Electronic Journal of Mathematical Analysis and Applications
Vol. 9(2) July 2021, pp. 158-165.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

UNIQUENESS PROBLEM FOR DIFFERENTIAL POLYNOMIALS OF FERMAT-WARING TYPE

RAJESHWARI S. , HUSNA V. AND NAVEEN KUMAR S.H

Abstract. In this paper, we discuss the uniqueness problem for differential polynomials $(P^n(f))^{(k)}, (Q^n(g))^{(k)}$ sharing the same values, where $P = f^d + a_1 f^{d-m} + b_1 f^{d-m+1} + c_1$ and $Q = g^d + a_2 g^{d-m} + b_2 g^{d-m+1} + c_2$ are polynomials of Fermat-Waring type. On non-Archimedian field, f and g are meromorphic functions.

1. Introduction, Notation and Main results

Let $\mathbb{H}$ be an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value. We denote by $A(\mathbb{H})$ the ring of entire functions in $\mathbb{H}$, by $M(\mathbb{H})$ the field of meromorphic functions, i.e., the field of fractions of $A(\mathbb{H})$, and $\widehat{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$. We assume that the reader is familiar with the notations in the non-Archimedean Nevanlinna theory (see [10]]). Let f be a non-constant meromorphic function on $\mathbb{H}$. For every $a \in \mathbb{H}$, define the function $d_f^a : \mathbb{H} \longrightarrow \mathbb{N}$ by

$$d_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ m & \text{if } f(z) = a \text{ with multiplicity } m, \end{cases}$$

and set $d_f^\infty = d_{\frac{1}{f}}^0$. For $f \in M(\mathbb{H})$ and $S \subset \mathbb{H} \cup \{\infty\}$, we define

$$E_f(S) = \cup_{a \in S} \{(z, d_f^a(z)) : z \in \mathbb{H}\}.$$

In this paper, we consider the differential operator $(P^n(f))^{(k)}$ and $(Q^n(g))^{(k)}$ sharing the same value where P and Q are Fermat-Waring type polynomials. Then we establish an uniqueness theorem for non-archimedian meromorphic functions and their differential polynomials.

Now let us describe main results of the paper. Let $d, m, n, k \in N^*$ and $a_1, b_1, c_1, a_2, b_2, c_2, k \in \mathbb{H}$; where $\mathbb{H}$ be an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value. $a_1, b_1, c_1, a_2, b_2, c_2 \neq 0$. We will let

$$P(z) = z^d + a_1 z^{d-m} + b_1 z^{d-m+1} + c_1 \;\; and \;\; Q(z) = z^d + a_2 z^{d-m} + b_2 z^{d-m+1} + c_2, \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 30D35.

Key words and phrases. Nevanlinna theory, non-archimedean meromorphic functions, Fermat-Waring polynomial.

Submitted April 30, 2020. Revised Nov. 27, 2020.

be a polynomials of degree d of Fermat-Waring type in $\mathbb{H}[z]$ without multiple zeros. We shall prove the following theorems.

Theorem I. *Let f and g be two non-constant meromorphic functions on $\mathbb{H}$ and let $P(z), Q(z)$ be defined in (1.1). Assume that $n \geq 3k+5, d \geq 2m+10$ and either $m \geq 2$ or $(d, m+1) = 1$ and $m \geq 1$. If $(P^n(f))^{(k)}$ and $(Q^n(g))^{(k)}$ share 1 CM, then $g = hf$ and for a constant h such that $h^d = \frac{c_2}{c_1}$, $h^{nd} = 1$, $h^m = \frac{b_2}{b_1}, h^{m+1} = \frac{a_2}{a_1}$.*

Theorem II. *Let f and g be two non-constant meromorphic functions on $\mathbb{H}$ and let $P(z), Q(z)$ be defined in (1.1). Assume that $d \geq 2m+10$ and either $m \geq 2$ or $(d, m+1) = 1$ and $m \geq 1$. If $(P(f)$ and $Q(f)$ share 0 CM, then $g = hf$ and for a constant h such that $h^d = \frac{c_2}{c_1}$, $h^m = \frac{b_2}{b_1}, h^{m+1} = \frac{a_2}{a_1}$.*

2. Preliminaries

In order to prove our results, we need the following Lemmas.

Lemma 2.1. *([10]) Let f be a non-constant meromorphic function on $\mathbb{H}$ and let $a_1, a_2, ..., a_q$, be distinct points of $\mathbb{H} \cup \{\infty\}$. Then*

$$(q-2)T(r,f) \leq \sum_{i=1}^{q} N_1(r, \frac{1}{f-a_i}) - logr + O(1).$$

Lemma 2.2. *([10]) Let f be a non-constant meromorphic function on $\mathbb{H}$ and let $a_1, a_2, ..., a_q$, be distinct points of $\mathbb{H} \bigcup \{\infty\}$. Suppose either $f - a_i$ has no zeros, or $f - a_i$ has zeros, in which case all the zeros of the functions $f - a_i$ have multiplicity at least $m_i, i = 1, ..., q$. Then*

$$\sum_{i=1}^{q} (1 - \frac{1}{m_i}) < 2.$$

Lemma 2.3. *([8]) Let f and g be non-constant meromorphic functions on $\mathbb{H}$. If $E_f(1) = E_g(1)$, then one of the following three cases holds:*

1. *$T(r,f) \leq N_2(r,f) + N_2(r, \frac{1}{f}) + N_2(r,f) + N_2(r, \frac{1}{g}) - logr + O(1)$, and the same inequality holds for $T(r,g)$;*
2. *$fg = 1$;*
3. *$f = g$.*

Lemma 2.4. *([1]) Let f be a non-constant meromorphic function on $\mathbb{H}$ and n, k be positive integers, $n > k$ and a be a pole of f. Then*

1. *$(f^n)^{(k)} = \frac{\varphi_k}{(z-a)^{np+k}}$, where $p = d_f^{\infty}, \varphi_k(a) \neq 0$.*
2. *$\frac{(f^n)^{(k)}}{f^{n-k}} = \frac{h_k}{(z-a)^{pk+k}}$, where $p = d_f^{\infty}, h_k(a) \neq 0$.*

Lemma 2.5. *([1]) Let f be a non-constant meromorphic function on $\mathbb{H}$ and n, k be positive integers, $n > 2k$, and let $P(z)$ be a polynomial of degree $d > 0$. Then*

1. $(n-2k)dT(r,f) + kN(r,P(f)) + N(r, \frac{1}{\frac{((P(f))^n)^{(k)}}{(P(f))^{n-k}}}) \leq T(r,((P(f))^n)^{(k)}) + O(1)$
$\leq (k+1)ndT(r,f) + O(1)$.
2. $N(r, \frac{1}{\frac{((P(f))^n)^{(k)}}{(P(f))^{n-k}}}) \leq kdT(r,f) + N_1(r,P(f)) + O(1)$
$= kdT(r,f) + kN_1(r,f) + O(1) \leq k(d+1)T(r,f) + O(1)$.

Lemma 2.6. *Let $d \geq 2m+5$ and either $m \geq 2$ or $(d, m+1) = 1$ and $m \geq 1, k \neq 0$, and let $P(z), Q(z)$be defined by (1.1). Assume that the equation $P(f) = kQ(g)$ has a non-constant meromorphic solution (f, g). Then $g = hf$ for a constant h such that $h^d = \frac{1}{k} = \frac{c_2}{c_1}$, $h^m = \frac{b_2}{b_1}, h^{m+1} = \frac{a_2}{a_1}$.*

Proof. Consider $P(f) = Q(g)$ we get $f^d + a_1 f^{d-m} + b_1 f^{d-m+1} + c_1 = k(g^d + a_2 g^{d-m} + b_2 g^{d-m+1} + c_2)$
$dT(r, f) + O(1) = dT(r, g)$,

$$T(r, f) + O(1) = T(r, g). \tag{2.1}$$

Equation (2.1) can be rewritten as $f_1 + f_2 = kc_2 - c_1$, where

$$f_1 = f^{d-m}(a_1 + b_1 f + f^m)$$
$$f_2 = -kg^{d-m}(a_2 + b_2 g + g^m).$$

If $kc_2 - c_1 \neq 0$, then by Lemma 2.1, we have

$$T(r, f_1) \leq N_1(r, f_1) + N_1(r, \frac{1}{f_1}) + N_1(r, \frac{1}{f_1 - (kc_2 - c_1)}) - logr + O(1),$$
$$dT(r, f) \leq N_1(r, f) + N_1(r, \frac{1}{f}) + N_1(r, \frac{1}{f^m + b_1 f + a_1}) + N_1(r, \frac{1}{g})$$
$$+ N_1(r, \frac{1}{g^m + b_1 g + a_1}) - logr + O(1)$$
$$dT(r, f) \leq (2m + 5)T(r, f) - logr + O(1)$$
$$(d - 2m - 5)T(r, f) \leq -logr + O(1),$$

which contradicts to $d \geq 2m + 5$. Hence $kc_2 - c_1 = 0$. Thus , (2.1) becomes

$$f^d + a_1 f^{d-m} + b_1 f^{d-m+1} = kg^d + ka_1 g^{d-m} + kb_1 g^{d-m+1}. \tag{2.2}$$

For simplicity, set $h = g/f$, and $\alpha = 1/k \neq 0, \beta_1 = \frac{b_1}{kb_2} \neq 0, \beta_2 = \frac{a_1}{ka_2} \neq 0$. Then we obtain

$$f^{m+1}(kh^d - 1) = -(ka_2 h^{d-m} - a_1) - (kb_2 h^{d-m+1} - b_1)$$
$$f^{m+1} = \frac{-a_2(h^{d-m} - \beta_1) - b_2(h^{d-m+1} - \beta_2)}{h^d - \alpha}. \tag{2.3}$$

Assume that h is not a constant. Consider the following possible cases:
CASE 1. $m \geq 1$, $(m + 1, d) = 1$. If $h^d - \alpha, h^{d-m} - \beta_1$ and $h^{d-m+1} - \beta_2$ have no common zeros, then all zeros of $h^d - \alpha$ have multiplicity $\geq m + 1$. Then

$$N_1(r, \frac{1}{h^d - \alpha}) \leq \frac{1}{m+1} N(r, \frac{1}{h^d - \alpha}).$$

By Lemma 2.1 we obtain

$$T(r, h^d) \leq N_1(r, h^d) + N_1(r, \frac{1}{h^d}) + N_1(r, \frac{1}{h^d - \alpha}) - logr + O(1),$$
$$dT(r, h) \leq 2T(r, h) + \frac{1}{m+1} N(r, \frac{1}{h^d - \alpha}) - logr + O(1),$$
$$\leq (2 + \frac{d}{m+1})T(r, h) - logr + O(1)$$
$$(d - 2 - \frac{d}{m+1})T(r, h) \leq -logr + O(1).$$

which leads to $dm < 2(m + 1)$, a contradiction to the condition $d \geq 2m + 5$.

If $h^d - \alpha$ and $h^{d-m} - \beta_1, h^{d-m-1} - \beta_2$ have common zeros, then there exists z_0 such that $h^d(z_0) = \alpha, h^{d-m}(z_0) = \beta_1$ and $h^{d-m-1} - \beta_2$.
From (2.3) we get

$$\alpha f^{m+1}((\frac{h}{h(z_0)})^d - 1) = -\beta_1 a_2((\frac{h}{h(z_0)})^{d-m} - 1) - \beta_2 b_2((\frac{h}{h(z_0)})^{d-m+1} - 1).$$

Since $(m+1, d) = 0$, the equations $z^d - 1 = 0, z^{d-m} - 1 = 0$ and $z^{d-m+1} = 0$ have different roots, except for $z = 1$. Let $r_i, i = 1, ..., 3d - 2m - 3$, be all the roots of them. Then all zeros of $\frac{h}{h(z_0)} - r_i$ have multiplicities $\geq m+1$. Therefore, by Lemma 2.2, we obtain

$$(1 - \frac{1}{m+1})(3d - 2m - 3) < 2, \; 3dm < 2m^2 + 6m + 3,$$

which contradicts $d \geq 2m + 5, m \geq 1$. Thus, h is a constant.
CASE 2. $m \geq 2$. Note that equation $z^d - \alpha = 0$ has d simple zeros, equation $z^{d-m} - \beta_1 = 0$ has $d-m$ simple zeros, and equation $z^{d-m+1} - \beta_2 = 0$ has $d-m+1$ common simple zeros. Therefore, the equation $z^d - \alpha$ has atleast m distinct roots, which are not roots of $z^{d-m} - \beta_1$ and $z^{d-m+1} - \beta_2 = 0$. Let $r_1, r_2, ..., r_m$ be all these roots. Then all zeros of $h - r_j, j = 1, ..., m$, have multiplicities $\geq m+1$. By Lemma 2.2, we have $(m+1)(1 - \frac{1}{m+1}) < 2$. Therefore, $m < 2$. From $m \geq 2$, we obtain a contradiction. Thus h is a constant. □

3. Proof of Theorem I

We have

$$\begin{aligned} P(f) &= (f - e_1)...(f - e_d), e_j \neq 0 \in \mathbb{H} \\ (P(f))^n &= (f - e_1)^n...(f - e_d)^n, \\ Q(g) &= (g - k_1)...(g - k_d), k_i \neq 0 \in \mathbb{H} \\ (Q(g))^n &= (g - k_1)^n...(g - k_d)^n \end{aligned}$$

Set

$$X_1 = (P^n(f))^{(k)}, \quad X_2 = (Q^n(g))^{(k)}, \quad Y_1 = P(f),$$

$$Y_2 = Q(g), \quad F = \frac{X_1}{Y_1^{n-k}}, \quad G = \frac{X_2}{Y_2^{n-k}}$$

Then

$$Y_1 = (f - e_1)...(f - e_d), \quad Y_2 = (g - k_1)...(g - k_d)$$

$$X_1 = (Y_1^n)^{(k)} = FY_1^{n-k}, \quad X_2 = (Y_2^n)^{(k)} = GY_2^{n-k}.$$

Applying Lemma 2.3 to $(Y_1^n)^{(k)}, (Y_2^n)^{(k)}$ we have one of the following possibilities:
CASE 1.

$$T(r, X_1) \leq N_2(r, X_1) + N_2(r, \frac{1}{X_1}) + N_2(r, \frac{1}{X_2}) + N_2(r, X_2) - logr + O(1),$$

$$T(r, X_2) \leq N_2(r, X_1) + N_2(r, \frac{1}{X_1}) + N_2(r, \frac{1}{X_2}) + N_2(r, X_2) - logr + O(1).$$

We see that, if a is a pole of X_1, then $Y_1(a)=\infty$ with $\nu^{\infty}_{X_1}(a)\geq n+k\geq 2$. Therefore

$$N_1(r,Y_1)=N_1(r,(f-e_1)...(f-e_d))=N_1(r,f)\leq T(r,f)+O(1),$$
$$N_1(r,\frac{1}{Y_1})=\Sigma_{i=1}^{d}N_1(r,\frac{1}{f-e_i})\leq dT(r,f)+O(1)$$
$$N_2(r,X_1)=2N_1(r,Y_1)\leq 2T(r,f)+O(1)$$
$$N_2(r,\frac{1}{X_1})\leq N_2(r,\frac{1}{Y_1^{n-k}})+N_1(r,\frac{1}{F})=2N_1(r,\frac{1}{Y_1})+N_1(r,\frac{1}{F})$$
$$\leq 2dT(r,f)+N(r,\frac{1}{F})\leq 2dT(r,f)+kN_1(r,Y_1)$$
$$+kdT(r,f)+O(1)=d(k+2)T(r,f)+kN_1(r,Y_1)+O(1).$$

Similarly

$$N_2(r,X_2)\leq 2T(r,g)+O(1)$$
$$N_2(r,\frac{1}{X_2})\leq 2dT(r,g)+N(r,\frac{1}{G})$$
$$=d(k+2)T(r,g)+kN_1(r,Y_2)+O(1).$$

Combining the above two inequalities, we get

$$T(r,X_1)\leq(2+2d+kd)T(r,f)+(2+2d)T(r,g)+kN_1(r,Y_1)+N(r,\frac{1}{G})-logr+O(1),$$
$$T(r,X_2)\leq(2+2d+kd)T(r,g)+(2+2d)T(r,f)+kN_1(r,Y_2)+N(r,\frac{1}{F})-logr+O(1),$$
$$T(r,X_1)+T(r,X_2)\leq(4+4d+kd)(T(r,f)+T(r,g))+KN_1(r,Y_1)+N(r,\frac{1}{G})$$
$$+kN_1(r,Y_2)+N(r,\frac{1}{F})-2logr+O(1).$$

By Lemma 2.5, we obtain

$$(n-2k)dT(r,f)+kN(r,Y_1)+N(r,\frac{1}{F})\leq T(r,X_1)+O(1),$$
$$(n-2k)dT(r,g)+kN(r,Y_2)+N(r,\frac{1}{G})\leq T(r,X_2)+O(1).$$

Thus

$$(n-2k)d[T(r,f)+T(r,g)]+kN(r,Y_1)+N(r,\frac{1}{F})+kN(r,Y_2)+N(r,\frac{1}{G})$$
$$\leq T(r,X_1)+T(r,X_2)+O(1),$$
$$(n-2k)d[T(r,f)+T(r,g)]+kN(r,Y_1)+N(r,\frac{1}{F})+kN(r,Y_2)+N(r,\frac{1}{G})$$
$$\leq(4+4d+kd)[T(r,f)+T(r,g)]+kN_1(r,Y_1)$$
$$+N(r,\frac{1}{G})+kN_1(r,Y_2)+N(r,\frac{1}{F})-2logr+O(1).$$

Therefore

$$(n-2k)d[T(r,f)+T(r,g)]\leq(4+4d+kd)(T(r,f)+T(r,g))-2logr+O(1),$$
$$((n-2k)d-4-4d-kd)(T(r,f)+T(r,g))\leq -2logr+O(1).$$

Since $n\geq 3k+5>2k+\frac{4+4d+kd}{d}$, we obtain a contradiction.
CASE 2. $(P(f)^n)^{(k)}(Q(g)^n)^{(k)}=1$. Then we have $Y_1=P(f)=(f-e_1)...(f-e_d)$.

$Y_1 = Y_1^{n-k}F, Y_2 = G(g)$. Therefore

$$(f-e_1)^{n-k}...(f-e_d)^{n-k}.X_1(Y_2^n)^{(k)} = (Y_1^n)^{(k)}(Y_2^n)^{(k)} = 1.$$

Because $n \geq 3k+5$ we see that, if z_0 is a zero of $f-e_i$ with $1 \leq i \leq d$, then z_0 is a zero of Y_1, and therefore, z_0 is a zero of $(Y_p^n)^{(k)}$ and then z_0 is a pole of $(Y_2^n)^{(k)}$ and $v^{\infty}_{(Y_2^n)^{(k)}}(z_0) = (n-k)v_f^{e_i}(z_0)$. Thus, z_0 is a pole of g and by Lemma 2.4 we get

$$v^{\infty}_{(Y_2^n)^{(k)}}(z_0) = ndv_g^{\infty}(z_0) + k \geq nd + k.$$

So, $v_f^{e_i}(z_0) = \frac{ndv_g^{\infty}(z_0)+k}{n-k} \geq \frac{nd+k}{n-k}$, $i = 1, 2, ...d$. Applying Lemma 2.2, we obtain

$$\sum_{i=1}^{d}(1 - \frac{n-k}{nd+k}) < 2.$$

From this we have $n(d^2-3d) < 2k(1-d)$, and so we obtain a contradiction to $d \geq 12$.

CASE 3. $(P(f)^n)^{(k)} = (Q(g)^n)^{(k)}$. Then $(P(f))^n - s = (Q(g))^n$, where s is a polynomial of degree $< k$. We prove $s \equiv 0$. If it is not the case, then

$$\frac{(P(f)^n)}{s} - 1 = \frac{(g-k_1)^n...(g-k_d)^n}{s},$$

$$\frac{(g-k_1)^n...(g-k_d)^n}{s} + 1 = \frac{(f-k_1)^n...(f-k_2)^n}{s}$$

Set $I = \frac{Y_1^n}{s}, J = \frac{Y_2^n}{s}$. Since f, g are not constants, and so are $Y_1, Y_2, Y_1^n, Y_2^n, I, J$. Applying Lemma 2.1 to I with values $\infty, 0, 1$, we get

$$T(r,I) \leq N_1(r,I) + N_1(r,\frac{1}{I}) + N_1(r,\frac{1}{I-1}) - logr + O(1).$$

On the other hand,

$$T(r,Y_1^n) = nT(r,Y_1) + O(1) \leq T(r,I) + T(r,s) \leq T(r,I) + (k-1)logr + O(1)$$

$$nT(r,Y_1) - (k-1)logr \leq T(r,I) + O(1), \;\; ndT(r,f) - (k-1)logr \leq T(r,I) + O(1)$$

$$N_1(r,I) \leq N_1(r,Y_1^n) + N_1(r,\frac{1}{s}) \leq N_1(r,f) + (k-1)logr \leq T(r,f) + (k-1)logr,$$

$$N_1(r,\frac{1}{I}) \leq N_1(r,\frac{1}{Y_1^n}) = N_1(r,\frac{1}{Y_1}) \leq T(r,Y_1) + O(1) = dT(r,f) + O(1),$$

$$N_1(r,\frac{1}{I-1}) = N_1(r,\frac{1}{J}) \leq N_1(r,\frac{1}{Y_2^n}) = N_1(r,\frac{1}{Y_2}) \leq T(r,Y_2) + O(1) = dT(r,g) + O(1),$$

$$ndT(r,f) - (k-1)logr \leq T(r,f) + (k-1)logr + d(T(r,f) + T(r,g)) + O(1).$$

From this, and noting that $logr \leq T(r,f)$, we get

$$(nd - 2(k-1))T(r,f) \leq T(r,f) + d(T(r,f) + T(r,g)) + O(1).$$

Applying Lemma 2.1 to J with values $\infty, 0, -1$, and noting that $logr \leq T(r,g)$, we obtain

$$T(r,J) \leq N_1(r,J) + N_1(r,\frac{1}{J}) + N_1(r,\frac{1}{J+1}) - logr + O(1).$$

we get

$$(nd - 2(k-1))T(r,g) \leq T(r,g) + d(T(r,f) + T(r,g)) - logr + O(1).$$

So

$$(nd-2(k-1))(T(r,f)+T(r,g)) \leq T(r,f)+T(r,g)+2d(T(r,f)+T(r,g))-2logr+O(1).$$

$$(nd-2d-2k+1)(T(r,f)+T(r,g))+2logr \leq O(1).$$

We obtain a contradiction to $n \geq 3k+5 > \frac{2d+2k-1}{d}$. So $s=0$. Then $(P(f))^n = (Q(g))^n$. Therefore $P(f) = kQ(g)$, $k^n = 1$. From this and by Lemma 2.6, we obtain the conclusion of Theorem I.

Proof of Theorem II. Set

$$Y_1 = P(f) = f^d + a_1 f^{d-m} + b_1 f^{d-m+1} + c_1.$$

$$Y_2 = Q(g) = g^d + a_2 g^{d-m} + b_2 g^{d-m+1} + c_2.$$

$$U = -\frac{f^{d-m}(f^m + b_1 f + a_1)}{c_1}, V = -\frac{g^{d-m}(g^m + b_2 g + a_1)}{c_2}$$

.

Since $P(f)$ and $Q(g)$ share 0 CM. we get $E_U(1) = E_V(1)$. Applying Lemma 2.3 to U, V, we have one of the following possibilities.

CASE 1.

$$T(r,U) \leq N_2(r,U) + N_2(r,\frac{1}{U}) + N_2(r,V) + N_2(r,\frac{1}{V}) - logr + O(1).$$

$$T(r,V) \leq N_2(r,V) + N_2(r,\frac{1}{V}) + N_2(r,U) + N_2(r,\frac{1}{U}) - logr + O(1).$$

More over

$$T(r,U) = dT(r,f) + O(1),$$
$$N_1(r,U) = N_1(r,f) \leq T(r,f) + O(1),$$
$$N_2(r,U) = 2N_1(r,f) \leq 2T(r,f) + O(1)$$
$$N_2(r,\tfrac{1}{U}) \leq 2N_1(r,\tfrac{1}{f}) + N_2(r,\tfrac{1}{f^m+b_1f+a_1}) \leq 2T(r,f) + (m+1)T(r,f) + O(1)$$

Similarly $N_2(r,V) \leq 2T(r,g)+O(1)$, $N_2(r,\frac{1}{V}) \leq 2T(r,g)+(m+1)T(r,g)+O(1)$.

Therefore

$$T(r,V) = dT(r,f) + O(1) \leq 4(T(r,f)+T(r,g)) + (m+1)(T(r,f)+T(r,g)) - logr + O(1).$$

Similarly

$$T(r,V) = dT(r,g) + O(1) \leq 4(T(r,f)+T(r,g)) + (m+1)(T(r,f)+T(r,g)) - logr + O(1)$$

Combining the above inequalities we get

$$d(T(r,f)+T(r,g)) \leq 8(T(r,f)+T(r,g)) + (2m+2)(T(r,f)+T(r,g)) - 2logr + O(1)$$

$$(d-2m-10)(T(r,f)+T(r,g)) + 2logr \leq O(1).$$

We obtain a contradiction to $d \geq 2m+10$.

CASE 2. $UV = 1$. i.e., $f^{d-m}(f^m + b_1 f + a_1) g^{d-m}(g^m + b_2 g + a_2) = \frac{c_1}{c_2}$.

Note that equation $z^m + b_1 z + a_1 = 0$ has (m+1) simple zeros. Let $r_1, r_2, ...r_m$ be all these roots. Therefore

$$f^{d-m}(f^m + b_1 f + a_1) g^{d-m}(g^m + b_2 g + a_2) = \frac{c_1}{c_2}. \tag{3.1}$$

From (3.1) it follows that all zeros of $f - r_j, j = 1, 2, ...m$, has multiplicities $\geq d$, and all zeros of f have multiplicities $\geq \frac{d}{d-m+1}$. By Lemma 2.2 we have $1 - \frac{d-m+1}{d} + (m+1)(1-\frac{1}{d}) < 2$. Then $m < 2$. Since $m \geq 1$, we obtain a contradiction.

CASE 3. $U = V$, i.e., $\frac{f^{d-m}(f^m+b_1f+a_1)}{c_1} = \frac{g^{d-m}(g^m+b_2g+a_1)}{c_2}$
then

$$f^d + a_1 f^{d-m} + b_1 f^{d-m+1} + C_1 = \frac{C_1}{C_2} g^d + a_1 g^{d-m} + b_1 g^{d-m+1} + C_2. \tag{3.2}$$

Applying Lemma 2.6 to (3.2), we obtain the conclusion of Theorem II.

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RAJESHWARI S.

Assistant Professor, School of Engineering, Presidency University, Itagalpura, Rajanakunte, Yelahanka, Bangalore-560 064, INDIA

E-mail address: <rajeshwari.s@presidencyuniversity.in, rajeshwaripreetham@gmail.com>

HUSNA V.

Assistant Professor, School of Engineering, Presidency University, Itagalpura, Rajanakunte, Yelahanka, Bangalore-560 064, INDIA

E-mail address: <husna@presidencyuniversity.in, husnav43@gmail.com>

NAVEEN KUMAR S.H.

Department of Mathematics, Gitam School of Technology, Gitam Bangalore-562163, INDIA

E-mail address: <naveenkumarsh.220@gmail.com>