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RESULTS ON CERTAIN DIFFERENTIAL POLYNOMIALS OF L-FUNCTIONS SHARING A FINITE VALUE

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Abstract. In this Article, we analyses a uniqueness of meromorphic functions of certain differential polynomials that share a nonzero finite value or have the same fixed points with the same of L-functions. The results in this paper extend the corresponding results from Xiao-Min Li, Fang Liu, Hong-Xun .

1. Introduction

In this Article, by L-functions with the Riemann zeta function as a prototype, are important objects in number theory, and value distribution of L-functions has been studied extensively. L-functions can be analytically continued as meromorphic functions in $\mathbb{C}$. It is well-known that a non-constant meromorphic function in $\mathbb{C}$ completely determined by five such pre-images five such pre-images, which is a famous theorem due to Nevanlinna and often referred to as Nevanlinna's uniqueness theorem. Two meromorphic functions f and g in the complex plane are said to share a value $c \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in C. Moreover, f and g are said to share a value c CM (counting multiplicities) if they share the value c and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities. Throughout the paper, an L-function always means an L-function L in the Selberg class, which includes the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and essentially those Dirichlet series where one might expect a Riemann hypothesis. Such an L-function is defined to be a Dirichlet series $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ satisfying the following axioms[[19][20]] :

(i) Ramanujan hypothesis:

$$a(n) \ll n^{\varepsilon}$$

for every $\varepsilon > 0$.

(ii) Analytic continuation: There is a non negative integer k such that $(s-1)^k L(s)$

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is an entire function of finite order.
(iii) Functional equation: L satisfies a functional equation of type

$$\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{k} \Gamma\left(\lambda_j s + v_j\right)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\nu_j \geq 0$ and $|\omega| = 1$.
(iv) Euler product hypothesis:

$$Ł(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b\left(p^k\right)}{p^{ks}}\right)$$

with suitable coefficients $b\left(p^k\right)$ satisfying $b\left(p^k\right) \ll p^{k\theta}$ for some $\theta < 1/2$, where the product is taken over all prime numbers p.
We first recall the following result due to Steuding [20], which actually holds without the Euler product hypothesis:

Theorem 1.1 [20] If two L-functions L_1 and L_2 with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $L_1 = L_2$.

Later on, Li [8] proved the following result to deal with a question posed by Chung-Chun Yang [22]:

Theorem 1.2 [8] Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a non-constant function L share a CM and b IM, then $L = f$.

In 1997, Lahiri [9] posed the following question: What can be said about the relationship between two meromorphic functions f and g, when two differential polynomials, generated by f and g respectively, share some nonzero finite value? In this direction, Fang [1] and Yang-Hua [22] respectively proved the following results:

Theorem 1.3 [1] Let f and g be two non-constant entire functions, and let n and k be two positive integers such that $n > 2k+4$. If $\left(f^n\right)^{(k)}$ and $\left(g^n\right)^{(k)}$ share 1 CM, then either $f(z) = c_l e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k \left(c_1 c_2\right)^n (nc)^{2k} = 1$, or $f =$ tg for a constant t such that $t^n = 1$.

Theorem 1.4 [22] Let f and g be two non-constant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_l e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $\left(c_1 c_2\right)^{n+1} c^2 = -1$, or $f =$ tg for a constant t such that $t^{n+1} = 1$.

Regarding Theorems 1.1-1.4, one may ask, what can be said about the relationship between a meromorphic function f and an L-function L, if $\left(f^n\right)^{(k)}$ and $\left(L^n\right)^{(k)}$ share

1 CM or that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ have the same fixed points, where n and k are positive integers? In this direction, Fang Liu, Xiao-Min Li and Hong-Xun Yi[10] proved the following two results respectively:

Theorem 1.5 [10] Let f be a non-constant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > 3k+6$. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.

Theorem 1.6 [10] Let f be a non-constant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > 3k+6$. If $(f^n)^{(k)}(z)-z$ and $(L^n)^{(k)}(z)-z$ share 0 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.

To prove Theorems 1.5 and 1.6 in the paper, authors applied Nevanlinna theory, which can be found in [[4],[11],[23],[24]]. In addition, we will use the lower order $\mu(f)$ and the order $\rho(f)$ of a meromorphic function f, which can be found, for example in [[4],[11],[24]], are in turn defined as follows:

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

We also need the following two definitions:

Definition 1 [12]. Let p be a positive integer and $a \in \mathbf{C}\bigcup\{\infty\}$. Next we denote by $N_{p)}\left(r, \frac{1}{f-a}\right)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, and denote by $N_{(p}\left(r, \frac{1}{f-a}\right)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p. We denote by $\overline{N}_{p)}\left(r, \frac{1}{f-a}\right)$ and $\overline{N}_{(p}\left(r, \frac{1}{f-a}\right)$ the reduced forms of $N_{p)}\left(r, \frac{1}{f-a}\right)$ and $N_{(p}\left(r, \frac{1}{f-a}\right)$ respectively. Here $N_{p)}\left(r, \frac{1}{f-\infty}\right), \overline{N}_{p)}\left(r, \frac{1}{f-\infty}\right), N_{(p}\left(r, \frac{1}{f-\infty}\right)$ and $\overline{N}_{(p}\left(r, \frac{1}{f-\infty}\right)$ mean $N_{p)}(r,f), \overline{N}_{p)}(r,f), N_{(p}(r,f)$ and $\overline{N}_{(p}(r,f)$ respectively.

Definition 2[12] Let a be an any value in the extended complex plane and let k be an arbitrary non negative integer. We define

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r,f)},$$

$$\delta_k(a,f) = 1 - \limsup_{r\to\infty} \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r,f)},$$

where

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$$

Remark 1. By Definition 2 , we have

$$0 \le \delta_k(a,f) \le \delta_{k-1}(a,f) \le \delta_1(a,f) \le \Theta(a,f) \le 1$$

Remark 2. Recently Hu and Li pointed out that Theorem 1.4 is false when $c = 1$. A counter example was given by Hu and Li, see [3].

In 2010, Li [13] introduced the following question posed by Chung-Chun Yang:

Question 1. [13]. If f is a meromorphic function in $\mathbb{C}$ that shares three distinct values a, b CM and c IM with the Riemann zeta function ζ, where $c \notin \{a, b, 0, \infty\}$, is f equal to ζ ?

Li [13] also proved the following result to deal with Question 1:

Theorem 1.7 [13] Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a non-constant L-function L share a CM and b IM, then $L = f$.

Remark 3. In 2012, Gao and Li completely solved Question 1, see [13].

Concerning the value distribution of nonlinear differential polynomials of meromorphic functions, we recall the following result proved by Fang in 2002:

Theorem 1.8 [2] Let f and g be two non-constant entire functions, and let n, k be two positive integers satisfying $n \ge 2k + 8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f = g$.

Regarding Theorem 1.8, one may ask, what can be said about the relationship between two meromorphic functions f and g, if $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM (IM), where n and k are positive integers ? which was also posed by Professor M. L. Fang in 2009. By now this question is still open. In this paper, Xiao-Min Li , Fang Liu , Hong-Xun [7] proved the following result by considering the nonlinear differential polynomials of L-functions.

Theorem 1.9 [7] Let f be a non-constant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > 3k + 9$ and $k \ge 2$. If $(f^n(f-1))^{(k)}$ and $(L^n(L-1))^{(k)}$ share 1 CM, then $f = L$.

Theorem 1.10 [7] Let f be a non-constant meromorphic function, let L be an L-function, and let n and k be two positive integers satisfying $n > 7k + 17$ and $k \ge 2$. If $(f^n(f-1))^{(k)}$ and $(L^n(L-1))^{(k)}$ share 1 IM, then $f = L$.

Regarding Theorems 1.9 and 1.10 one may ask the following question. What happens if $(f^n(f-1))^{(k)}$ and $(L^n(L-1))^{(k)}$ is replaced by $((f^n)^s P(f))^{(k)}$ and $((L^n)^s P(L))^{(k)}$ in Theorems 1.9 and 1.10 ?

We obtain analogous results to answer the above question affirmatively, we prove the following results which is the main results of this article.

2. Main Result

Theorem 2.1 Let f be a non-constant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > \frac{k+m+6+2s(k+1)}{s}$. If $((f^n)^s P(f))^{(k)}$ and $((L^n)^s P(L))^{(k)}$ share 1 CM, then $f = L$.

In the same manner as in the proof of Theorem 2.1 in Section 4 of this paper, we can get the following result by Lemma 3.3 in Section 3 of this paper:

Theorem 2.2 Let f be a non-constant meromorphic function, let L be an L-function, and let n and k be two positive integers satisfying $n > \frac{2k+7+5s(k+1)}{s}$. If $((f^n)^s P(f))^{(k)}$ and $((L^n)^s P(L))^{(k)}$ share 1 IM, then $f = L$.

3. **Some Lemmas**

In this section, we present some lemmas which will be needed later on to prove main results .

Lemma 3.1 [4] Let f be a non-constant meromorphic function, let $k \geq 1$ be a positive integer, and let c be a nonzero finite complex number. Then

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \end{aligned}$$

Lemma 3.2 [16] (Valiron-Mokhonoko,). Let f be a non-constant meromorphic function, and let

$$F = \frac{\sum_{k=0}^{p} a_k f^k}{\sum_{j=0}^{q} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p \neq 0$ and $b_q \neq 0$. Then $T(r,F) = dT(r,f) + O(1)$, where $d = \max\{p,q\}$.

Lemma 3.3 [14] Let F and G be two non-constant meromorphic functions such that $F^{(k)} - P$ and $G^{(k)} - P$ share 0 CM, where $k \geq 1$ is a positive integer, $P \not\equiv 0$ is a polynomial. If

$$(k+2)\Theta(\infty,F) + 2\Theta(\infty,G) + \Theta(0,F) + \Theta(0,G) + \delta_{k+1}(0,F) + \delta_{k+1}(0,G) > (k+7)$$

and

$$(k+2)\Theta(\infty,G) + 2\Theta(\infty,F) + \Theta(0,G) + \Theta(0,F) + \delta_{k+1}(0,G) + \delta_{k+1}(0,F) > (k+7),$$

then either $F^{(k)}G^{(k)} = P^2$ or $F = G$.

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$$(2k+3)\Theta(\infty,F) + (2k+4)\Theta(\infty,G) + \Theta(0,F) + \Theta(0,G) + 2\delta_{k+1}(0,F) + 3\delta_{k+1}(0,G) > (4k+13)$$

and

$$(2k+3)\Theta(\infty,G)+(2k+4)\Theta(\infty,F)+\Theta(0,G)+\Theta(0,F)+2\delta_{k+1}(0,G)+3\delta_{k+1}(0,F) > (4k+13),$$

then either $F^{(k)}G^{(k)} = P^2$ or $F = G$.

Lemma 3.5 [15] Suppose that f is a meromorphic of finite order in the plane, and that $f^{(k)}$ has finitely many zeros for some $k \geq 2$. Then f has finitely many poles in the complex plane.

Lemma 3.6 [25] Let f_1 and f_2 be two non-constant meromorphic functions such that

$$\overline{N}(r, f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) = S(r)$$

for $1 \leq j \leq 2$ Then, either $\overline{N}_0(r, 1; f_1, f_2) = S(r)$ or that there exist two integers p and q satisfying $|p| + |q| > 0$, such that $f_1^p f_2^q = 1$, where $\overline{N}_0(r, 1; f_1, f_2)$ denotes the reduced counting function of the common 1-points of f_1 and f_2 in $|z| < r, T(r) = T(r, f_1) + T(r, f_2)$ and $S(r) = o(T(r))$, as $r \notin E$ and $r \to \infty$. Here $E \subset (0, +\infty)$ is a subset of finite linear measure.

Lemma 3.7 [15] Let f be a transcendental meromorphic function in the complex plane. Then, for each $K > 1$, there exists a set $M(K) \subset (0, +\infty)$ of the lower logarithmic density at most $d(K) = 1 - \left(2e^{K-1} - 1\right)^{-1} > 0$, that is

$$\underline{\log \operatorname{dens}} M(K) = \liminf_{r\to\infty} \frac{1}{\log r} \int_{M(K)\cap[1,r]} \frac{dt}{t} \leq d(K),$$

such that, for every positive integer k, we have

$$\limsup_{\substack{r\to\infty \\ r\notin M(K)}} \frac{T(r,f)}{T\left(r, f^{(k)}\right)} \leq 3eK.$$

Lemma 3.8 [25] Let $s > 0$ and t be relatively prime integers, and let c be a finite complex number such that $c^s = 1$, then there exists one and only one common zero of $\omega^s - 1$ and $\omega^t - c$.

4. Proof of Main Results

Proof of Theorem 2.1.

First of all, we denote by d the degree of L. Then $d = 2\sum_{j=1}^{K} \lambda_j > 0$ (cf.[22],p.113) where K and λ_j are respectively the positive integer and the positive real number in the functional equation of the axiom (iii) of the definition of L-functions. Therefore, by Steuding (cf.[20],p.150) we have

$$T(r, L) = \frac{d}{\pi} r \log r + O(r). \tag{1}$$

Noting that an L-function at most has one pole $z = 1$ in the complex plane, we deduce by Lemma 3.1 and Lemma 3.2 that

$$\begin{aligned} T(r, L^n) &= nT(r, L) + O(1) \\ &\leq \overline{N}(r, (L^n)^s P(L)) + N_{k+1}\left(r, \frac{1}{(L^n)^s P(L)}\right) \\ &+ \overline{N}\left(r, \frac{1}{((L^n)^s P(L))^{(k)} - 1}\right) \\ &- N_0\left(r, \frac{1}{((L^n)^s P(L))^{(k+1)}}\right) + O(\log r) \\ &\leq (s+m)\overline{N}(r, L) + (k+1+s+m)\overline{N}\left(r, \frac{1}{L}\right) \\ &+ \overline{N}\left(r, \frac{1}{((f^n)^s P(f))^{(k)} - 1}\right) + O(\log r) \\ &\leq (k+1+s+m)T(r, L) + T\left(r, ((f^n)^s P(f))^{(k)}\right) + O(\log r) \end{aligned}$$

$$(n-k-1-s-m)T(r, L) \leq T\left(r, ((f^n)^s P(f))^{(k)}\right) + O(\log r) \tag{2}$$

Let

$$F = ((f^n)^s P(f)), G = ((L^n)^s P(L)) \tag{3}$$

now we let

$$\Delta_1 = (k+2)\Theta(\infty, F) + 2\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G) \tag{4}$$

and

$$\Delta_2 = (k+2)\Theta(\infty, G) + 2\Theta(\infty, F) + \Theta(0, G) + \Theta(0, F) + \delta_{k+1}(0, G) + \delta_{k+1}(0, F). \tag{5}$$

by Lemma 3.2 we have

$$\Theta(\infty, F) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, F)}{T(r, F)} = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{(n+1)T(r, f) + O(1)} \geq 1 - \frac{1}{ns+m}, \tag{6}$$

$$\begin{aligned} \delta_{k+1}(0, F) &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}\left(r, \frac{1}{F}\right)}{T(r, F)} \\ &\geq 1 - \frac{s(k+1)+m}{ns+m} \end{aligned} \tag{7}$$

and

$$\Theta(0, F) \geq 1 - \frac{2}{ns+m}, \quad \Theta(0, G) \geq 1 - \frac{2}{ns+m}, \quad \delta_{k+1}(0, G) \geq 1 - \frac{s(k+1)+m}{ns+m}. \tag{8}$$

by noting that an L-function has at most one pole $z = 1$ in the complex plane, we have by equation 1 that

$$\Theta(\infty, G) = 1. \tag{9}$$

by equation 4-9 we have

$$\Delta_1 \geq k + 8 - \frac{k + 2s(k+1) + 2m + 6}{ns+m}, \quad \Delta_2 \geq k + 8 - \frac{2s(k+1) + 2m + 6}{ns+m}. \tag{10}$$

By equation 10 and the assumption $n > \frac{k+m+6+2s(k+1)}{s}$, we have $\Delta_1 > k+7$ and $\Delta_2 > k+7$. This together with equations 4 and 5, Lemma 3.3 and the assumption that $F^{(k)}$ and $G^{(k)}$ share 1 CM gives $F^{(k)}G^{(k)} = 1$ or $F = G$. We consider the following two cases:

Case 1. Suppose that $F^{(k)}G^{(k)} = 1$. Then, by equation 3 we have

$$(((f^n)^s P(f))^{(k)})(((L^n)^s P(L))^{(k)}) = 1. \tag{11}$$

On the other hand, by equations 1 and 11, Lemma 3.2, a result from Whittaker(cf.[25],p.82) and the definition of the order of a meromorphic function we have

$$\begin{aligned}\rho(f) &= \rho\left((f^n)^s P(f)\right) = \rho\left(((f^n)^s P(f))^{(k)}\right) = \rho\left(((L^n)^s P(L))^{(k)}\right) = \rho\left((L^n)^s P(L)\right) \\ &= \rho(L) = 1.\end{aligned} \tag{12}$$

By equation 12 we can see that f is a transcendental meromorphic function. Since an L-function at most has one pole $z = 1$ in the complex plane, we deduce by equation 11 that $((f^n)^s P(f))^{(k)}$ at most has one zero $z = 1$ in the complex plane. Combining this with equation 12, Lemma 3.5 and the assumption $k \geq 2$, we have that $(f^n)^s P(f)$, and so f has at most finitely many poles in the complex plane. This together with equation 11 implies that $((L^n)^s P(L))^{(k)}$ has at most finitely many zeros in the complex plane. Therefore, by equation 3 we have

$$\overline{N}\left(r, F^{(k)}\right) + \overline{N}\left(r, \frac{1}{F^{(k)}}\right) \leq O(\log r) \tag{13}$$

and

$$\overline{N}\left(r, G^{(k)}\right) + \overline{N}\left(r, \frac{1}{G^{(k)}}\right) \leq O(\log r). \tag{14}$$

We now set

$$f_1 = \frac{F^{(k)}}{G^{(k)}}, \quad f_2 = \frac{F^{(k)} - 1}{G^{(k)} - 1}. \tag{15}$$

By equation 15 and the assumption that f and L are transcendental meromorphic functions, we have $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$. Suppose that one of f_1 and f_2 is a nonzero constant. Then, by equation 15 we see that $F^{(k)}$ and $G^{(k)}$ share ∞ CM. Combining this with $F^{(k)}G^{(k)} = 1$ we deduce that ∞ is a Picard exceptional value of f and L. Next we suppose that f_1 and f_2 are non-constant meromorphic functions. We set

$$F_1 = F^{(k)}, \quad G_1 = G^{(k)}. \tag{16}$$

Then, by equations 15-16 we have

$$F_1 = \frac{f_1(1 - f_2)}{f_1 - f_2}, \quad G_1 = \frac{1 - f_2}{f_1 - f_2}. \tag{17}$$

By equation 17 we can find that there exists a subset $I \subset (0, +\infty)$ with infinite linear measure such that $S(r) = o(T(r))$ and

$$T(r, F_1) \leq 2(T(r, f_1) + T(r, f_2)) + S(r) \leq 8T(r, F_1) + S(r) \tag{18}$$

or

$$T(r, G_1) \leq 2(T(r, f_1) + T(r, f_2)) + S(r) \leq 8T(r, G_1) + S(r), \tag{19}$$

as $r \in I$ and $r \to \infty$, where $T(r) = T(r, f_1) + T(r, f_2)$. Without loss of generality, we suppose that equation 18 holds. Then we have $S(r) = S(r, F_1)$, as $r \in I$ and $r \to \infty$. By $F_1 G_1 = 1$ we see that F_1 and G_1 share 1 and -1 CM. By noting that

F_1 and G_1 are transcendental meromorphic functions such that F_1 and G_1 share 1 CM, we deduce by equations 13-15 that

$$\overline{N}\left(r, \frac{1}{f_j}\right) + \overline{N}\left(r, f_j\right) = o(T(r)), (j = 1, 2), \tag{20}$$

as $r \in I$ and $r \to \infty$. By noting that F_1 and G_1 share -1 CM, we deduce by equations 15 and 16 and the second fundamental theorem that

$$\begin{aligned} T\left(r, F_1\right) &\leq \overline{N}\left(r, F_1\right) + \overline{N}\left(r, \frac{1}{F_1}\right) + \overline{N}\left(r, \frac{1}{F_1+1}\right) + O\left(T\left(r, F_2\right)\right) \\ &\leq \overline{N}\left(r, \frac{1}{F_1+1}\right) + O(\log r) + o\left(T\left(r, F_1\right)\right) \\ &\leq \overline{N}_0\left(r, 1; f_1, f_2\right) + o\left(T\left(r, F_1\right)\right), \end{aligned} \tag{21}$$

as $r \in I$ and $r \to \infty$. By equations 18 and 21 we have

$$T\left(r, f_1\right) + T\left(r, f_2\right) \leq \overline{N}_0\left(r, 1; f_1, f_2\right) + o(T(r)), \tag{22}$$

By equations 13-16, 20, 22 and Lemma 3.6 we find that there exist two relatively prime integers x and y satisfying $|x| + |y| > 0$, such that $f_1^x f_2^y = 1$. Combining this with equations 15-16, we have

$$\left(\frac{F_1}{G_1}\right)^x \left(\frac{F_1-1}{G_1-1}\right)^y = 1. \tag{23}$$

we consider the following two subcases:

Subcase 1.1 Suppose that $xy < 0$, say $x > 0$ and $y < 0$, say $y = -y_1$, where y_1 is some positive integer. Then, equation 23 can be rewritten as

$$\left(\frac{F_1}{G_1}\right)^x = \left(\frac{F_1-1}{G_1-1}\right)^{y_1}. \tag{24}$$

Let $z_1 \in \mathbb{C}$ be a pole of F_1 of multiplicity $p_1 \geq 1$. Then, by $F_1 G_1 = 1$ we can see that z_1 be a zero of G_1 of multiplicity p_1. Therefore, by equation 24 we deduce that $2x = y_1 = -y$. Combining this with the assumption that x and y are two relatively prime integers, we have $x = 1$ and $y = -y_1 = -2$. Therefore, equation 24 can be rewritten as $F_1\left(G_1 - 1\right)^2 = \left(F_1 - 1\right)^2 G_1$, this equivalent to the obtained result $F_1 G_1 = 1$. Next we can deduce a contradiction by using the other method. Indeed, by equations 12 and 14, the right equality of equation 3 and the fact that L, and so $((L^n)^s P(L))^{(k)}$ has at most one pole $z = 1$ in the complex plane, we deduce

$$((L^n)^s(z)P(L))^{(k)} = \frac{P_1(z)}{(z-1)^{p_2}} e^{A_1 z + B_1}, \tag{25}$$

where P_1 is a nonzero polynomial, $p_2 \geq 0$ is an integer, $A_1 \neq 0$ and B_1 are constants. By equation 25, Hayman[6], Lemma 3.2 and Lemma 3.7 we deduce that there exists a subset $I \subset (0, +\infty)$ with logarithmic measure $I = \int_I \frac{dt}{t} = \infty$ such that for some given sufficiently large positive number $K > 1$, we have

$$\begin{aligned} (ns+m)T(r, L) &= T\left(r, ((L^n)^s P(L))\right) \\ &\leq 3eKT\left(r, ((L^n)^s P(L))^{(k)}\right) = \frac{3eK\left|A_1\right| r}{\pi}(1 + O(1)) + O(\log r), \end{aligned} \tag{26}$$

as $r \in I$ and $r \to \infty$. By equations 1 and 26 we have a contradiction.

Subcase 1.2 Suppose that $st = 0$, say $s = 0$ and $t \neq 0$. Then, by equation 23 we can see that F_1 and G_1 share ∞ CM. This together with equations 3 and 16 and the assumption $F_1G_1 = 1$ implies that ∞ is a Picard exceptional value of f and L.

Subcase 1.3 Suppose that $st > 0$, say $s > 0$ and $t > 0$. Then, by equation 23 we can see that F_1 and G_1 share ∞ CM. This together with equations 3 and 16 and the assumption $F_1G_1 = 1$ implies that ∞ is a Picard exceptional value of f and L.

By equations 3 and 14 and the assumption $n > \frac{k+m+6+2s(k+1)}{s}$ we deduce that L has at most finitely many zeros in the complex plane. This together with the obtained result that ∞ is a Picard exceptional value of f and L gives

$$L(z) = P_2(z)e^{A_2z+B_2}, \tag{27}$$

where P_2 is a nonzero polynomial, $A_2 \neq 0$ and B_2 are constants. By equation 27 and Hayman [6] we have

$$T(r, L(z)) = T\left(r, P_3(z)e^{A_2z+B_2}\right) = \frac{|A_2|\, r}{\pi}(1 + O(1)) + O(\log r),$$

which contradicts equation 1.

Case 2. Suppose that $F = G$. Then by equation 3 we have

$$(f^n)^s P(f) = (L^n)^s P(L) \tag{28}$$

now we set

$$H = \frac{f}{L} \tag{29}$$

If H is a non-constant meromorphic function, then we get equation 28.
Suppose H is a constant. Then from equation 29, we get

$$[a_nf^m + a_{n-1}f^{m-1} + \cdots + a_1f][(f^n)^s] = [a_nL^m + a_{n-1}L^{m-1} + \cdots + a_1L][(L^n)^s]$$

i.e,

$$a_nL^{(m+ns)}[H^{m+ns}-1]+a_{n-1}L^{(m+ns-1)}[H^{m+ns-1}-1]+\cdots+a_1L^{(1+ns)}[H^{1+ns}-1] = 0$$

which implies $H^{\chi_n} = 1$, where

$$\chi_n = \begin{cases} 1 & \sum\limits_{j=1}^{n-1} |a_{n-j}| \neq 0 \ ; \\ d_1 & a_j = 0, \forall j = 1, 2, \cdots, n-1 \ , \end{cases}$$

$$d_1 = gcd(m + ns, m + ns - 1, \cdots, ns + 1),$$

Therefore, $f = tL$, for a constant t satisfies $t^{\chi_n} = 1$. We get the conclusion of Theorem 2.1. This completes the proof of Theorem 2.1.

Proof of Theorem 2.2.
First of all, we denote by d the degree of L. Then $d = 2\sum_{j=1}^{K} \lambda_j > 0$ [22], where K and λ_j are respectively the positive integer and the positive real number in the functional equation of the axiom (iii) of the definition of L-function. Therefore, by

the results of Steuding equations we have equation 1. Now we let equation 3, and let

$$\begin{aligned}\Delta_3 =&(2k+3)\Theta(\infty, F_1) + (2k+4)\Theta(\infty, G_1) + \Theta(0, F_1) + \Theta(0, G_1)\\ &+ 2\delta_{k+1}(0, F_1) + 3\delta_{k+1}(0, G_1)\end{aligned} \tag{30}$$

and

$$\begin{aligned}\Delta_4 =&(2k+3)\Theta(\infty, G_1) + (2k+4)\Theta(\infty, F_1) + \Theta(0, G_1) + \Theta(0, F_1)\\ &+ 2\delta_{k+1}(0, G_1) + 3\delta_{k+1}(0, F_1).\end{aligned} \tag{31}$$

In the same manner as in the proof of Theorem 2.1 by equations 6-9 and by equations 30-31 we have

$$\Delta_3 \geq 4k+14-\frac{2k+7+5m+5s(k+1)}{ns+m}, \quad \Delta_4 \geq 4k+14-\frac{2k+8+5m+5s(k+1)}{ns+m}. \tag{32}$$

By equation 32 and the assumption $n > \frac{2k+7+5s(k+1)+4m}{s}$ we deduce $\Delta_3 > 4k+13$ and $\Delta_4 > 4k+13$. This together with Lemma 3.4 gives $F^{(k)}G^{(k)} = 1$ or $F = G$. We consider the following two cases:

Case 1. Suppose that $F^{(k)}G^{(k)} = 1$. Then, in the same manner as in Case 1 of the proof of Theorem 2.1 we have a contradiction.

Case 2. Suppose that $F = G$. Then, in the same manner as in Case 2 of the proof of Theorem 2.1 we get the conclusion of Theorem 2.2 This completely proves Theorem 2.2.

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