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NOTE ON RESULTS IN C^*-ALGEBRA-VALUED METRIC SPACES

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Abstract. In this note, with the help of examples, we point out that C^*-algebra-valued metric space is more general and results in this space are proper generalizations\ extensions of the corresponding results in the literature in standard metric spaces. Hence, results of C^*-algebra-valued metric spaces do not coincide with (can not be derived from) the results in other related spaces.

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1. Introduction and Preliminaries

A C^*-algebra [13] is being continually used to explain a physical system in quantum field theory and statistical mechanics and subsequently appeared as a significant area of research. Recently Ma et al. [11] familiarised with the idea of C^*-algebra valued metric space. Kadelburg and Radenović [8], Alsulami et al. [1], Dung et. al [7], and Senapati and Dey [14] witnessed that the results in this framework can be deduced as consequences of different results in standard metric and other related spaces. The aim of this note is to point out that functions have different nature in different spaces and the results in C^*-algebra valued metric space can not be reduced to their metric counterparts unless C^*-algebra $\mathbb{A}$ is the set of real number $\mathbb{R}$. Also it is worth mentioning here that C^*-algebra is a Banach algebra (over the field $\mathbb{C}$ of complex numbers) together with an involution satisfying the properties of the adjoint and any complex number may be a real number however a real number can never be a complex number.

Example 1 Let $\mathcal{X} = \mathbb{C}$ be a set of complex numbers and $\mathbb{A} = M_n(\mathbb{C})$ be the C^*-algebra of complex matrices. If $A = [a_{ij}] \in \mathbb{A}$ then $A^* = [\bar{a_{ji}}]$ is non-zero element of $\mathbb{A}$. Norm is defined as, $\|A\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|$, where $\|.\|_\infty$ is the usual $l^\infty-$norm on $\mathbb{C}^n$. Define $d : \mathcal{X}\times\mathcal{X} \longrightarrow \mathbb{A}$ as $d(u,v) =$

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$\begin{bmatrix} a\,|u-v| & 0 & 0 \\ 0 & b\,|u-v| & 0 \\ 0 & 0 & c\,|u-v| \end{bmatrix}$, where $a, b, c > 0$ and $u, v \in \mathbb{C}$. Here, d is a C^*-algebra valued metric but not a standard metric. Now, a self map T on $\mathcal{X}$ defined by $Tu = \frac{u}{2}, u \in \mathcal{X}$, satisfies all the conditions of Theorem 2.1 [11] and has a unique fixed point at $u = 0$. It is interesting to see that T does not satisfy the celebrated Banach Contraction principle [3].

Example 2 Let $\mathcal{X} = \{x \in \mathbb{C} : |x| \leq 1\}$ be a compact Hausdorff space and $B(\mathcal{X})$ be the set of all bounded Borel functions on $\mathcal{X}$. With $\|f\| = \sup_{x\in\mathcal{X}} |f(x)|$ and $f^*(x) = \overline{f(x)}$, $B(X)$ becomes a C^*-algebra. Define $d : \mathcal{X} \times \mathcal{X} \longrightarrow B(\mathcal{X})$ as $d(u, v) = |u - v|\, ff^*$, where $u, v \in \mathcal{X}$. Here, d is a C^*-algebra valued metric but not a standard metric. Now, a self map T on $\mathcal{X}$ defined by $Tu = \frac{i}{2}u + \frac{1}{2}$, where $i = \sqrt{-1}$ and $u \in \mathcal{X}$, satisfies all the conditions of Theorem 2.3 [11] and has a unique fixed point $u = \frac{2+i}{5}$ in $\mathcal{X}$. However, T is not a quasi contraction in the sense of Ćirić [5] as a space under consideration is not a standard metric space.

Example 3 Let $\mathcal{X} = \{0, 1, i\}$ and $\mathbb{A} = M_n(\mathbb{C})$ be the C^*-algebra of complex matrices. Define $d : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{A}$ as $d(0, 1) = d((1, i) = d(1, 0) = d(i, 1) = I$ and $d(i, 0) = d(0, i) = mI$, where $m \geq 2, i = \sqrt{-1}$. Then $d(x, y) \preceq \frac{mI}{2}[d(u, v)+d(v, w)]$, for all $u, v, w \in \mathcal{X}$. Here, d is a C^*-algebra-valued $b-$metric but not a standard $b-$metric. If $m > 2$, triangle inequality does not hold and it is not even a C^*-algebra-valued $b-$metric. Now, a self map T on $\mathcal{X}$ defined by $Tu = \frac{u}{2}, u \in \mathcal{X}$, satisfies all the conditions of Theorem 2.1 [12] and has a unique fixed point $u = 0$ in $\mathcal{X}$. However it does not cover similar results ([2], [6], and so on) in a standard $b-$metric space. Also, it contradicts the claim of Dung et al. [7] that $b-$metric is a special type of C^*-algebra-valued $b-$metric and there is an equivalence between them. Consequently, neither the fixed point results nor the topological properties in C^*-algebra-valued $b-$metric spaces can be deduced from any result in $b-$metric spaces and C^*-algebra-valued $b-$metric space is not metrizable unless $\mathbb{A} = \mathbb{R}$.

Senapati and Dey [14] noticed that in the common fixed point results on C^*-algebra-valued metric spaces established in Xin et al. [16], maps T and S are identical and consequently, Theorem 2.1 [16](see Theorem 2.1 [14]) is not a new result as it coincides with the result of Ma et al. [11]. Here, we wish to provide an example wherein Theorem 2.1 [14] remains valid even if maps T and S are not identical. Also, it is worth mentioning here that this example can not be covered by results in other related spaces.

Example 4 Taking $\mathcal{X}$ and $\mathbb{A}$ as in Example 1 , define $d : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{A}$ by $d(u, v) = |u - v|AA^*$. Here, d is a C^*-algebra valued metric but not a standard metric. Now, define self maps T and S on $\mathcal{X}$ as $Tu = \frac{u}{2}$ and $Su = \frac{u}{4}$, then
$d(Tu, Sv) = d(\frac{u}{2}, \frac{v}{4}) = \left|\frac{u}{2} - \frac{v}{4}\right| AA^* \preceq \frac{1}{2}\,|u - v|\,AA^* = \frac{1}{2}d(u, v)$,
i.e., $d(Tu, Sv) \preceq Bd(u, v)B^*, \forall u, v \in \mathbb{C}$, where $\|B\| = \frac{1}{2}$ and $B \in \mathbb{A}$. Hence, T and S satisfy all the conditions of Theorem 2.1 [16] and have a unique coincidence point at $u = 0$ in $\mathbb{C}$. But $Tu \neq Su,\ \forall u \in \mathbb{C}$.

Remark 1 One may observe that none of the examples given above verifies the hypotheses of the analogous theorems in standard metric spaces or equivalents of standard metric spaces.

2. Conclusion

It is interesting to see that the function $Tu = u\,|u|$ have three fixed points 0, 1, and -1 in a real valued metric space $(\mathbb{R}, d)$ whereas five fixed points $0, 1, -1, i$, and $-i$ in a complex valued metric space $(\mathbb{C}, d)$. We conclude that the functions have different natures in different spaces. Consequently, the results in C^*-algebra valued metric space can not be reduced to their metric counterparts unless $\mathbb{A} = \mathbb{R}$. This remains true for all results in [1], [4], [9]-[12], [14]-[16], and references therein.

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