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GENERALIZED SUBCLASSES OF QUASI-CONVEX FUNCTIONS DEFINED WITH SUBORDINATION

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Abstract. In this paper, certain generalized subclasses of quasi-convex functions in the open unit disc $E = \{z : |z| < 1\}$ are introduced. Various geometric properties such as the coefficient estimates, distortion theorems, growth theorems, radius of quasi convexity and relationship with other classes have been studied for these classes. The results so obtained generalize the results of several earlier works.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$ and further normalized specifically by $f(0) = f'(0) - 1 = 0$.
By S, we denote the subclass of A consisting of functions of the form (1) and which are univalent in E.

Let U be the class of Schwarzian functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k,$$

which are regular in the unit disc E and satisfying the conditions

$$w(0) = 0, |w(z)| < 1.$$

For the functions f and g analytic in E, we say that f is subordinate to g (symbolically $f \prec g$) if a Schwarzian function $w(z) \in U$ can be found for which $f(z) = g(w(z))$. This result is known as principle of subordination.

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The well known classes S^* and K, the classes of starlike and convex functions respectively are defined as

$$S^* = \left\{ f : f \in A, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$$

and

$$K = \left\{ f : f \in A, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

Further, Janowski [7] defined with the help of subordination, the following subclasses of starlike and convex functions respectively as

$$S^*(A,B) = \left\{ f : f \in A, \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in E \right\}$$

and

$$K(A,B) = \left\{ f : f \in A, \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in E \right\}.$$

The classes $S^*(A,B)$ and $K(A,B)$ were studied further by Goel and Mehrok [5].
In particular, $S^*(1,-1) \equiv S^*$ and $K(1,-1) \equiv K$.
Subsequently, Noor [10] introduced the class of quasi-convex functions as

$$C^* = \left\{ f : f \in A, Re\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, h \in K, z \in E \right\}.$$

Note that every quasi-convex function is convex and so univalent. Various subclasses of quasi-convex functions were studied by several authors from time to time. Some recently studied classes relevant to the present work are mentioned below.
By C_s^*, we denote the subclass of quasi-convex functions defined as

$$C_s^* = \left\{ f : f \in A, Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0, g \in S^*, z \in E \right\}.$$

Selvaraj and Stelin [12] studied the class $C^*(\alpha,\beta)$, a subclass of quasi-convex functions defined as below:

$$C^*(\alpha,\beta) = \left\{ f : f \in A, \frac{(zf'(z))'}{h'(z)} \prec \frac{1+(2\alpha-1)\beta z}{1+\beta z}, h \in K, 0 \leq \alpha < 1, 0 < \beta \leq 1, z \in E \right\}.$$

In particular, $C^*(0,-1) \equiv C^*$.
Further Selvaraj et al. [13] introduced and studied the class $C_s^*(\alpha,\beta)$, a subclass of quasi-convex functions defined as

$$C_s^*(\alpha,\beta) = \left\{ f : f \in A, \frac{(zf'(z))'}{g'(z)} \prec \frac{1+(2\alpha-1)\beta z}{1+\beta z}, g \in S^*, 0 \leq \alpha < 1, 0 < \beta \leq 1, z \in E \right\}.$$

Particularly, $C_s^*(0,-1) \equiv C_s^*$.
Xiong and Liu [15] established the class $C^*(A,B)$ given below:

$$C^*(A,B) = \left\{ f : f \in A, \frac{(zf'(z))'}{h'(z)} \prec \frac{1+Az}{1+Bz}, h \in K, -1 \leq B < A \leq 1, z \in E \right\}.$$

It is obvious that $C^*((2\alpha-1)\beta,\beta) \equiv C^*(\alpha,\beta)$ and $C^*(1,-1) \equiv C^*$.
By $C_s^*(A,B)$, we denote a subclass of quasi-convex functions defined as

$$C_s^*(A,B) = \left\{ f : f \in A, \frac{(zf'(z))'}{g'(z)} \prec \frac{1+Az}{1+Bz}, g \in S^*, -1 \leq B < A \leq 1, z \in E \right\}.$$

Obviously $C_s^*((2\alpha-1)\beta,\beta)\equiv C_s^*(\alpha,\beta)$ and $C_s^*(1,-1)\equiv C_s^*$.

Apart from the classes defined above, some more interesting subclasses of quasi convex functions have also been studied recently by Altintas and Kilic [2], Altintas and Aydogan [3] and Mahmood et al. [8, 9].
To avoid repetition, it is laid down once for all that

$$-1\le D\le B<A\le C\le 1, z\in E.$$

Motivated by the above work, we introduce the following generalized subclasses of quasi-convex functions:
Definition 1 $C^*(A,B;C,D)$ be the class of functions $f\in A$ of the form (1) which satisfy the condition

$$\frac{(zf'(z))'}{h'(z)}\prec\frac{1+Cz}{1+Dz},$$

where $h(z)=z+\sum_{k=2}^{\infty}b_kz^k\in K(A,B)$.
The following observations are obvious:
(i) $C^*(1,-1;C,D)\equiv C^*(C,D)$.
(ii) $C^*(1,-1;(2\alpha-1)\beta,\beta)\equiv C^*(\alpha,\beta)$.
(iii) $C^*(1,-1;1,-1)\equiv C^*$.
Definition 2 Let $C_s^*(A,B;C,D)$ denote the class of functions $f\in A$ of the form (1) and satisfying the condition that

$$\frac{(zf'(z))'}{g'(z)}\prec\frac{1+Cz}{1+Dz},$$

where $g(z)=z+\sum_{k=2}^{\infty}d_kz^k\in S^*(A,B)$.
We have the following observations:
(i) $C_s^*(1,-1;C,D)\equiv C_s^*(C,D)$.
(ii) $C_s^*(1,-1;(2\alpha-1)\beta,\beta)\equiv C_s^*(\alpha,\beta)$.
(iii) $C_s^*(1,-1;1,-1)\equiv C_s^*$.

The paper is concerned with the study of the classes $C^*(A,B;C,D)$ and $C_s^*(A,B;C,D)$. We obtain the coefficient estimates, distortion theorems, growth theorems, radius of quasi convexity and relationship with other classes for the functions in these classes. By giving the particular values to the parameters A, B, C and D, the results proved by various authors follows as special cases.

2. Preliminary Results

Lemma 1 [6] If $P(z)=\dfrac{1+Cw(z)}{1+Dw(z)}=1+\sum_{k=1}^{\infty}p_kz^k$, then

$$|p_n|\le(C-D), n\ge 1.$$

Lemma 2 [5] If $g(z)\in S^*(A,B)$, then for $A-(n-1)B\ge(n-2), n\ge 3$,

$$|d_n|\le\frac{1}{(n-1)!}\prod_{j=2}^{n}(A-(j-1)B).$$

Lemma 3 [5] If $g(z)\in S^*(A,B)$, then for $|z|=r<1$,

$$r(1-Br)^{\frac{A-B}{B}}\le|g(z)|\le r(1+Br)^{\frac{A-B}{B}}, B\ne 0;$$
$$re^{-Ar}\le|g(z)|\le re^{Ar}, B=0.$$

Lemma 4 [14] If $h(z) \in K(A,B)$, then for $A-(n-1)B \geq (n-2), n \geq 3$,

$$|b_n| \leq \frac{1}{n!}\prod_{j=2}^{n}(A-(j-1)B).$$

Lemma 5 [14] If $h(z) \in K(A,B)$, then for $|z| = r < 1$,

$$\frac{1}{A}\left[1-(1-Br)^{\frac{A}{B}}\right] \leq |h(z)| \leq \frac{1}{A}\left[(1+Br)^{\frac{A}{B}}-1\right], B \neq 0;$$

$$\frac{1}{A}\left[1-e^{-Ar}\right] \leq |h(z)| \leq \frac{1}{A}\left[e^{Ar}-1\right], B = 0.$$

Lemma 6 [4] If $P(z) = \dfrac{1+Cw(z)}{1+Dw(z)}, -1 \leq D < C \leq 1, w(z) \in U$, then for $|z| = r < 1$, we have

$$Re\frac{zP'(z)}{P(z)} \geq \begin{cases} -\dfrac{(C-D)r}{(1-Cr)(1-Dr)}, & \text{if } R_1 \leq R_2, \\ 2\dfrac{\sqrt{(1-D)(1-C)(1+Cr^2)(1+Dr^2)}-(1-CDr^2)}{(C-D)(1-r^2)} \\ +\dfrac{C+D}{C-D}, & \text{if } R_1 \geq R_2, \end{cases}$$

where $R_1 = \sqrt{\dfrac{(1-C)(1+Cr^2)}{(1-D)(1+Dr^2)}}$ and $R_2 = \dfrac{1-Cr}{1-Dr}$.

Lemma 7 [1, 11] Let N and D be analytic in E, D maps E onto a many sheeted starlike region, $N(0) = 0 = D(0)$. Then

$$\frac{N'(z)}{D'(z)} \prec \frac{1+Az}{1+Bz} \Rightarrow \frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}.$$

Lemma 8 Let $h(z) \in K(A,B)$ and define

$$G(z) = \int_0^z \frac{h(t)}{t}dt.$$

Then $G(z) \in K(A,B)$.

Proof As

$$G(z) = \int_0^z \frac{h(t)}{t}dt,$$

so we have

$$\frac{(zG'(z))'}{G'(z)} = \frac{zh'(z)}{h(z)}. \tag{2}$$

But $h(z) \in K(A,B)$, so

$$\frac{(zh'(z))'}{h'(z)} \prec \frac{1+Az}{1+Bz}. \tag{3}$$

Using (2), (3) and Lemma 7, it yields

$$\frac{(zG'(z))'}{G'(z)} = \frac{zh'(z)}{h(z)} \prec \frac{1+Az}{1+Bz},$$

which proves Lemma 8.

3. The class $C^*(A,B;C,D)$

Theorem 1 Let $f(z) \in C^*(A,B;C,D)$, then for $A-(n-1)B \geq (n-2), n \geq 2$,

$$|a_n| \leq \frac{1}{n(n!)} \prod_{j=2}^{n} (A-(j-1)B) + \frac{(C-D)}{n^2} \left[1 + \sum_{k=2}^{n-1} \frac{1}{(k-1)!} \prod_{j=2}^{k} (A-(j-1)B) \right]. \tag{4}$$

The bounds are sharp.
Proof. In Definition 1, using Principle of subordination, we have

$$(zf'(z))' = h'(z) \left(\frac{1+Cw(z)}{1+Dw(z)} \right), w(z) \in U. \tag{5}$$

On expanding (5), it yields
$1 + 4a_2 z + 9a_3 z^2 + ... + n^2 a_n z^{n-1} + ...$

$$= (1 + 2b_2 z + 3b_3 z^2 + ... + nb_n z^{n-1} + ...)(1 + p_1 z + p_2 z^2 + ... + p_{n-1} z^{n-1} + ...). \tag{6}$$

Equating the coefficients of z^{n-1} in (6), we have

$$n^2 a_n = nb_n + (n-1)p_1 b_{n-1} + (n-2)p_2 b_{n-2} ... + 2p_{n-2} b_2 + p_{n-1}. \tag{7}$$

Applying triangle inequality and Lemma 1 in (7), it gives

$$n^2 |a_n| \leq n|b_n| + (C-D) \left[(n-1)|b_{n-1}| + (n-2)|b_{n-2}| ... + 2|b_2| + 1 \right]. \tag{8}$$

Using Lemma 4 in (8), the result (4) is obvious.
For $n = 2$, equality sign in (4) hold for the functions $f_n(z)$ defined as

$$(zf_n'(z))' = (1 + B\delta_1 z)^{\frac{(A-B)}{B}} \left(\frac{1 + C\delta_2 z^n}{1 + D\delta_2 z^n} \right), |\delta_1| = 1, |\delta_2| = 1. \tag{9}$$

On putting $A = 1, B = -1$ in Theorem 1, we get the following result due to Xiong and Liu [15].
Corollary 1 Let $f(z) \in C^*(C,D)$, then

$$|a_n| \leq \frac{1}{n} + \frac{(n-1)(C-D)}{2n}.$$

For $A = 1, B = -1, C = (2\alpha - 1)\beta, D = \beta$, Theorem 1 agrees with the following result due to Selvaraj and Stelin [12].
Corollary 2 Let $f(z) \in C^*(\alpha, \beta)$, then

$$|a_n| \leq \frac{1}{n} \left[1 + \beta(1-\alpha)(n-1) \right].$$

For $A = 1, B = -1, C = 1, D = -1$, Theorem 1 coincides with the following result due to Noor [10].
Corollary 3 Let $f(z) \in C^*$, then

$$|a_n| \leq 1.$$

Theorem 2 If $f(z) \in C^*(A,B;C,D)$, then for $|z| = r, 0 < r < 1$, we have
for $D \neq -1, B \neq 0$,

$$\frac{1}{r} \int_0^r \left(\frac{1-Ct}{1-Dt} \right) (1-Bt)^{\frac{A-B}{B}} dt \leq |f'(z)| \leq \frac{1}{r} \int_0^r \left(\frac{1+Ct}{1+Dt} \right) (1+Bt)^{\frac{A-B}{B}} dt; \tag{10}$$

$$\int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1-Ct}{1-Dt} \right) (1-Bt)^{\frac{A-B}{B}} dt \right] ds \leq |f(z)| \leq \int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1+Ct}{1+Dt} \right) (1+Bt)^{\frac{A-B}{B}} dt \right] ds, \tag{11}$$

and for $D=-1, B\neq 0$,

$$\frac{1}{r}\int_0^r\left(\frac{1-Ct}{1+t}\right)(1-Bt)^{\frac{A-B}{B}}dt\leq|f'(z)|\leq\frac{1}{r}\int_0^r\left(\frac{1+Ct}{1-t}\right)(1+Bt)^{\frac{A-B}{B}}dt; \quad (12)$$

$$\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1-Ct}{1+t}\right)(1-Bt)^{\frac{A-B}{B}}dt\right]ds\leq|f(z)|\leq\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1+Ct}{1-t}\right)(1+Bt)^{\frac{A-B}{B}}dt\right]ds. \quad (13)$$

Estimates are sharp.

Proof. From (5), we have

$$|(zf'(z))'|=|h'(z)|\left|\frac{1+Cw(z)}{1+Dw(z)}\right|, w(z)\in U. \quad (14)$$

It is easy to show that the transformation

$$\frac{(zf'(z))'}{h'(z)}=\frac{1+Cw(z)}{1+Dw(z)}$$

maps $|w(z)|\leq r$ onto the circle

$$\left|\frac{(zf'(z))'}{h'(z)}-\frac{1-CDr^2}{1-D^2r^2}\right|\leq\frac{(C-D)r}{(1-D^2r^2)}, |z|=r.$$

This implies that

$$\frac{1-Cr}{1-Dr}\leq\left|\frac{1+Cw(z)}{1+Dw(z)}\right|\leq\frac{1+Cr}{1+Dr}. \quad (15)$$

Let $F(z)=zf'(z)$.

As $h(z)\in K(A,B)$, so from Lemma 5, we have

$$(1-Br)^{\frac{A-B}{B}}\leq|h'(z)|\leq(1+Br)^{\frac{A-B}{B}}, B\neq 0. \quad (16)$$

Using (15) and (16) in (14), it yields

$$\left(\frac{1-Cr}{1-Dr}\right)(1-Br)^{\frac{A-B}{B}}\leq|F'(z)|\leq\left(\frac{1+Cr}{1+Dr}\right)(1+Br)^{\frac{A-B}{B}}, B\neq 0. \quad (17)$$

On integrating (17) from 0 to r, the results (10) and (12) are obvious.

Again integrating (10) and (12) from 0 to r, the results (11) and (13) can be easily obtained.

Sharpness follows if we take $f_n(z)$ defined in (9).

On putting $A=1, B=-1$ in Theorem 2, it gives the following result due to Xiong and Liu [15].

Corollary 4 Let $f(z)\in C^*(C,D)$, then

for $D\neq -1$,

$$\frac{C-D}{r(1+D)^2}log\frac{1-Dr}{1+r}+\frac{1+C}{(1+D)(1+r)}\leq|f'(z)|\leq\frac{C-D}{r(1+D)^2}log\frac{1-r}{1+Dr}+\frac{1+C}{(1+D)(1-r)};$$

$$\int_0^r\left[\frac{C-D}{t(1+D)^2}log\frac{1-Dt}{1+t}+\frac{1+C}{(1+D)(1+t)}\right]dt\leq|f(z)|$$
$$\leq\int_0^r\left[\frac{C-D}{t(1+D)^2}log\frac{1-t}{1+Dt}+\frac{1+C}{(1+D)(1-t)}\right]dt$$

and for $D=-1$,

$$-\frac{1+C}{2r(1+r)^2}+\frac{C}{r(1+r)}+\frac{1}{2r}(1-C)\leq|f'(z)|\leq\frac{1+C}{2r(1-r)^2}-\frac{C}{r(1-r)}+\frac{1}{2r}(C-1);$$

$$\int_0^r \left[-\frac{1+C}{2t(1+t)^2}+\frac{C}{t(1+t)}+\frac{1}{2t}(1-C)\right]dt \leq |f(z)|$$
$$\leq \int_0^r \left[\frac{1+C}{2t(1-t)^2}-\frac{C}{t(1-t)}+\frac{1}{2t}(C-1)\right]dt.$$

For $A=1, B=-1, C=(2\alpha-1)\beta, D=\beta$, Theorem 2 agrees with the following result due to Selvaraj and Stelin [12].

Corollary 5 Let $f(z)\in C^*(\alpha,\beta)$, then
for $\beta\neq 1$,

$$L_1 \leq |f'(z)| \leq L_2$$

and

$$L_3 \leq |f(z)| \leq L_4,$$

where

$$L_1 = \frac{-2\beta(1-\alpha)}{(1-\beta)^2 r} log\left(\frac{1+r}{1+\beta r}\right)+\frac{1+(1-2\alpha)\beta}{(1-\beta)(1+r)},$$
$$L_2 = \frac{2\beta(1-\alpha)}{(1-\beta)^2 r} log\left(\frac{1-r}{1-\beta r}\right)+\frac{1+(1-2\alpha)\beta}{(1-\beta)(1-r)},$$
$$L_3 = \frac{-2\beta(1-\alpha)}{(1-\beta)^2}\int_0^r \frac{1}{t} log\left(\frac{1+t}{1+\beta t}\right)dt+\frac{1+(1-2\alpha)\beta}{(1-\beta)} log(1+r),$$
$$L_4 = \frac{2\beta(1-\alpha)}{(1-\beta)^2}\int_0^r \frac{1}{t} log\left(\frac{1-t}{1-\beta t}\right)dt-\frac{1+(1-2\alpha)\beta}{(1-\beta)} log(1-r),$$

and for $\beta=1$,

$$\frac{1+\alpha r}{(1+r)^2} \leq |f'(z)| \leq \frac{1-\alpha r}{(1-r)^2};$$
$$(1-\alpha)\frac{r}{1+r}+\alpha log(1+r) \leq |f(z)| \leq (1-\alpha)\frac{r}{1-r}-\alpha log(1-r).$$

For $A=1, B=-1, C=1, D=-1$, Theorem 2 coincides with the following result due to Noor [10].

Corollary 6 Let $f(z)\in C^*$, then

$$\frac{1}{(1+r)^2} \leq |f'(z)| \leq \frac{1}{(1-r)^2}$$

and

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}.$$

Theorem 3 Let $F(z)=zf'(z)$, where $f(z)\in C^*(A,B;C,D)$, then

$$Re\frac{(zF'(z))'}{F'(z)} \geq \begin{cases} \frac{1-Ar}{1-Br}-\frac{(C-D)r}{(1-Cr)(1-Dr)}, & \text{if } R_1\leq R_2, \\ \frac{1-Ar}{1-Br}+2\frac{\sqrt{(1-D)(1-C)(1+Cr^2)(1+Dr^2)}-(1-CDr^2)}{(C-D)(1-r^2)} & \\ +\frac{C+D}{C-D}, & \text{if } R_1\geq R_2, \end{cases} \tag{18}$$

where R_1 and R_2 are defined in Lemma 6.

Proof. As $f(z)\in C^*(A,B;C,D)$, we have

$$\frac{(zf'(z))'}{h'(z)}=\frac{1+Cw(z)}{1+Dw(z)}=P(z).$$

Here $F(z) = zf'(z)$. So on differentiating it logarithmically, we get

$$\frac{(zF'(z))'}{F'(z)} = \frac{(zh'(z))'}{h'(z)} + \frac{zP'(z)}{P(z)}. \tag{19}$$

Now for $h \in K(A, B)$, we have

$$Re\left(\frac{(zh'(z))'}{h'(z)}\right) \geq \frac{1 - Ar}{1 - Br}. \tag{20}$$

So using Lemma 6 and inequality (20) in equation (19), the result (18) is obvious. Sharpness follows if we take $f_n(z)$ to be same as in (9).
On putting $A = 1, B = -1$ in Theorem 3, it gives the following result due to Xiong and Liu [15].
Corollary 7 Let $F(z) = zf'(z)$, where $f(z) \in C^*(C, D)$, then

$$Re\frac{(zF'(z))'}{F'(z)} \geq \begin{cases} \dfrac{1-r}{1+r} - \dfrac{(C-D)r}{(1-Cr)(1-Dr)}, & \text{if } R_1 \leq R_2, \\ \dfrac{1-r}{1+r} + 2\dfrac{\sqrt{(1-D)(1-C)(1+Cr^2)(1+Dr^2)} - (1-CDr^2)}{(C-D)(1-r^2)} \\ +\dfrac{C+D}{C-D}, & \text{if } R_1 \geq R_2, \end{cases}$$

where R_1 and R_2 are defined in Lemma 6.
For $A = 1, B = -1, C = (2\alpha - 1)\beta, D = \beta$, Theorem 3 gives the following result due to Selvaraj and Stelin [12].
Corollary 8 Let $F(z) = zf'(z)$, where $f(z) \in C^*(\alpha, \beta)$, then

$$Re\frac{(zF'(z))'}{F'(z)} \geq \begin{cases} \dfrac{1-r}{1+r} - \dfrac{2(1-\alpha)\beta r}{(1+\beta r)[1+(2\alpha-1)\beta r]}, & \text{if } 0 < r \leq r^*, \\ \dfrac{1-r}{1+r} + \gamma - \dfrac{\alpha}{1-\alpha}, & \text{if } r^* < r < 1, \end{cases}$$

where

$$\gamma = \frac{\sqrt{(1+\beta)[1+(2\alpha-1)\beta](1-\beta r^2)[1-(2\alpha-1)\beta r^2]} - [1+(1-2\alpha)\beta^2 r^2]}{(1-\alpha)\beta(1-r^2)}$$

and r^* is the unique root of the equation

$$(2\alpha - 1)\beta^2 r^4 - 2(2\alpha - 1)\beta^2 r^3 - [1 + 4\alpha\beta + (2\alpha - 1)\beta^2]r^2 - 2r + 1 = 0$$

in the interval $(0, 1]$.
Theorem 4 If $f(z) \in C^*(A, B; C, D)$ with respect to the function $h(z) \in K(A, B)$ and let

$$F(z) = \int_0^z \frac{f(t)}{t} dt, G(z) = \int_0^z \frac{h(t)}{t} dt.$$

Then $F(z) \in C^*(A, B; C, D)$ with respect to the function $G(z)$.
Proof. Since $f(z) \in C^*(A, B; C, D)$ with respect to the function $h(z) \in K(A, B)$, so

$$\frac{(zf'(z))'}{h'(z)} \prec \frac{1 + Cz}{1 + Dz}. \tag{21}$$

From Lemma 8, it is clear that $G(z) \in K(A, B)$. Again, we have

$$\frac{(zF'(z))'}{G'(z)} = \frac{zf'(z)}{h(z)}. \tag{22}$$

Following (21), (22) and Lemma 8, we have

$$\frac{(zF'(z))'}{G'(z)} \prec \frac{1+Cz}{1+Dz},$$

which proves the theorem.

4. The class $C_s^*(A,B;C,D)$

Theorem 5 Let $f(z) \in C_s^*(A,B;C,D)$, then for $A-(n-1)B \geq (n-2), n \geq 2$,

$$|a_n| \leq \frac{1}{n!}\prod_{j=2}^{n}(A-(j-1)B)+\frac{(C-D)}{n^2}\left[1+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\prod_{j=2}^{k}(A-(j-1)B)\right]. \quad (23)$$

The results are sharp.

Proof. From Definition 2, using Principle of subordination, we have

$$(zf'(z))' = g'(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right), w(z) \in U. \quad (24)$$

On expanding (24), it yields

$$1+4a_2z+9a_3z^2+...+n^2a_nz^{n-1}+...$$

$$=(1+2d_2z+3d_3z^2+...+nd_nz^{n-1}+...)(1+p_1z+p_2z^2+...+p_{n-1}z^{n-1}+...) \quad (25)$$

Equating the coefficients of z^{n-1} in (25), we have

$$n^2a_n = nd_n+(n-1)p_1d_{n-1}+(n-2)p_2d_{n-2}...+2p_{n-2}d_2+p_{n-1}. \quad (26)$$

Applying triangle inequality and Lemma 1 in (26), it gives

$$n^2|a_n| \leq n|d_n|+(C-D)\left[(n-1)|d_{n-1}|+(n-2)|d_{n-2}|...+2|d_2|+1\right]. \quad (27)$$

Using Lemma 2 in (27), the result (23) is obvious.

For $n=2$, equality sign in (23) hold for the functions $f_n(z)$ defined by

$$(zf_n'(z))' = (1+B\delta_1 z)^{\frac{(A-B)}{B}}\left(\frac{1+A\delta_1 z^n}{1+B\delta_1 z^n}\right)\left(\frac{1+C\delta_2 z^n}{1+D\delta_2 z^n}\right), |\delta_1|=1, |\delta_2|=1. \quad (28)$$

On putting $A=1, B=-1$ in Theorem 5, we get the following result:

Corollary 9 Let $f(z) \in C_s^*(C,D)$, then

$$|a_n| \leq 1+\frac{(C-D)(n-1)(2n-1)}{6n}.$$

For $A=1, B=-1, C=(2\alpha-1)\beta, D=\beta$, Theorem 5 gives the following result due to Selvaraj et al. [13].

Corollary 10 Let $f(z) \in C_s^*(\alpha,\beta)$, then

$$|a_n| \leq [1-2(1-\alpha)\beta]+\frac{[(1-\alpha)\beta(n+1)(2n+1)]}{3n}.$$

For $A=1, B=-1, C=1, D=-1$, Theorem 5 gives the following result:

Corollary 11 Let $f(z) \in C_s^*$, then

$$|a_n| \leq \frac{2n^2+1}{3n}.$$

Theorem 6 If $f(z) \in C_s^*(A,B;C,D)$, then for $|z|=r, 0<r<1$, we have for $D \neq -1, B \neq 0$,

$$\frac{1}{r}\int_0^r \left(\frac{1-Ct}{1-Dt}\right)\left(\frac{1-At}{1-Bt}\right)(1-Bt)^{\frac{A-B}{B}}dt \leq |f'(z)|$$

$$\leq \frac{1}{r}\int_0^r \left(\frac{1+Ct}{1+Dt}\right)\left(\frac{1+At}{1+Bt}\right)(1+Bt)^{\frac{A-B}{B}}dt; \tag{29}$$

$$\int_0^r \left[\frac{1}{s}\int_0^s \left(\frac{1-Ct}{1-Dt}\right)\left(\frac{1-At}{1-Bt}\right)(1-Bt)^{\frac{A-B}{B}}dt\right]ds \leq |f(z)|$$
$$\leq \int_0^r \left[\frac{1}{s}\int_0^s \left(\frac{1+Ct}{1+Dt}\right)\left(\frac{1+At}{1+Bt}\right)(1+Bt)^{\frac{A-B}{B}}dt\right]ds, \tag{30}$$

and for $D=-1, B\neq 0$,

$$\frac{1}{r}\int_0^r \left(\frac{1-Ct}{1+t}\right)\left(\frac{1-At}{1-Bt}\right)(1-Bt)^{\frac{A-B}{B}}dt \leq |f'(z)|$$
$$\leq \frac{1}{r}\int_0^r \left(\frac{1+Ct}{1-t}\right)\left(\frac{1+At}{1+Bt}\right)(1+Bt)^{\frac{A-B}{B}}dt; \tag{31}$$

$$\int_0^r \left[\frac{1}{s}\int_0^s \left(\frac{1-Ct}{1+t}\right)\left(\frac{1-At}{1-Bt}\right)(1-Bt)^{\frac{A-B}{B}}dt\right]ds \leq |f(z)|$$
$$\leq \int_0^r \left[\frac{1}{s}\int_0^s \left(\frac{1+Ct}{1-t}\right)\left(\frac{1+At}{1+Bt}\right)(1+Bt)^{\frac{A-B}{B}}dt\right]ds. \tag{32}$$

Estimates are sharp.

Proof. From (24), we have

$$|(zf'(z))'| = |g'(z)|\left|\frac{1+Cw(z)}{1+Dw(z)}\right|, w(z)\in U. \tag{33}$$

As in Theorem 2, we have

$$\frac{1-Cr}{1-Dr} \leq \left|\frac{1+Cw(z)}{1+Dw(z)}\right| \leq \frac{1+Cr}{1+Dr}. \tag{34}$$

Let $F(z) = zf'(z)$.
As $g(z)\in S^*(A,B)$, so from Lemma 3, we have

$$\left(\frac{1-Ar}{1-Br}\right)(1-Br)^{\frac{A-B}{B}} \leq |g'(z)| \leq \left(\frac{1+Ar}{1+Br}\right)(1+Br)^{\frac{A-B}{B}}, B\neq 0. \tag{35}$$

Therefore from (34) and (35), it yields

$$\left(\frac{1-Cr}{1-Dr}\right)\left(\frac{1-Ar}{1-Br}\right)(1-Br)^{\frac{A-B}{B}} \leq |F'(z)| \leq \left(\frac{1+Cr}{1+Dr}\right)\left(\frac{1+Ar}{1+Br}\right)(1+Br)^{\frac{A-B}{B}}, B\neq 0. \tag{36}$$

On integrating (36) from 0 to r, the result (29) and (31) are obvious.
Again integrating (29) and (31) from 0 to r, the results (30) and (32) can be easily obtained.
Sharpness follows if we take $f_n(z)$ defined in (28).
On putting $A=1, B=-1$ in Theorem 6, it gives the following result:

Corollary 12 Let $f(z)\in C_s^*(C,D)$, then
for $D\neq -1$,
$L_1 \leq |f'(z)| \leq L_2$ and $L_3 \leq |f(z)| \leq L_4$, where

$$L_1 = \frac{(D-1)}{r(D+1)^3}log\left|\frac{1+r}{1-Dr}\right| + \left[\frac{(D-1)}{D(D+1)^2} - \frac{C}{D}\right]\frac{1}{1+r} + \frac{1}{2}\left[1+\frac{C}{D} - \frac{(D-1)}{D(D+1)}\right]\frac{(2+r)}{(1+r)^2},$$

$$L_2 = \frac{(D-1)}{r(D+1)^3}log\left|\frac{1+Dr}{1-r}\right| + \left[\frac{(D-1)}{D(D+1)^2} - \frac{C}{D}\right]\frac{1}{1-r} + \frac{1}{2}\left[1+\frac{C}{D} - \frac{(D-1)}{D(D+1)}\right]\frac{(2-r)}{(1-r)^2},$$

$$L_3 = \int_0^r \left[\frac{(D-1)}{t(D+1)^3} log\left|\frac{1+t}{1-Dt}\right| + \left[\frac{(D-1)}{D(D+1)^2} - \frac{C}{D}\right]\frac{1}{1+t} + \frac{1}{2}\left[1+\frac{C}{D} - \frac{(D-1)}{D(D+1)}\right]\frac{(2+t)}{(1+t)^2}\right] dt,$$

$$L_4 = \int_0^r \left[\frac{(D-1)}{t(D+1)^3} log\left|\frac{1+Dt}{1-t}\right| + \left[\frac{(D-1)}{D(D+1)^2} - \frac{C}{D}\right]\frac{1}{1-t} + \frac{1}{2}\left[1+\frac{C}{D} - \frac{(D-1)}{D(D+1)}\right]\frac{(2-t)}{(1-t)^2}\right] dt,$$

and for $D = -1$,

$$\frac{C}{1+r} - \frac{(3C+1)(2+r)}{2(1+r)^2} + \frac{2(C+1)(3+3r+r^2)}{3(1+r)^3}$$

$$\leq |f'(z)| \leq \frac{C}{1-r} - \frac{(3C+1)(2-r)}{2(1-r)^2} + \frac{2(C+1)(3-3r+r^2)}{3(1-r)^3},$$

$$\int_0^r \left[\frac{C}{1+t} - \frac{(3C+1)(2+t)}{2(1+t)^2} + \frac{2(C+1)(3+3t+t^2)}{3(1+t)^3}\right] dt \leq |f(z)|$$

$$\leq \int_0^r \left[\frac{C}{1-t} - \frac{(3C+1)(2-t)}{2(1-t)^2} + \frac{2(C+1)(3-3t+t^2)}{3(1-t)^3}\right] dt.$$

For $A = 1, B = -1, C = (2\alpha - 1)\beta, D = \beta$, Theorem 6 gives the following result due to Selvaraj et al. [13].

Corollary 13 Let $f(z) \in C_s^*(\alpha, \beta)$, then for $\beta \neq 1$,

$$L_1 \leq |f'(z)| \leq L_2$$

and

$$L_3 \leq |f(z)| \leq L_4,$$

where

$$L_1 = \frac{2\beta(1-\alpha)(1+\beta)}{(1-\beta)^3 r} log\left[\frac{1+r}{1+\beta r}\right] + \frac{(1-2\alpha)\beta^2 + 2\beta(3\alpha-2) - 1}{(1-\beta)^2}\frac{1}{1+r} + \frac{[1+(1-2\alpha)\beta]}{(1-\beta)}\frac{r+2}{(1+r)^2},$$

$$L_2 = \frac{-2\beta(1-\alpha)(1+\beta)}{(1-\beta)^3 r} log\left[\frac{1-r}{1-\beta r}\right] + \frac{(1-2\alpha)\beta^2 + 2\beta(3\alpha-2) - 1}{(1-\beta)^2}\frac{1}{1-r} - 2\frac{[1+(1-2\alpha)\beta]}{(1-\beta)}\frac{r-2}{(1-r)^2},$$

$$L_3 = \frac{2\beta(1-\alpha)(1+\beta)}{(1-\beta)^3}\int_0^r \frac{1}{t} log\left[\frac{1+t}{1+\beta t}\right] dt + \frac{[1+(1-2\alpha)\beta]}{(1-\beta)}\frac{r}{1+r} - \frac{4(1-\alpha)\beta}{(1-\beta)^2} log(1+r),$$

$$L_4 = \frac{-2\beta(1-\alpha)(1+\beta)}{(1-\beta)^3}\int_0^r \frac{1}{t} log\left[\frac{1+t}{1+\beta t}\right] dt + \frac{[1+(1-2\alpha)\beta]}{(1-\beta)}\frac{r}{1-r} + \frac{4(1-\alpha)\beta}{(1-\beta)^2} log(1-r),$$

and for $\beta = 1$,

$$M_1 \leq |f'(z)| \leq M_2,$$

where

$$M_1 = \frac{4(1-\alpha)(r^2+3r+3)}{3(1+r)^3} - (2-3\alpha)\frac{r+2}{(1+r)^2} + (1-2\alpha)\frac{1}{1+r},$$

$$M_2 = \frac{4(1-\alpha)(r^2-3r+3)}{3(1-r)^3} - (2-3\alpha)\frac{r-2}{(1-r)^2} + (1-2\alpha)\frac{1}{1-r},$$

and

$$\frac{(1-\alpha)}{3} log(1+r) + \frac{(5\alpha-2)r}{3(1+r)} + \frac{2(1-\alpha)r(r+2)}{3(1+r)^2}$$

$$\leq |f(z)| \leq -\frac{(1-\alpha)}{3} log(1-r) + \frac{(5\alpha-2)r}{3(1-r)} - \frac{2(1-\alpha)r(r-2)}{3(1-r)^2}.$$

For $A = 1, B == -1, C = 1, D = -1$, Theorem 6 gives the following result:
Corollary 14 Let $f(z) \in C_s^*$, then

$$\frac{r^2+3}{3(1+r)^3} \le |f'(z)| \le \frac{r^2+3}{3(1-r)^3}$$

and

$$\frac{1}{3}\left[log|1+r| + \frac{2r}{(1+r)^2}\right] \le |f(z)| \le \frac{1}{3}\left[-log|1-r| + \frac{2r}{(1-r)^2}\right].$$

Theorem 7 Let $F(z) = zf'(z)$, where $f(z) \in C_s^*(A,B;C,D)$, then

$$Re\frac{(zF'(z))'}{F'(z)} \ge \begin{cases} \frac{1-Ar}{1-Br} - \frac{(A-B)r}{1-B^2r^2} - \frac{(C-D)r}{(1-Cr)(1-Dr)}, & \text{if } R_1 \le R_2, \\ \frac{1-Ar}{1-Br} - \frac{(A-B)r}{1-B^2r^2} \\ +2\frac{\sqrt{(1-D)(1-C)(1+Cr^2)(1+Dr^2)}-(1-CDr^2)}{(C-D)(1-r^2)} + \frac{C+D}{C-D}, & \text{if } R_1 \ge R_2, \end{cases} \tag{37}$$

where R_1 and R_2 are defined in Lemma 6.
Proof. As $f(z) \in C_s^*(A,B;C,D)$, we have

$$\frac{(zf'(z))'}{g'(z)} = \frac{1+Cw(z)}{1+Dw(z)} = P(z).$$

Here $F(z) = zf'(z)$. So on differentiating it logarithmically, we get

$$\frac{(zF'(z))'}{F'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zP'(z)}{P(z)}. \tag{38}$$

Now for $g \in S^*(A,B)$, we have

$$Re\left(\frac{(zg'(z))'}{g'(z)}\right) \ge \frac{1-Ar}{1-Br} - \frac{(A-B)}{1-B^2r^2}. \tag{39}$$

So using Lemma 6 and inequality (39) in equation (38), the result (37) is obvious. Sharpness follows if we take $f_n(z)$ to be same as in (28).
On putting $A = 1, B = -1$ in Theorem 7, it gives the following result:
Corollary 15 Let $F(z) = zf'(z)$, where $f(z) \in C_s^*(C,D)$, then

$$Re\frac{(zF'(z))'}{F'(z)} \ge \begin{cases} \frac{1-r}{1+r} - \frac{2r}{1-r^2} - \frac{(C-D)r}{(1-Cr)(1-Dr)}, & \text{if } R_1 \le R_2, \\ \frac{1-r}{1+r} - \frac{2r}{1-r^2} \\ +2\frac{\sqrt{(1-D)(1-C)(1+Cr^2)(1+Dr^2)}-(1-CDr^2)}{(C-D)(1-r^2)} + \frac{C+D}{C-D}, & \text{if } R_1 \ge R_2, \end{cases}$$

where R_1 and R_2 are defined in Lemma 6.
For $A = 1, B = -1, C = (2\alpha-1)\beta, D = \beta$, Theorem 7 gives the following result due to Selvaraj et al. [13].
Corollary 16 Let $F(z) = zf'(z)$, where $f(z) \in C^*(\alpha,\beta)$, then

$$Re\frac{(zF'(z))'}{F'(z)} \ge \begin{cases} \frac{1-r}{1+r} - \frac{2r}{1-r^2} - \frac{2(1-\alpha)\beta r}{(1+\beta r)[1+(2\alpha-1)\beta r]}, & \text{if } 0 \le r \le r^*, \\ \frac{1-r}{1+r} - \frac{2r}{1-r^2} + \gamma - \frac{\alpha}{1-\alpha}, & \text{if } r^* < r < 1, \end{cases}$$

where $\gamma = \dfrac{\sqrt{(1+\beta)[1+(2\alpha-1)\beta](1-\beta r^2)[1-(2\alpha-1)\beta r^2]}-[1+(1-2\alpha)\beta^2 r^2]}{(1-\alpha)\beta(1-r^2)}$
and and r^* is the unique root of the equation

$$(2\alpha-1)\beta^2 r^4 - 2(2\alpha-1)\beta^2 r^3 - [1+4\alpha\beta+(2\alpha-1)\beta^2]r^2 - 2r + 1 = 0$$

in the interval $(0, 1]$.

Theorem 8 If $f(z) \in C_s^*(A, B; C, D)$ with respect to the function $g(z) \in S^*(A, B)$ and let

$$F(z) = \int_0^z \frac{f(t)}{t}dt, G(z) = \int_0^z \frac{h(t)}{t}dt.$$

Then $F(z) \in C_s^*(A, B; C, D)$ with respect to the function $G(z)$.

Proof. Since $f(z) \in C_s^*(A, B; C, D)$ with respect to the function $g(z) \in S^*(A, B)$, so

$$\frac{(zf'(z))'}{g'(z)} \prec \frac{1+Cz}{1+Dz}. \tag{40}$$

From Lemma 8, it is easy to show that $G(z) \in S^*(A, B)$. Again, we have

$$\frac{(zF'(z))'}{G'(z)} = \frac{zf'(z)}{g(z)}. \tag{41}$$

Following (40), (41) and Lemma 8, we have

$$\frac{(zF'(z))'}{G'(z)} \prec \frac{1+Cz}{1+Dz},$$

which proves the theorem.

References

[1] R. Aghalary and S. R. Kulkarni, Some properties of the integral operators in univalent function, Studia Univ Babes-Bolyat Math. Vol. 46, 3-9, 2001.
[2] Osman Altintas and Ozur Ozkan Kilic, Coefficient estimates for a class containing quasi-convex functions, Turkish Journal of Mathematics, Vol. 42, 2819-2825, 2018.
[3] Osman Altintas and Melika Aydogan, Coefficient estimates for the class of quasi q-convex functions, Turkish Journal of Mathematics, Vol. 44, 342-347, 2020.
[4] V. V. Anh and P. D. Tuan, On β-convexity of certain starlike functions, Rev. Roum. Math. Pures et Appl. Vol. 25, 1413-1424, 1979.
[5] R. M. Goel and B. S. Mehrok, On a class of close-to-convex functions, Ind. J. Pure Appl. Math. Vol. 12, No. 5, 648-658, 1981.
[6] R. M. Goel and B. S. Mehrok, A subclass of univalent functions, Houston J. Math. Vol. 8, No. 3, 343-357, 1982.
[7] W. Janowski, Some extremal problems for certain families of analytic functions, Ann. Pol. Math. Vol. 28, 297-326, 1973.
[8] S. Mahmood, I. Khan, S.N. Malik and S.Z.H. Bukhari, On subclass of alpha-quasi convex functions associated with conic regions, Journal of inequalities and special functions, Vol. 8, No. 2, 53-64, 2017.
[9] S. Mahmood, Jansuz Sokol, H.M. Srivastava and Sarfraz Nawaz Malik, Some reciprocal classes of close-to-convex and quasi-convex analytic functions, Mathematics, Vol. 7, 1-13, 2019.
[10] K.I. Noor, On quasi-convex functions and related topics, Int. J. Math. and Math. Sci. Vol. 10, 241-258, 1987.
[11] R. Parvatham and T. N. Shanmugham, On analytic functions with reference to an integral operator, Bull. Austral. Math. Soc. Vol. 28, 207-215, 1983.
[12] C. Selvaraj and S. Stelin, On a generalized class of quasi-convex functions, Int. J. Pure and Appl. Math. Vol. 101, No. 6, 1051-1061, 2015.
[13] C. Selvaraj, S. Stelin and S. Logu, On a subclass of close-to-convex functions associated with fixed second coefficient, Applied and Computational Math. Vol. 4, No. 5, 342-345, 2015.

[14] Harjinder Singh and B. S. Mehrok, Subclasses of close-to-convex functions, Tamkang J. Math. Vol. 44, No. 4, 377-386, 2013.
[15] Liangpeng Xiong and Xiaoli Liu, A general subclass of close-to-covex functions, Kragujevac Journal of Mathematics, Vol. 36, No. 2, 251-260, 2012.

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