmi
Department of Mathematics, Periyar University, Salem-636 011, Tamilnadu, India
E-mail address: vmuthupu@gmail.com

R. Manjuram
Department of Mathematics, Periyar University, Salem-636 011, Tamilnadu, India
E-mail address: ariyanmanju@gmail.com