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EXISTENCE OF ITERATED PROXIMATE ORDER AND ITERATED PROXIMATE TYPE OF AN ENTIRE FUNCTION

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Abstract. In this paper we introduced iterated proximate order (iterated lower proximate order), iterated proximate type (iterated lower proximate type) of an entire function and proved the corresponding existence theorems.

1. Introduction

If $f(z)$ is an entire function of finite order ρ and $M_f(r) = \sup\limits_{|z|=r} |f(z)|$, it is proved (Valiron [4]) that there exists a positive continuous function $\rho(r)$ with the following properties:

(i) $\rho(r)$ is differentiable for sufficiently large values of r except at isolated points where $\rho'(r-0), \rho'(r+0)$ exist;

(ii) $\lim\limits_{r\to\infty} \rho(r) = \rho$;

(iii) $\lim\limits_{r\to\infty} \rho'(r) r \log r = 0$;

(iv) $\limsup\limits_{r\to\infty} \frac{\log M_f(r)}{r^{\rho}} = 1$.

Such a function is called a proximate order for the entire function $f(z)$. Shah [2] gave a simple proof of the existence of proximate order of an entire function. Lahiri [1] generalised the idea for a meromorphic function.

There are two other indicators of growth of an entire function $f(z)$,the type T and the lower type t. They are defined for all ρ, $0 < \rho < \infty$ as

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho}} &= T, \\ \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\rho}} &= t. \end{aligned}$$

Definition 1 [3] A function $T(r)$ is said to be a proximate type of an entire function $f(z)$ of order $\rho(0 < \rho < \infty)$ and finite type T if it satisfies the following properties:

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(i) $T(r)$ is real valued, continuous and piecewise differentiable for sufficiently large values of r;

(ii) $\lim\limits_{r\to\infty} T(r) = T$;

(iii) $\lim\limits_{r\to\infty} rT'(r) = 0$, where $T'(r)$ is either the right or the left hand derivative at points where they are different;

(iv) $\limsup\limits_{r\to\infty} \frac{M_f(r)}{\exp\{r^{\rho}T(r)\}} = 1$.

Srivastava and Juneja [3] gave the proof of the existence of proximate type of an entire function.

Definition 2 [5] Tu, Chen, Zheng introduced the definition of iterated p order ρ_p of an entire function f for $p \in \mathbb{N}$ as

$$\rho_p = \limsup_{r\to\infty} \frac{\log_{p+1} M_f(r)}{\log r}. \tag{1}$$

Similarly, the iterated p lower order λ_p of an entire function f for $p \in \mathbb{N}$ as

$$\lambda_p = \liminf_{r\to\infty} \frac{\log_{p+1} M_f(r)}{\log r}. \tag{2}$$

Definition 3 [5] The finiteness degree of the order of an entire function f is defined by

$$i(f) = \begin{cases} 0 & \text{for } f \text{ polynomial,} \\ \min\{p \in \mathbb{N} : \rho_p < \infty\} & \text{for } f \text{ transcendental for which some } p \in \mathbb{N} \text{ with } \rho_p < \infty \text{ exists.} \\ \infty & \text{for } f \text{ with } \rho_p = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Then it is clear that $i(f)$ and $i(g)$ are positive integers.

Definition 4 Also one can define the iterated p type T_p of an entire function f as

$$T_p = \limsup_{r\to\infty} \frac{\log_p M_f(r)}{r^{\rho_p}}.$$

Similarly the iterated p lower type t_p of an entire dunction f as

$$t_p = \liminf_{r\to\infty} \frac{\log_p M_f(r)}{r^{\rho_p}}.$$

In this paper we want to prove the existence of iterated proximate p order and the existence of iterated proximate p type of an entire function.

2. Main Results

In this section we first introduce the definitions of iterated proximate p order and iterated proximate p type of an entire function. Then we prove their existence.

Definition If $f(z)$ is an entire function of iterated p order ρ_p. A function $\rho_p(r)$ is said to be finite iterated proximate p order of $f(z)$ if the following properties hold:

(i) $\rho_p(r)$ is differentiable for sufficiently large values of r except at isolated points where $\rho_p'(r-0), \rho_p'(r+0)$ exist;

(ii) $\lim\limits_{r\to\infty} \rho_p(r) = \rho_p$;

(iii) $\lim\limits_{r\to\infty} \rho_p'(r) \prod_{i=0}^{p} \log_i(r) = 0$;

(iv) $\limsup\limits_{r\to\infty} \frac{\log_{p-1} M_f(r)}{r^{\rho}} = 1$.

Similarly finite iterated proximate p lower order of $f(z)$ can be defined.

Definition If $f(z)$ is an entire function of iterated p order ρ_p.Then the function $T_p(r)$ is said to be iterated proximate p type of $f(z)$ of order $\rho(0<\rho<\infty)$ and finite type T_p if it satisfies the following properties:

(i) $T_p(r)$ is differentiable for sufficiently large values of r except at isolated points where $T_p'(r-0), T_p'(r+0)$ exist;

(ii) $\lim\limits_{r\to\infty} T_p(r) = T_p$;

(iii) $\lim\limits_{r\to\infty} T_p'(r)\prod_{i=0}^{p-1}\log_i r = 0$;

(iv) $\limsup\limits_{r\to\infty} \frac{M_f(r)}{\exp_p\{r^{\rho_p}T_p(r)\}} = 1$.

Similarly finite iterated proximate p lower type of $f(z)$ can be defined.

Theorem 1 For every entire function $f(z)$ of finite iterated order ρ_p, with $i(f)=p$ $(0<\rho_p<\infty)$, there exists a proximate iterated order $\rho_p(r)$.

Proof. Let

$$\sigma_p(r) = \frac{\log_p M_f(r)}{\log r}$$

then

$$\limsup_{r\to\infty} \sigma_p(r) = \rho_p.$$

We consider two cases:

Case I: Let $\sigma_p(r) > \rho_p$ for atleast a sequence of values of r tending to infinity.

We define

$$\phi_p(r) = \max_{x\geq r}\{\sigma_p(x)\}.$$

Note that $\phi_p(r)$ exists and is nonincreasing.

Let $R_1 > \exp_{p+1}(1)$ and $\sigma_p(R) > \rho_p$.

Then for $r \geq R_1 > R$, we get

$$\sigma_p(r) \leq \sigma_p(R).$$

Since $\sigma_p(r)$ is continuous, there exists $r_1 \in [R, R_1]$ such that

$$\sigma_p(r_1) = \max_{R\leq x\leq R_1}\{\sigma_p(x)\}.$$

Obviously $r_1 > \exp_{p+1}(1)$ and $\phi_p(r_1) = \sigma_p(r_1)$.

Note that $r = r_1$ exists for a sequence of values of r tending to infinity.

Let $\rho_p(r_1) = \phi_p(r_1)$ and t_1 be the smallest integer not less than $1 + r_1$ such that $\phi_p(r_1) > \phi_p(t_1)$.

We define $\rho_p(r) = \rho_p(r_1)$ for $r_1 < r \leq t_1$.

Clearly $\phi_p(r)$ and $\rho_p(r_1) - \log_{p+1} r + \log_{p+1} t_1$ are continuous functions of r and

$$\lim_{r\to\infty} \rho_p(r_1) - \log_{p+1} r + \log_{p+1} t_1 = -\infty.$$

Further $\lim\limits_{r\to\infty} \rho_p(r_1) - \log_{p+1} r + \log_{p+1} t_1 > \phi_p(t_1)$ for $r\,(>t_1)$ sufficiently close to t_1 and $\phi_p(r)$ is nonincreasing.

We can define u_1 as follows

$$\begin{aligned} u_1 &> t_1 \\ \rho_p(r) &= \rho_p(r_1) - \log_{p+1} r + \log_{p+1} t_1, \text{ for } t_1 \leq r \leq u_1 \\ \rho_p(r) &= \phi_p(r), \text{ for } r = u_1 \\ \rho_p(r) &> \phi_p(r), \text{ for } t_1 \leq r < u_1. \end{aligned}$$

Let r_2 be the smallest value of r for which $r \geq u_1$ and $\phi_p(r_2) = \sigma_p(r_2)$. If $r_2 > u_1$ then let $\rho_p(r) = \phi_p(r)$ for $u_1 \leq r \leq r_2$.

One can be easily checked that $\phi_p(r)$ is constant in $u_1 \leq r \leq r_2$. Thus $\rho_p(r)$ is constant in $u_1 \leq r \leq r_2$.

Repeating this process infinitely and we obtain that $\rho_p(r)$ is differentiable in adjacent intervals.

Further, $\rho_p'(r) = 0$ or $\frac{-1}{\prod_{i=0}^{p+1} \log_i(r)}$ and $\rho_p(r) \geq \phi_p(r) \geq \sigma_p(r)$ for all $r \geq r_1$.

Also $\rho_p(r) = \sigma_p(r)$ for a sequence of values of r tending to infinity, $\rho_k(r)$ is nonincreasing for $r \geq r_1$ and

$$\begin{aligned} \rho_p &= \limsup_{r\to\infty} \sigma_p(r) \\ &= \lim_{r\to\infty} \phi_p(r). \end{aligned}$$

So

$$\begin{aligned} \limsup_{r\to\infty} \rho_p(r) &= \liminf_{r\to\infty} \rho_p(r) \\ &= \lim_{r\to\infty} \rho_p(r) \\ &= \rho_p \end{aligned}$$

and

$$\lim_{r\to\infty} \rho_k'(r) \prod_{i=0}^{p} \log_i(r) = 0.$$

Further we have

$$\begin{aligned} \log_{p-1} M_f(r) &= r^{\sigma_p(r)} \\ &= r^{\rho_p(r)} \end{aligned}$$

for a sequence of values of r tending to ∞ and

$$\log_{p-1} M_f(r) < r^{\rho_p(r)}$$

for remaining r's. Therefore

$$\limsup_{r\to\infty} \frac{\log_{p-1} M_f(r)}{r^{\rho_p(r)}} = 1.$$

Finally note that $\rho_p(r)$ is continuous for $r \geq r_1$. It proves Case I.

Case II: Let $\sigma_p(r) \leq \rho_p$ for all sufficiently large values of r.

In case II we have two Subcases

Subcase A: Let $\sigma_p(r) = \rho_p$ for atleast a sequence of values of r tending to infinity.

We take $\rho_p(r) = \rho_p$ for all values of r.

Subcase B: Let $\sigma_p(r) < \rho_p$ for all sufficiently large values of r.

Let

$$\xi_p(r) = \max_{R_2 \leq x \leq r} \sigma_p(x)$$

where $R_2 > \exp_{p+1}(1)$ is such that $\sigma_p(x) < \rho_p$ whenever $x \geq R_2$.

Note that $\xi_p(r)$ is increasing and for all sufficiently large $x \geq R_2$,the roots of $\xi_p(x) = \rho_p + \log_{p+1} x - \log_{p+1} r$ are less than r.

For a suitable large value $u_2 > R_2$, we define

$$\begin{aligned} \rho_p(u_2) &= \rho_p, \\ \rho_p(r) &= \rho_p + \log_{p+1} r - \log_{p+1} u_2, \end{aligned}$$

for $t_2 \leq r \leq u_2$ where $t_2 < u_2$ is such that $\xi_p(t_2) = \rho_p(t_2)$.

In fact t_2 is given by the largest positive root of $\xi_p(x) = \rho_p + \log_{p+1} x - \log_{p+1} u_2$.

If $\xi_p(t_2) \neq \sigma_p(t_2)$, let $v_1\ (< t_2)$ be the upper bound of point v at which $\xi_p(v) = \sigma_p(v)$ and $v < t_2$.

Note that $\xi_p(v_1) = \sigma_p(v_1)$.

We define

$$\rho_p(r) = \xi_p(r)$$

for $v_1 \leq r \leq t_2$.

One can check that $\xi_p(r)$ is constant in $v_1 \leq r \leq t_2$. Thus $\rho_p(r)$ is constant in $[v_1, t_2]$.

If $\xi_p(t_2) = \sigma_p(t_2)$, we take $v_1 = t_2$.

We choose $u_3 > u_2$ suitably large and let

$$\begin{aligned} \rho_p(u_2) &= \rho_p, \\ \rho_p(r) &= \rho_p + \log_{p+1} r - \log_{p+1} u_3, \end{aligned}$$

for $t_3 \leq r \leq u_3$ where $t_3 < u_3$ is such that $\xi_p(t_3) = \rho_p(t_3)$.

If $\xi_p(t_3) \neq \rho_p(t_3)$, let $\rho_p(r) = \xi_p(r)$ for $v_2 \leq r \leq t_3$, where v_2 has a similar property as that of v_1.

Similarly as before $\rho_p(r)$ is constant in $[v_2, t_3]$.

If $\xi_p(t_3) = \sigma_p(t_3)$, we take $v_2 = t_3$.

Let

$$\rho_p(r) = \rho_p(v_2) + \log_{p+1} v_2 - \log_{p+1} r$$

for $t_4 \leq r \leq v_2$ where $t_4\ (< v_2)$ is the point of intersection of $y = \rho_p$ and $y = \rho_p(v_2) + \log_{p+1} v_2 - \log_{p+1} x$.

We can choose u_3 so large that $u_2 < t_4$.

Let $\rho_p(r) = \rho_p$ for $u_2 \leq r \leq t_4$.

We repeat this process.

Now we can show that for all $r \geq u_2, \rho_p \geq \rho_p(r) \geq \xi_p(r) \geq \sigma_p(r)$ and $\rho_p(r) = \sigma_p(r)$ for $r = v_1, v_2, ...$

So we obtain

$$\begin{aligned} \limsup_{r\to\infty} \rho_p(r) &= \liminf_{r\to\infty} \rho_p(r) \\ &= \lim_{r\to\infty} \rho_p(r) \\ &= \rho_p. \end{aligned}$$

Since

$$\begin{aligned} \log_{p-1} M_f(r) &= r^{\sigma_p(r)} \\ &= r^{\rho_p(r)} \end{aligned}$$

for a sequence of values of r tending to infinity and

$$\log_{p-1} M_f(r) < r^{\rho_p(r)}$$

for remaining r's.

Therefore

$$\limsup_{r\to\infty} \frac{\log_{p-1} M_f(r)}{r^{\rho_p(r)}} = 1.$$

Also $\rho_p(r)$ is differentiable in adjacent intervals.

Further $\rho_p'(r) = 0$ or $\frac{-1}{\prod_{i=0}^{p+1} \log_i(r)}$ and then

$$\lim_{r\to\infty} \rho_p'(r) \prod_{i=0}^{p} \log_i(r) = 0.$$

Finally note that $\rho_p(r)$ is continuous. Hence it proves Case II.

Theorem 2 For every entire function $f(z)$ of finite iterated lower order λ_p, with $i(f) = p$ $(0 < \lambda_p < \infty)$, there exists a proximate iterated lower order $\lambda_p(r)$.

Proof. We can prove this theorem in the same line of the previous theorem.

Example For an entire function $f(z)$, its maximum modulus is $M_f(r) = \sup_{|z|=r} |f(z)|$.

Set $\phi(r) = \log_p M_f(r) > 0$ for sufficiently large values of r.

Obviously $\rho_p = \limsup_{r\to\infty} \frac{\log \phi(r)}{\log r} < \infty$.

Then it can be found (lengthy process) proximate iterated p order $\rho_p(r)$ such that

$$\phi(r) \leq r^{\rho_p(r)}$$

for sufficiently large values of r, and

$$\phi(r_n) \geq r^{\rho_p(r_n)}$$

for a sequence of values of $\{r_n\}$, $r_n \to \infty$.

Theorem 3 For every entire function $f(z)$ of finite iterated order ρ_p, with $i(f) = p$ $(0 < \rho_p < \infty)$ and finite iterated type T_p, there exists a proximate iterated type $T(r)$.

Proof.

$$\begin{aligned} \rho_p &= \limsup_{r\to\infty} \frac{\log_{p+1} M_f(r)}{\log r}, \\ T_p &= \limsup_{r\to\infty} \frac{\log_p M_f(r)}{r^{\rho_p}}, \\ t_p &= \liminf_{r\to\infty} \frac{\log_p M_f(r)}{r^{\rho_p}}. \end{aligned}$$

Let

$$S_p(r) = \frac{\log_p M_f(r)}{r^{\rho_p}}.$$

Then two cases arise.

Case I: $S_p(r) > T_p$ for a sequence of values of r tending to infinity.

Define

$$Q_p(r) = \max_{x\geq r_1} \{S_p(x)\}.$$

Since $S_p(x)$ is continuous, $\limsup_{x\to\infty} S_p(x) = T_p$ and $S_p(x) > T_p$ for a sequence of values of x tending to infinity, $Q_p(r)$ exists and is a nonincreasing function of r.

Let r_1 be a number such that $r_1 > \exp_p(1)$ and $Q_p(r_1) = \max_{x\geq r_1} \{S_p(x)\} = S_p(r_1)$. Such values exists for a sequence of values of r tending to infinity.

Next, suppose that $T_p(r_1) = Q_p(r_1)$ and choose t_1 be the smallest integer not less than $1 + r_1$ such that $Q_p(r_1) > Q_p(t_1)$.

We set, $T_p(r) = T_p(r_1) = Q_p(r_1)$ for $r_1 < r \leq t_1$.

Define u_1 as

$$\begin{aligned} u_1 &> t_1 \\ T_p(r) &= T(r_1) - \log_p r + \log_p t_1 \quad \text{for } t_1 \leq r \leq u_1, \\ T_p(r) &= Q_p(r) \quad \text{for } r = u_1, \end{aligned}$$

but

$$T_p(r) > Q_p(r) \text{ for } t_1 \leq r \leq u_1.$$

Let r_2 be the smallest value of r for which $r_2 \geq u_1$ and $Q_p(r_2) = S_p(r_2)$.

If $r_2 > u_1$ then let $T_p(r) = Q_p(r)$ for $u_1 \leq r \leq r_2$. One can be easily checked that $Q_p(r)$ is constant in $u_1 \leq r \leq r_2$.

Repeating the argument we obtain that $T_p(r)$ is differentiable in adjacent intervals.

Further $T_p'(r) = 0$ or $-\left(\prod_{i=0}^{p-1} \log_i r\right)$ and $T_p(r) \geq Q_p(r) \geq S_p(r)$ for all $r \geq r_1$.

Further $T_p(r) = S_p(r)$ for an infinite number of values of r, also $T_p(r)$ is nonincreasing and $T_p = \limsup\limits_{r\to\infty} S_p(r) = \lim\limits_{r\to\infty} Q_p(r)$.

So,

$$\limsup_{r\to\infty} T_p(r) = \liminf_{r\to\infty} T_p(r) = \lim_{r\to\infty} T_p(r) = T_p$$

and

$$\lim_{r\to\infty} T_p'(r) \prod_{i=0}^{p-1} \log_i r = 0.$$

Further we have,

$$M_f(r) = \exp_p \{r^{\rho_p} S_p(r)\} = \exp_p \{r^{\rho_p} T_p(r)\}$$

for sufficiently large values of r,

$$M_f(r) < \exp_p \{r^{\rho_p} T_p(r)\}$$

for the remaining r's.

Therefore

$$\limsup_{r\to\infty} \frac{M_f(r)}{\exp_p \{r^{\rho_p} T_p(r)\}} = 1.$$

Case II: Let $S_p(r) \leq T_p$ for sufficiently large values of r. There are two Subcases.

Subcase A:

$$S_p(r) = T_p$$

for atleast a sequence of values of r tending to infinity.

We take $T_p(r) = T_p$ for all values of r.

Subcase B:

$$S_p(r) < T_p$$

for sufficiently large values of r.

Let $L_p(r) = \max_{X \leq x \leq r} \{S_p(x)\}$, where $X > \exp_p(1)$ is such that $S_p(x) < T_p$ whenever $x \geq X$.

Note that $L_p(r)$ is nondecreasing. Take a suitably large value of $r_1 \geq X$ and let

$$\begin{aligned} T_p(r_1) &= T_p, \\ T_p(r) &= T_p + \log_p r - \log_p r_1, \text{ for } s_1 \leq r \leq r_1, \end{aligned}$$

where $s_1 < r_1$ is such that $L_p(s_1) = T_p(s_1)$ upto the nearest point $t_1 < s_1$, at which $L_p(t_1) = S_p(t_1)$.

$T_p(r)$ is then constant for $t_1 \leq r \leq s_1$. If $L_p(s_1) = S_p(s_1)$, then let $t_1 = s_1$.

Choose $r_2 > r_1$ suitably large and let

$$\begin{aligned} T_p(r_2) &= T_p, \\ T_p(r) &= T + \log_p r - \log_p r_2, \text{ for } s_2 \leq r \leq r_2 \end{aligned}$$

where $s_2\,(< r_2)$ is such that $L_p(s_2) = T_p(s_2)$.

If $L_p(s_2) \neq S_p(s_2)$ then $L_p(r) = T_p(r)$ for $t_2 \leq r \leq s_2$ where $t_2\,(< s_2)$ is the nearest point to s_2 at which $L_p(t_2) = S_p(t_2)$.

If $L_p(s_2) = S_p(s_2)$, then let $t_2 = s_2$.

For $r < t_2$, let

$$T_p(r) = T_p(t_2) + \log_p(t_2) - \log_p r \text{ , for } u_1 \leq r \leq t_2$$

where $u_1(< t_2)$ is the point of intersection of $y = T_p$ with

$$y = T_p(t_2) + \log_p (t_2) - \log_p r.$$

Let $T_p(r) = T_p$ for $r_1 \leq r \leq u_1$. It is always possible to choose r_2 so large that $r_1 < u_1$.

Repeating the procedure and note that

$$T_p(r) \geq L_p(r) \geq S_p(r)$$

and $T_p(r) = S_p(r)$ for $r = t_1, t_2, t_3, \ldots$.

Hence

$$\lim_{r\to\infty} T_p(r) = T_p$$

and

$$\limsup_{r\to\infty} \frac{M_f(r)}{\exp_p \{r^{\rho_p} T_p(r)\}} = 1.$$

Theorem 4 For every entire function $f(z)$ of finite iterated lower order λ_p, with $i(f) = p$ $(0 < \lambda_p < \infty)$ and finite iterated lower type t_p, there exists a proximate iterated lower type $t_p(r)$.

Proof. We can prove this theorem in the same line of the previous theorem.

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