UNIQUENESS OF ENTIRE FUNCTIONS CONCERNING THE
SHIFT DIFFERENTIAL POLYNOMIALS

B. SAHA AND T. BISWAS

ABSTRACT. In this paper we consider the uniqueness problem of the shift
differential polynomial \((f^n(z)(f(z) - 1)^m \prod_{j=1}^{s} f(z + c_j)^{\mu_j})^{(k)}\), where \(f(z)\) is
a transcendental entire function of finite order, \(c_j (j = 1, 2, \ldots, s)\) are distinct
finite complex numbers and \(n \geq 1, m \geq 1, k \geq 0, s\) and \(\mu_j (j = 1, 2, \ldots, s)\)
are integers. The results of the paper improve and extend some results given
by K. Zhang and H. X. Yi [Acta Mathematica Scientis Series Manuscript,
34B(3)(2014), 719-728] and P. Sahoo and the present first author [Applied
Mathematics E-Notes, 16(2016) 33-44].

1. Introduction, Definitions and Results
In this paper, a meromorphic function \(f(z)\) means meromorphic in the complex
plane. We shall adopt the standard notations in Nevanlinna’s value distribution
theory of meromorphic functions as explained in [6], [8] and [14]. For a nonconstant
meromorphic function \(h\), we denote by \(T(r, h)\) the Nevanlinna characteristic of \(h\)
and by \(S(r, h)\) any quantity satisfying \(S(r, h) = o\{T(r, h)\}\) \((r \to \infty, r \notin E)\).

Let \(f\) and \(g\) be two nonconstant meromorphic functions and \(a \in \mathbb{C} \cup \{\infty\}\). If
the zeros of \(f - a\) and \(g - a\) coincide in locations and multiplicity, we say that \(f\)
and \(g\) share the value \(a\) CM (counting multiplicities). On the other hand, if the
zeros of \(f - a\) and \(g - a\) coincide only in their locations, then we say that \(f\)
and \(g\) share the value \(a\) IM (ignoring multiplicities). For a positive integer \(p\), we denote by
\(N_p(r, a; f)\) the counting function of \(a\)-points of \(f\), where an \(a\)-point of multiplicity
\(m\) is counted \(m\) times if \(m \leq p\) and \(p\) times if \(m > p\). A meromorphic function
\(\alpha(\neq 0, \infty)\) is called a small function with respect to \(f\), if \(T(r, \alpha) = S(r, f)\).

Recently, the topic of difference equation and difference product in the complex
plane \(\mathbb{C}\) has attracted many mathematicians, a large number of papers have focused
on value distribution of differences and differences operator analogues of Nevanlinna
theory (including [3], [4], [5], and [9]), and many people paid their attention to the
uniqueness of differences and difference polynomials of meromorphic function and

2010 Mathematics Subject Classification. Primary: 30D35; Secondary: 39A10.
Key words and phrases. Uniqueness, Entire function, difference-differential polynomial,
Weighted Sharing.
obtained many interesting results. K. Liu and L.Z. Yang [10] also considered the zeros of \( f^n(z)f(z + c) - p(z) \) and \( f^n \Delta_c f \), where \( p(z) \) is a nonzero polynomial and obtain the following theorem.

**Theorem A.** Let \( f \) be a transcendental entire function of finite order and \( p(z) \) be a polynomial. If \( n \geq 2 \), then \( f^n(z)f(z + c) - p(z) \) has infinitely many zeros. If \( f \) is not a periodic function with period \( c \) and \( n \geq 2 \), then \( \Delta_c f = f(z + c) - f(z) \) has infinitely many zeros.

In 2010, X.G. Qi, L.Z. Yang and K. Liu [12] proved the following uniqueness result which corresponded to Theorem A.

**Theorem B.** Let \( f \) and \( g \) be two transcendental entire functions of finite order, and \( c \) be a nonzero complex constant, and let \( n \geq 6 \) be an integer. If \( f^n(z)f(z + c) \) and \( g^n(z)g(z + c) \) share the value 1 CM, then either \( fg = t_1 \) or \( f = t_2g \) for some constants \( t_1 \) and \( t_2 \) satisfying \( t_1^{n+1} = t_2^{n+1} = 1 \).

In the same year J.L. Zhang [15] considered the zeros of one certain type of difference polynomial and obtained the following result.

**Theorem C.** Let \( f \) be a transcendental entire function of finite order, \( \alpha(z)(\neq 0) \) be a small function with respect to \( f \) and \( c \) be a nonzero complex constant. If \( n \geq 2 \) is an integer, then \( f^n(z)(f(z) - 1)f(z + c) - \alpha(z) \) has infinitely many zeros.

In the same paper the author also proved the following uniqueness result which corresponds to Theorem C.

**Theorem D.** Let \( f \) and \( g \) be two transcendental entire functions of finite order, and \( \alpha(z)(\neq 0) \) be a small function with respect to both \( f \) and \( g \). Suppose that \( c \) is a nonzero complex constant and \( n \geq 7 \) is an integer. If \( f^n(z)(f(z) - 1)f(z + c) \) and \( g^n(z)(g(z) - 1)g(z + c) \) share \( \alpha(z) \) CM, then \( f = g \).

In 2012, M.R. Chen and Z.X. Chen [2] considered zeros of one certain type of difference polynomials and obtain the following theorem.

**Theorem E.** Let \( f \) be transcendental entire function of finite order and \( \alpha(z)(\neq 0) \) be a small function with respect to \( f(z), c_j (j = 1, 2,..., s) \), \( n, m \), \( s \) and \( \mu_j (j = 1, 2,..., s) \) be integers. If \( n \geq 2 \) then \( f^n(z)(f^m(z) - 1) \prod_{j=1}^{s} f(z + c_j)^{\mu_j} - \alpha(z) \) has infinitely many zeros.

In the same paper the author also proved the following uniqueness result which corresponds to Theorem E.

**Theorem F.** Let \( f \) and \( g \) be two transcendental entire functions of finite order, \( \alpha(z)(\neq 0) \) be a common small function with respect to \( f \) and \( g \), \( c \) be nonzero finite complex numbers. If \( n \geq m + 8\sigma \), \( n, m \), \( s \), \( \mu_j (j = 1, 2,..., s) \) and \( \sigma = \sum_{j=1}^{s} \mu_j \) are integers, and \( f^n(z)(f^m(z) - 1) \prod_{j=1}^{s} f(z + c_j)^{\mu_j} \) and \( g^n(z)(g^m(z) - 1) \prod_{j=1}^{s} g(z + c_j)^{\mu_j} \) share \( \alpha(z) \) CM, then \( f = tg \) where \( t^m = t^{m+\sigma} = 1 \).
In 2014, K. Zhang and H.X. Yi [17] investigated the difference-differential polynomial of the form $(f^n(z)(f(z) - 1)^m \prod_{j=1}^s f(z + c_j)^{\mu_j})^{(k)}$, where $f$ is transcendental entire function of finite order, $c_j (j = 1, 2, ..., s)$, $n$, $m$, $s$ and $\mu_j (j = 1, 2, ..., s)$ are nonnegative integers, and $\sigma = \sum_{j=1}^s \mu_j$ and obtained the following theorem.

**Theorem G.** Let $f$ and $g$ be transcendental entire functions of finite order, $\alpha(z)(\neq 0)$ be a common small function with respect to $f$ and $g$, $c_j (j = 1, 2, ..., s)$ be distinct finite complex numbers and $n$, $m$, $s$, and $\mu_j (j = 1, 2, ..., s)$ are nonnegative integers. If $n \geq 4k - m + \sigma + 9$, and the differential-difference polynomial $(f^n(z)(f(z) - 1)^m \prod_{j=1}^s f(z + c_j)^{\mu_j})^{(k)}$ and $(g^n(z)(g(z) - 1)^m \prod_{j=1}^s g(z + c_j)^{\mu_j})^{(k)}$ share $\alpha(z)$ CM, then $f = g$.

An increment to uniqueness theory has been to considering weighted sharing instead of sharing IM or CM, this implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2001, which measure how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

**Definition 1.1.** ([7]) Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of $f$ where an a-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f$, $g$ share the value $a$ with weight $k$.

The definition implies that if $f$, $g$ share a value $a$ with weight $k$, then $z_0$ is an a-point of $f$ with multiplicity $m(\leq k)$ if and only if it is an a-point of $g$ with multiplicity $m(\leq k)$ and $z_0$ is an a-point of $f$ with multiplicity $m(> k)$ if and only if it is an a-point of $g$ with multiplicity $m(> k)$, where $m$ is not necessarily equal to $n$.

We write $f$, $g$ share $(a,k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a,k)$ then $f$, $g$ share $(a,p)$ for any integer $p$, $0 \leq p < k$. Also we note that $f$, $g$ share a value a IM or CM if and only if $f$, $g$ share $(a,0)$ or $(a,\infty)$ respectively.

If $\alpha$ is a small function of $f$ and $g$, then $f$, $g$ share $\alpha$ with weight $k$ means that $f - \alpha$, $g - \alpha$ share the value 0 with weight $k$.

In 2016, P. Sahoo and the present author [13] proved the following theorems.

**Theorem H.** Let $f$ and $g$ be two transcendental entire functions of finite order and $\alpha(z)(\neq 0)$ be a small function with respect to $f$ and $g$. Suppose that $c$ is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 2k + m + 6$ when $m \leq k + 1$ and $n \geq 4k - m + 10$ when $m > k + 1$. If $(f^n(z)(f(z) - 1)^m f(z + c))^{(k)}$ and $(g^n(z)(g(z) - 1)^m g(z + c))^{(k)}$ share $(\alpha,2)$ then either $f = g$ or $f$ and $g$ satisfy the algebraic equation $R(f,g) = 0$ where $R(f,g)$ is given by

$$R(w_1, w_2) = w_1^n (w_1 - 1)^m w_1 (z + c) - w_2^m (w_2 - 1)^m w_2 (z + c)$$

**Theorem I.** Let $f$ and $g$ be two transcendental entire functions of finite order and $\alpha(z)(\neq 0)$ be a small function with respect to $f$ and $g$. Suppose that $c$ is
a nonzero complex constant, \( n(\geq 1) \), \( m(\geq 1) \) and \( k(\geq 0) \) are integers satisfying
\( n \geq 5k + 4m + 12 \) when \( m \leq k + 1 \) and \( n \geq 10k - m + 19 \) when \( m > k + 1 \).
If \( (f^n(z)(f(z) - 1)^m f(z + c))^{(k)} \) and \( (g^n(z)(g(z) - 1)^m g(z + c))^{(k)} \) share \( \alpha(z) \) IM, then the conclusions of Theorem H hold.

Now it is natural to ask the following questions which are the motivation of the paper.

**Question 1.1.** Is it possible to relax in any way the nature of sharing the small function in Theorem G keeping the lower bound of \( n \) fixed?

**Question 1.2.** What can be said if we consider the difference-differential polynomial \( (f^n(z)(f(z) - 1)^m \prod_s f(z + c_j)^{\mu_j})^{(k)} \), where \( f(z) \) is a transcendental entire function of finite order, \( c_j (j = 1, 2, \ldots, s) \), \( n(\geq 1) \), \( m(\geq 1) \), \( k(\geq 0) \), \( s \) and \( \mu_j(j = 1, 2, \ldots, s) \) are integers, \( \sigma = \sum_{j=1}^{s} \mu_j \) in theorem H and theorem I?

In the paper, our main concern is to find the possible answer of the above questions. The following are the main results of the paper.

**Theorem 1.1.** Let \( f \) and \( g \) be two transcendental entire functions of finite order and \( \alpha(z) \neq 0 \) be a small function with respect to \( f \) and \( g \). Suppose that \( c_j (j = 1, 2, \ldots, s) \) be distinct finite complex numbers and \( n(\geq 1) \), \( m(\geq 1) \), \( s \), \( \mu_j(j = 1, 2, \ldots, s) \) and \( k(\geq 0) \) are nonnegative integers satisfying \( n \geq 2k + m + \sigma + 5 \) when \( m \leq k + 1 \) and \( n \geq 4k - m + \sigma + 9 \), when \( m > k + 1 \). If the difference-differential polynomial \( (f^n(z)(f(z) - 1)^m \prod_s f(z+c_j)^{\mu_j})^{(k)} \) and \( (g^n(z)(g(z) - 1)^m \prod_s g(z+c_j)^{\mu_j})^{(k)} \) share \( \alpha(z) \), then either \( f = g \) or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where \( R(f, g) \) is given by
\[
R(w_1, w_2) = w_1^n(w_1 - 1)^m \prod_{j=1}^{s} w_1(z + c_j)^{\mu_j} - w_2^n(w_2 - 1)^m \prod_{j=1}^{s} w_2(z + c_j)^{\mu_j}
\]

**Remark 1.1.** Theorem 1.1 improves Theorem G.

**Remark 1.2.** Theorem 1.1 extends Theorem H.

**Theorem 1.2.** Let \( f \) and \( g \) be two transcendental entire functions of finite order and \( \alpha(z) \neq 0 \) be a small function with respect to \( f \) and \( g \). Suppose that \( c_j (j = 1, 2, \ldots, s) \) be distinct finite complex numbers and \( n(\geq 1) \), \( m(\geq 1) \), \( s \), \( \mu_j(j = 1, 2, \ldots, s) \) and \( k(\geq 0) \) are nonnegative integers satisfying \( n \geq 5k + 4m + 4\sigma + 8 \) when \( m \leq k + 1 \) and \( n \geq 10k - m + 4\sigma + 15 \), when \( m > k + 1 \). If the difference-differential polynomial \( (f^n(z)(f(z) - 1)^m \prod_s f(z+c_j)^{\mu_j})^{(k)} \) and \( (g^n(z)(g(z) - 1)^m \prod_s g(z+c_j)^{\mu_j})^{(k)} \) share \( \alpha(z) \) IM, then the conclusions of Theorem 1.1 hold.

**Remark 1.3.** Theorem 1.2 extends Theorem I.
2. Lemmas

Let $F$ and $G$ be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by $H$ the function as follows:

$$H = \left( \frac{F''}{G'} - 2 \frac{F'}{F - 1} \right) - \left( \frac{G''}{G'} - 2 \frac{G'}{G - 1} \right).$$

**Lemma 2.1.** [11] Let $f$ be a meromorphic function of finite order $\rho$ and let $c(\neq 0)$ be a fixed nonzero complex constant. Then

$$N(r, 1, f(z + c)) \leq N(r, 1, f) + S(r, f),$$

outside a possible exceptional set of finite logarithmic measure.

**Lemma 2.2.** [17] Let $f$ be an entire function of finite order and $(f^n(z)(f(z) - 1)m \prod_{j=1}^{s} f(z + c_j)^{\nu_j})^{(k)}$. Then

$$T(r, F) = (n + m + \sigma)T(r, f) + S(r, f).$$

**Lemma 2.3.** [16] Let $f$ be a nonconstant meromorphic function, and $p$, $k$ be two positive integers. Then

$$N_p \left( r, 0; f^{(k)} \right) \leq T \left( r, f^{(k)} \right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f). \quad (2.1)$$

and

$$N_p \left( r, 0; f^{(k)} \right) \leq kN(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \quad (2.2)$$

**Lemma 2.4.** [7] Let $f$ and $g$ be two nonconstant meromorphic functions sharing (1, 2). Then one of the following three cases hold:

(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r),$  

(ii) $f = g,$  

(iii) $fg = 1,$

Where $T(r) = \max\{T(r, f), T(r, g)\}$ and $S(r) = o\{T(r)\}.$

**Lemma 2.5.** [1] Let $F$ and $G$ be two nonconstant meromorphic functions sharing the value $1$ IM and $H \neq 0$. Then

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2N(r, 0; F) + 2N(r, \infty; F) + 2N(r, \infty; G) + S(r, F) + S(r, G),$$

and the same inequality holds for $T(r, G)$.

**Lemma 2.6.** Let $f$ and $g$ be two entire functions and $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$, be integers, and let $F = (f^n(z)(f(z) - 1)m \prod_{j=1}^{s} f(z + c_j)^{\nu_j})^{(k)}$, $G = (g^n(z)(g(z) - 1)m \prod_{j=1}^{s} g(z + c_j)^{\nu_j})^{(k)}$. If there exists nonzero constants $c_1$ and $c_2$ such that $N_1(r, c_1; F) = N_1(r, 0; F) \text{ and } N_1(r, c_2; G) = N_1(r, 0; F)$, then $n \leq 2k + m + \sigma + 2$ for $m \leq k + 1$ and $n \leq 4k - m + \sigma + 4$ for $m > k + 1.$
Proof. We put $F_1 = f^n(z)(f(z) - 1)^m \prod_{j=1}^s f(z + c_j)^{\mu_j}$, $G_1 = g^n(z)(g(z) - 1)^m \prod_{j=1}^s g(z + c_j)^{\mu_j}$. By the second fundamental theorem of Nevanlinna we have

$$T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, c_1; F) + S(r, F)$$
$$\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F). \quad (2.3)$$

Using (2.3), Lemmas 2.2 and 2.3, we obtain

$$(n + m + \sigma)T(r, f) \leq (k + m + \sigma + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \quad (2.5)$$

Similarly,

$$(n + m + \sigma)T(r, g) \leq (k + m + \sigma + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \quad (2.6)$$

Combining (2.5) and (2.6) we obtain

$$(n - 2k - m - \sigma - 2)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g),$$

which gives

$$n \leq 2k + m + \sigma + 2.$$ 

If $m \leq k + 1$, we deduce from (2.4) that

$$(n + m + \sigma)T(r, f) \leq (k + m + \sigma + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \quad (2.7)$$

Similarly,

$$(n + m + \sigma)T(r, g) \leq (k + m + \sigma + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \quad (2.8)$$

Combining (2.7) and (2.8) we obtain

$$(n - 2k - m - \sigma - 2)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g),$$

which gives $n \leq 2k + m + \sigma + 4$. This proves the lemma. \qed

3. Proof of the Theorems

Proof of Theorem 1.1. Let $F_1 = f^n(z)(f(z) - 1)^m \prod_{j=1}^s f(z + c_j)^{\mu_j}$, $G_1 = g^n(z)(g(z) - 1)^m \prod_{j=1}^s g(z + c_j)^{\mu_j}$, $F = \frac{f^{(k)}}{\alpha(z)}$ and $G = \frac{g^{(k)}}{\alpha(z)}$. Then $F$ and $G$ are transcendental meromorphic functions that share (1, 2) except the zeros and poles of $\alpha(z)$. Using (2.1) and Lemma 2.2 we get

$$N_2(r, 0; F) \leq N_2(r, 0; (F_1)^{k}) + S(r, f)$$
$$\leq T(r, (F_1)^{k}) - (n + m + \sigma)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f)$$
$$\leq T(r, F) - (n + m + \sigma)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f).$$

From this we get

$$(n + m + \sigma)T(r, f) \leq T(r, F) + N_{k+2}(r, 0; F_1) - N_2(r, 0; F) + S(r, f). \quad (3.1)$$
Again by (2.2) we have
\[ N_2(r, 0; F) \leq N_2(r, 0; F_1^{(k)}) + S(r, f) \]
\[ \leq N_{k+2}(r, 0; F_1) + S(r, f). \]  
(3.2)

Suppose, if possible, that (i) of Lemma 2.4 holds. Then using (3.2) we obtain from (3.1)
\[ (n + m + \sigma)T(r, f) \leq N_2(r, 0; G) + N_2(r, 1; F) + N_2(r, 1; G) + N_{k+2}(r, 0; F_1) \]
\[ + S(r, f) + S(r, g) \]
\[ \leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + S(r, f) + S(r, g). \]  
(3.3)

If \( m \leq k + 1 \) we deduce from (3.3) that
\[ (n + m + \sigma)T(r, f) \leq (k + m + \sigma + 2)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g). \]  
(3.4)

In a similar manner we obtain
\[ (n + m + \sigma)T(r, g) \leq (k + m + \sigma + 2)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g). \]  
(3.5)

(3.4) and (3.5) together give
\[ (n - 2k - m - \sigma - 4)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g), \]
contradicting with the fact that
\[ n \leq 2k + m + \sigma + 5. \]

If \( m > k + 1 \) we deduce from (3.3) that
\[ (n + m + \sigma)T(r, f) \leq (2k + \sigma + 4)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g). \]  
(3.6)

In a similar manner we obtain
\[ (n + m + \sigma)T(r, g) \leq (2k + \sigma + 4)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g). \]  
(3.7)

(3.6) and (3.7) together give
\[ (n - 4k + m - \sigma - 8)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g), \]
contradicting with the fact that
\[ n \leq 4k - m + \sigma + 9. \]

Therefore, by Lemma 2.4 we have either \( FG = 1 \) or \( F = G \). Let \( FG = 1 \). Then
\[ (f^n(z)(f(z) - 1)^m \prod_{j=1}^{s} f(z + c_j)^{\mu_j})^{(k)} \]
\[ (g^n(z)(g(z) - 1)^m \prod_{j=1}^{s} g(z + c_j)^{\mu_j})^{(k)} = \alpha^2 \]

It can be easily viewed from above that \( N(r, 0; f) = S(r, f) \) and \( N(r, 1; f) = S(r, f) \).
Thus we obtain
\[ \delta(0, f) + \delta(1, f) + \delta(1, f) = 3, \]
which is not possible. Therefore, we must have \( F = G \), and then
\[ (f^n(z)(f(z) - 1)^m \prod_{j=1}^{s} f(z + c_j)^{\mu_j})^{(k)} = (g^n(z)(g(z) - 1)^m \prod_{j=1}^{s} g(z + c_j)^{\mu_j})^{(k)}, \]
Integrating above we obtain
\[(f^n(z)(f(z) - 1)^m \prod_{j=1}^{s} f(z + c_j)^{\mu_j} = (g^n(z)(g(z) - 1)^m \prod_{j=1}^{s} g(z + c_j)^{\mu_j} + \epsilon_{k-1},\]

where \(\epsilon_{k-1}\) is a constant. If \(\epsilon_{k-1} \neq 0\), using Lemma 2.6 it follows that \(n \leq 2k + m + \sigma + 2\), when \(m \leq k + 1\) and \(n \leq 4k - m + \sigma + 4\) when \(n > k + 1\), a contradiction. Hence \(\epsilon_{k-1} = 0\). Repeating the process \(k\)-times, we deduce that

\[f^n(z)(f(z) - 1)^m \prod_{j=1}^{s} f(z + c_j)^{\mu_j} = g^n(z)(g(z) - 1)^m \prod_{j=1}^{s} g(z + c_j)^{\mu_j}, \tag{3.8}\]

Set \(h = \frac{f}{g}\). If \(h\) is a constant, then substituting \(f = gh\) in (3.8), we deduce that

\[
\prod_{j=1}^{s} g(z + c_j)^{\mu_j} [g^m(h^{n+m+\sigma} - 1) - m C_1 g^{m-1}(h^{n+m+\sigma-1} - 1) + \ldots + (-1)^m(h^{n+\sigma} - 1)] = 0.
\]

Since \(g\) is a transcendental entire function, we have \(\prod_{j=1}^{s} g(z + c_j)^{\mu_j} \neq 0\). So from above we obtain

\[g^n(h^{n+m+\sigma} - 1) - m C_1 g^{m-1}(h^{n+m+\sigma-1} - 1) + \ldots + (-1)^m(h^{n+\sigma} - 1) = 0,
\]

which implies \(h = 1\) and hence \(f = g\). If \(h\) is not a constant, then it follows from (3.8) that \(f\) and \(g\) satisfy the algebraic equation \(R(f, g) = 0\) where \(R(f, g)\) is given by

\[R(w_1, w_2) = w_1^n(w_1 - 1)^m \prod_{j=1}^{s} w_1(z + c_j)^{\mu_j} - w_2^n(w_2 - 1)^m \prod_{j=1}^{s} w_2(z + c_j)^{\mu_j}.
\]

This proves Theorem 1.1. \(\square\)

**Proof of Theorem 1.2.** Let \(F, G, F_1\) and \(G_1\) be defined as in the proof of Theorem 1.1. Then \(F\) and \(G\) are transcendental meromorphic functions that share the value \(1\) IM except the zeros and poles of \(\alpha(z)\). We assume, if possible, that \(H \neq 0\). Using Lemma 2.5 and (3.2) we obtain from (3.1)

\[(n + m + \sigma)T(r, f) \leq N_2(r, 0; G) + N_2(r, 1; F) + N_2(r, 1; G) + 2N(r, 0; F) + N(r, 0; G) + N_{k+2}(r, 0; F_1) + 2N(r, \infty; F) + N(r, \infty; G) + S(r, f) + S(r, g)
\]

\[\leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + 2N_{k+1}(r, 0; F_1) + N_{k+1}(r, 0; G_1) + S(r, f) + S(r, g)\]

\((3.9)\)

If \(m \leq k + 1\) we deduce from (3.9) that

\[n + m + \sigma)T(r, f) \leq (3k + 3m + 3\sigma + 4)T(r, f) + (2k + 2m + 2\sigma + 3)T(r, g) + S(r, f) + S(r, g)
\]

\[\leq (5k + 5m + 5\sigma + 7)T(r) + S(r).\]

\((3.10)\)

In a similar manner we obtain

\[(n + m + \sigma)T(r, f) \leq (5k + 5m + 5\sigma + 7)T(r) + S(r).\]

\((3.11)\)

(3.10) and (3.11) together give

\[(n - 5k - 4m - 4\sigma - 7)T(r) \leq S(r),\]
contradicting with the fact that
\[ n \leq 5k + 4m + 4\sigma + 8. \]
If \( m > k + 1 \) we deduce from (3.9) that
\[
(n + m + \sigma)T(r, f) \leq (6k + 3\sigma + 8)T(r, f) + (4k + 2\sigma + 6)T(r, g) + S(r, f) + S(r, g)
\leq (10k + 5\sigma + 14)T(r) + S(r).
\]
(3.12)
In a similar manner we obtain
\[
(n + m + \sigma)T(r, f) \leq (10k + 5\sigma + 14)T(r) + S(r).
\]
(3.13)
(3.12) and (3.13) together give
\[
(n - 10k + m - 4\sigma - 14)T(r) \leq S(r),
\]
contradicting with the fact that
\[ n \leq 10k - m + 4\sigma + 15. \]
We now assume that \( H \equiv 0 \). Then
\[
\left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right) = 0.
\]
Integrating both sides of the above equality twice we get
\[
\frac{1}{F - 1} = \frac{A}{G - 1} + B,
\]
(3.14)
where \( A(\neq 0) \) and \( B \) are constants. From (3.14) it is obvious that \( F, G \) share the value 1 CM and hence they share (1, 2). Therefore \( n \geq 2k + m + \sigma + 5 \) when \( m \leq k + 1 \) and \( n \geq 4k - m + \sigma + 9 \) when \( m > k + 1 \). We now discuss the following three cases separately.

**Case 1.** Suppose that \( B \neq 0 \) and \( A = B \). Then from (3.14) we obtain
\[
\frac{1}{F - 1} = \frac{BG}{G - 1}.
\]
(3.15)
If \( B = -1 \), then from (3.15) we obtain
\[ FG = 1, \]
which is a contradiction as in the proof of Theorem 1.1.

If \( B \neq -1 \), from (3.15), we have \( \frac{1}{F} = \frac{BG}{(1 + B)(G - 1)} \) and so \( N(r, \frac{1}{1 + BG}) = N(r, 0; F) \).

Using (2.1), (2.2) and the second fundamental theorem of Nevanlinna, we deduce that
\[
T(r, G) \leq N(r, 0; G) + N\left(r, \frac{1}{B + 1}; G\right) + N(r, \infty; F) + S(r, G)
\leq N(r, 0; F) + N(r, 0; G) + N(r, \infty; G) + S(r, G)
\leq N_{k+1}(r, 0; F_1) + T(r, G) + N_{k+1}(r, 0; G_1)
- (n + m + \sigma)T(r, g) + S(r, g).
\]
(3.16)
If \( m \leq k + 1 \) we deduce from (3.16) that
\[
(n + m + \sigma)T(r, g) \leq (k + m + \sigma + 1)\{T(r, f) + T(r, g)\} + S(r, g).
\]
Thus we obtain
\[
(n - 2k - m - \sigma - 2)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g).
\]
which is a contradiction as $n \geq 2k + m + \sigma + 5$.

If $m > k + 1$ we deduce from (3.16) that

$$(n + m + \sigma)T(r, g) \leq (2k + \sigma + 2)\{T(r, f) + T(r, g)\} + S(r, g).$$

Thus we obtain

$$(n - 4k + m - \sigma - 4)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which is a contradiction as $n \geq 4k - m + \sigma + 9$.

**Case 2.** Let $B \neq 0$ and $A \neq B$. Then from (3.14) we get $F = \frac{(B + 1)G - (B - A + 1)}{BG(A - B)}$ and so $N(r, \frac{B - A + 1}{B}; G) = N(r, 0; F)$. Proceeding in a manner similar to case 1 we can arrive at a contradiction.

**Case 3.** Let $B = 0$ and $A \neq 0$. Then from (3.14) we get $F = \frac{G + A - 1}{A}$ and $G = AF - (A - 1)$. If $A \neq 1$, it follows that $N(r, \frac{A - 1}{A}; F) = N(r, 0; G)$ and $N(r, 1 - A; G) = N(r, 0; F)$. Now applying Lemma 6 it can be shown that $n \leq 2k + m + \sigma + 2$ for $m \leq k + 1$ and $n \leq 4k - m + \sigma + 4$ for $m > k + 1$, which is a contradiction. Thus $A = 1$ and then $F = G$. Now the result follows from the proof of Theorem 1.1. This completes the proof of Theorem 1.2.

**References**


B. Saha
Department of Mathematics, Govt. General Degree College Muragachha, Nadia, West Bengal, 741154, India.
E-mail address: sahaanjan11@gmail.com, sahabiswajitku@gmail.com

T. Biswas
Rajbari, Rabindrapally, R. N. Tagore Road P.O. Krishnagar, Dist-Nadia, West Bengal, 741101, India.
E-mail address: tanmaybiswas_math@rediffmail.com