Electronic Journal of Mathematical Analysis and Applications
Vol. 11(2) July 2023, No. 15
ISSN: 2090-729X (online)
http://ejmaa.journals.ekb.eg/

*U*NIQUENESS RESULTS ON DIFFERENTIAL POLYNOMIALS GENERATED BY A MEROMORPHIC FUNCTION AND A L-FUNCTION

PREETHAM N. RAJ AND HARINA P. WAGHAMORE

Abstract. The Riemann zeta function and its various generalizations have been extensively studied by mathematicians worldwide. The L-functions are Selberg class functions with Riemann zeta function as the prototype and since L-functions are analytically continued as meromorphic functions, it is convenient to study the value distribution and uniqueness problems on L-functions and arbitrary meromorphic functions. Further, the fact that L-functions neither have a pole nor zero at the origin, but is having only possible pole at $s = 1$ helps us to study some of the classical results of Boussaf et al. [3] in terms of a L-function and an arbitrary meromorphic function. In this paper, by using the concept of weighted sharing and least multiplicity, we study the value distribution of a L-function and an arbitrary meromorphic function when certain type of differential polynomials generated by them share a non-zero small function with finite weight. Our results extends and improves the classical results due to Boussaf et al. (Indagationes Mathematicae 24(1):15-41, 2013).

1. Introduction and main results

The Nevanlinna theory is one of the several branches of complex analysis that has seen extensive research work. For the standard definitions and notations of the Nevanlinna theory one can refer ([10], [32], [33]).

Let $f(z)$ and $g(z)$ be two meromorphic functions in the complex plane $\mathbb{C}$. Suppose if $f(z) - a$ and $g(z) - a$ assumes the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicity), and if we do not consider the multiplicity, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicity), where a is a complex number. Around 2001, Lahiri ([12], [13]) introduced the idea of weighted sharing which is in between CM and IM

2010 *Mathematics Subject Classification.* 30D35(Primary), 30D30(Secondary).
Key words and phrases. Nevanlinna theory, Weighted sharing, Differential polynomials, Value sharing, p-Adic analysis.
Submitted April 14, 2023. Revised August 1, 2023. Accepted September 5, 2023.

sharing. If we say that f and g share the value a with the weight k, then it implies that, z_0 is a zero of $f(z) - a$ with multiplicity $m(\leq k)$ if and only if z_0 is a zero of $g(z) - a$ with multilpicity $m(\leq k)$, and z_0 is a zero of $f(z) - a$ with multiplicity $m(> k)$ if and only if z_0 is a zero of $g(z) - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We denote by f, g share (a, k) to mean f and g share the value a with the weight k.

Many mathematicians around the world have found the uniqueness problem of meromorphic functions in the field $\mathbb{C}$ or in the p-adic field to be an intriguing research topic and we can see significant number of research works regarding the polynomial of uniqueness in $\mathbb{C}$ (see [6], [8], [14], [16], [30], [31]) as well as in p-adic fields (see [1], [4], [5], [19], [20], [21], [22]). After studying the uniqueness problems of the form $P(f) = P(g)$, where P is a polynomial, the studies were extended to problems of the form $f'P'(f) = g'P'(g)$, in both complex and p-adic contexts. Boussaf et al. [3] conducted one such studies in which they re-examined several crucial lemmas obtained on a p-adic field to complex field and obtained similar results to those obtained in p-adic analysis.

For the sake of convenience we retain the same notations used earlier in p-adic analysis by many authors including Boussaf et al. [3]. Let $\mathcal{A}(\mathbb{C})$ represent the $\mathbb{C}$-algebra of entire functions in $\mathbb{C}$, $\mathcal{M}(\mathbb{C})$ represent the field of meromorphic functions in $\mathbb{C}$, and $\mathbb{C}[x]$ represent the field of rational functions. A polynomial $P \in \mathbb{C}[x]$ is called a polynomial of uniqueness for a class of functions $\mathcal{T}$, if the property $P(f) = P(g)$ implies $f = g$, for any two functions $f, g \in \mathcal{T}$ (see [1], [15], [29], [31]).

Let $f \in \mathcal{M}(\mathbb{C})$ be a function such that f has no pole or zero at 0. Then we define the positive logarithm, the proximity function $m(r, f)$, the counting function $Z(r, f)$ for the zeros of f (respectively the reduced counting function $\overline{Z}(r, f)$), the counting function $N(r, f)$ for the poles of f (respectively the reduced counting function $\overline{N}(r, f)$), the Nevanlinna characteristic function $T(r, f)$ and the small function α in a similar manner as defined in the previous paper by Boussaf et al. [3].

Suppose, if f assumes a zero or a pole of order t at 0, then we can either make a change of origin, or count the zero or pole at 0 by respectively adding or subtracting $t \log r$ to the counting functions. We denote by $\mathcal{M}_f(\mathbb{C})$ the set of all small meromorphic functions with respect to f in $\mathbb{C}$ and by $S(r, f)$ a function in $r \in (0, +\infty)$ such that $\lim\limits_{r\to+\infty} \frac{S(r,f)}{T(r,f)} = 0$ outside a subset of $(0, +\infty)$ of finite measure. For $a \in \mathbb{C}$, the deficiency of a with respect f is defined by $\delta(a, f) = 1 - \overline{\lim\limits_{r\to\infty}} \frac{Z(r,f-a)}{T(r,f)}$. Clearly $0 \leq \delta(a, f) \leq 1$. We also have $\delta(\infty, f) = 1 - \overline{\lim\limits_{r\to\infty}} \frac{N(r,f)}{T(r,f)}$.

Now, Boussaf et al. [3] obtained the following theorems and corollaries for the meromorphic functions $f, g \in \mathcal{M}(\mathbb{C})$ in the complex field.

Theorem A. [3, Theorem 3] *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$, let $P' = b(x - a_1)^n \prod_{i=2}^{l} (x - a_i)^{k_i}$ with $b \in \mathbb{C}\backslash\{0\}$, and $l \geq 2$, $k_i \geq k_{i+1}, 2 \leq i \leq l - 1$ when $l > 2$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$n \geq 10 + \sum_{i=3}^{l} max(0, 4 - k_i) + max(0, 5 - k_2)$,

$n \geq k + 3$,

if $l = 2$, then $n \neq 2k, 2k + 1, 3k + 1$,

if $l = 3$, then $n \neq 2k + 1, 3k_i - k, \forall i = 2, 3$.

Let f, $g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α CM, then $f = g$.

Corollary B. [3, Corollary 3.2] *Let $P \in \mathbb{C}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $k \geq 2$. Suppose that P satisfies the further conditions:*
$n \geq 10 + max(0, 5-k)$,
$n \geq k+3$,
$n \neq 2k, 2k+1, 3k+1$.
Let f, $g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α CM, then $f = g$.

Corollary C. [3, Corollary 3.3] *Let f, $g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. Let $a \in \mathbb{C}\backslash\{0\}$. If $f'f^n(f-a)^2$ and $g'g^n(g-a)^2$ share the function α CM and if $n \geq 13$, then $f = g$.*

Let L be an algebraically closed field and let $P \in L[x]\backslash\mathbb{K}$ (where $\mathbb{K}$ is algebraically closed field of characteristic zero, complete for an ultrametric absolute value) and let $\Xi(P)$ be the set of zeros c of P' such that $P(c) \neq P(d)$ for every zero d of P' other than c. Let $\phi(P)$ denote its cardinal.

Corollary D. [3, Corollary 3.4] *Let $P \in \mathbb{C}[x]$ be such that $P'(x) = b(x-a_1)^n(x-a_2)^{k_2}(x-a_3)^{k_3}$ with $b \in \mathbb{C}\backslash\{0\}$, $\phi(P) = 3$, $k_i \geq k_{i+1}$ and $i = 2, 3$. Suppose, P satisfies the following conditions:*
$n \geq 10 + \sum_{i=3}^{l} max(0, 4-k_2) + max(0, 5-k_3)$,
$n \geq k_2 + k_3 + 3$.
Let f, $g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α CM, then $f = g$.

By taking all k_i's equal to 1 in Theorem A, Boussaf et al. [3] obtained the following theorem.

Theorem E. [3, Theorem 4] *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$ such that P' is of the form $b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geq 3$, $b \in \mathbb{C}\backslash\{0\}$, with $n \geq l+10$. Let f, $g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α CM, then $f = g$.*

Boussaf et al. [3] also proved the following result.

Theorem F. [3, Theorem 5] *Let f, $g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. Let $a \in \mathbb{C}\backslash\{0\}$. If $f'f^n(f-a)$ and $g'g^n(g-a)$ share the function α CM and if $n \geq 12$, then either $f = g$ or there exists $h \in \mathcal{M}(\mathbb{C})$ such that $f = \frac{a(n+2)}{n+1}\left(\frac{h^{n+1}-1}{h^{n+2}-1}\right)h$ and $g = \frac{a(n+2)}{n+1}\left(\frac{h^{n+1}-1}{h^{n+2}-1}\right)$.*

Following Riemann's groundbreaking result [24], the Riemann zeta function as well as its various generalizations have been extensively studied by mathematicians for over a century. These functions are commonly referred to as L-functions. Over time, significant links have been established between the properties of the L-functions and other theories. Towards the end of twentieth century, in an effort to summarize the core properties of classical L-functions, Selberg [27] gave an axiomatic characterization of what would be called general L-functions. A L-function $\mathcal{F}$ means a Selberg class function with the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ as the prototype and the Selberg class $\mathcal{S}$ of L-function is defined as follows:

Definition 1.1. [27] *The Selberg class $\mathcal{S}$ consists of the functions $\mathcal{F}$ satisfying the following axioms:*

(1) *(Dirichlet series)* $\mathcal{F}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, *absolutely convergent for* $\sigma > 1$.
(2) *(Analytic continuation) There exists an integer m such that $(s-1)^m \mathcal{F}(s)$ is an entire function of finite order.*
(3) *(Functional equation) There exist an integer $r \geq 0$, positive real numbers Q, λ_j, complex numbers μ_j with $Re\,\mu_j \geq 0$ and ω with $|\omega| = 1$, such that the function $\Lambda(s)$ defined by*

$$\Lambda(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)\mathcal{F}(s) = \gamma(s)\mathcal{F}(s),$$

satisfies the functional equation $\Lambda(s) = \omega\overline{\Lambda}(1-s)$. We would call the function $\gamma(s)$ the γ-factor.
(4) *(Ramanujan conjecture) For every $\epsilon > 0$, $a(n) = O(n^{\epsilon})$.*
(5) *(Euler product) $a(1) = 1$, and $\log \mathcal{F}(s) = \sum_{n \geq 1} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a prime power, and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.*

By the comment on the order of a function, we can choose m in axiom (2) to be the order of the pole of $\mathcal{F}$ at $s = 1$.

Now, the main motivation to this paper is that, it is easy to compare the value distribution and uniqueness outcomes between the L-functions and any arbitrary meromorphic functions since L-functions are analytically continued as meromorphic functions (see [9], [11], [18], [23], [25], [26]). Additionally, the fact that L-function is not having any zero or pole at $s = 0$, but is having only possible pole at $s = 1$, makes it interesting to analyse the results from Boussaf et al. [3] by taking a L-function $\mathcal{F}$ and an arbitrary meromorphic function f, in the form $f'P'(f)$ and $\mathcal{F}'P'(\mathcal{F})$. Consideration of the concept of least multiplicity makes the study further interesting.

Main Results

Following are the main results of our paper which builds upon and extends the classical results of Boussaf et al. [3].

Theorem 1.1. *Let f be a non-constant meromorphic function, $\mathcal{F}$ be a L-function and P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$ such that P' is of the form $b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ where $b \in \mathbb{C}\backslash\{0\}$ and let $n > 0$, $l \geq 2$, $k_j \geq k_{j+1}$ and $k = \sum_{i=2}^{l} k_i$ be integers with $2 \leq j \leq l-1$ when $l > 2$. Let s, t be positive integers. Suppose that f, $\mathcal{F} \in \mathcal{M}(\mathbb{C})$ whose poles and a_1-points have multiplicities at least t and s respectively and $\alpha(\not\equiv 0) \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_{\mathcal{F}}(\mathbb{C})$, then if $f'P'(f)$ and $\mathcal{F}'P'(\mathcal{F})$ share (α, w), where $w \in \mathbb{N}\cup\{0\}$ and one of the following three conditions is satisfied:*
(1) $n > max\Big\{k+2,\ \Big(1-\delta(\infty,f)\Big)\left(\frac{11}{t}\right)+\left(\frac{7}{s}\right)+\sum_{i=3}^{l} max(0, 7-k_i)+max(0, 11-k_2)\Big\}$ *when $w = 0$;*
(2) $n > max\Big\{k+2,\ \Big(1-\delta(\infty,f)\Big)\left(\frac{6}{t}\right)+\left(\frac{9}{2s}\right)+\sum_{i=3}^{l} max(0, \frac{9}{2}-k_i)+max(0, 6-k_2)\Big\}$ *when $w = 1$;*
(3) $n > max\Big\{k+2,\ \Big(1-\delta(\infty,f)\Big)\left(\frac{5}{t}\right)+\left(\frac{4}{s}\right)+\sum_{i=3}^{l} max(0, 4-k_i)+max(0, 5-k_2)\Big\}$

when $w \geq 2$*;*
and also n *is such that*
if $l = 2$*, then* $n \neq 2k, 2k+1, 3k+1$*;*
if $l = 3$*, then* $n \neq 2k+1, 3k_i - k$ *for* $i = 2, 3$*,*
then $f = \mathcal{F}$*.*

Example 1.1. *Let* $f = \frac{1}{z}$ *and* $\mathcal{F} = \zeta(z)$*, where* $\zeta(z)$ *is the Riemann zeta function. Then* f *has a simple pole at* $z = 0$ *and a simple zero at* $z = \infty$*. Also,* $\mathcal{F}$ *has a simple pole at* $z = 1$ *and a zero at* $z = \infty$*. Now,* f *and* $\mathcal{F}$ *share* 0 *IM. Let* P *be a polynomial such that* P' *is of the form,* $P'(x) = x^{21}(x-1)^{11}(x-2)^7$*. Then* $f'P'(f)$ *and* $\mathcal{F}'P'(\mathcal{F})$ *share* $(0, 0)$*, but* $f \neq \mathcal{F}$*.*

Example 1.2. *Let* $f = -\zeta(z)$ *and* $\mathcal{F} = \zeta(z)$*, where* $\zeta(z)$ *is the Riemann zeta function. Now,* f *and* $\mathcal{F}$ *share* 0 *CM. Let* P *be a polynomial such that* P' *is of the form,* $P'(x) = x^{12}(x-1)^5(x-2)^4$*. Then* $f'P'(f)$ *and* $\mathcal{F}'P'(\mathcal{F})$ *share* $(0, \infty)$*, but* $f \neq \mathcal{F}$*.*

These examples shows that the conditions given in the theorem are necessary.

Corollary 1.1. *Let* f *be a non-constant meromorphic function,* $\mathcal{F}$ *be a L-function and let* $P \in \mathbb{C}[x]$ *be such that* P' *is of the form* $b(x-a_1)^n(x-a_2)^k$ *with* $b \in \mathbb{C}\backslash\{0\}$ *and* $k \geq 2$*. Let* f*,* $\mathcal{F} \in \mathcal{M}(\mathbb{C})$ *whose poles and* a_1*-points have multiplicities at least* t *and* s *respectively and* $\alpha(\not\equiv 0) \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_{\mathcal{F}}(\mathbb{C})$*. Suppose* P *satisfies one of the following conditions:*
(1) $n > max\Big\{k+2,\ \Big(1-\delta(\infty, f)\Big)\left(\frac{11}{t}\right) + \left(\frac{7}{s}\right) + max(0, 11-k)\Big\}$ *when* $w = 0$*;*
(2) $n > max\Big\{k+2,\ \Big(1-\delta(\infty, f)\Big)\left(\frac{6}{t}\right) + \left(\frac{9}{2s}\right) + max(0, 6-k)\Big\}$ *when* $w = 1$*;*
(3) $n > max\Big\{k+2,\ \Big(1-\delta(\infty, f)\Big)\left(\frac{5}{t}\right) + \left(\frac{4}{s}\right) + max(0, 5-k)\Big\}$ *when* $w \geq 2$*;*
and also n *is such that* $n \neq 2k, 2k+1, 3k+1$*. Then if* $f'P'(f)$ *and* $\mathcal{F}'P'(\mathcal{F})$ *share* (α, w)*, then* $f = \mathcal{F}$*.*

Corollary 1.2. *Let* f *be a non-constant meromorphic function,* $\mathcal{F}$ *be a L-function and let* $P \in \mathbb{C}[x]$ *be such that* P' *is of the form* $b(x-a_1)^n(x-a_2)^{k_2}(x-a_3)^{k_3}$ *with* $b \in \mathbb{C}\backslash\{0\}$*,* $\phi(P) = 3$*,* $k_i \geq k_{i+1}$ *and* $i = 2, 3$*. Let* f*,* $\mathcal{F} \in \mathcal{M}(\mathbb{C})$ *whose poles and* a_1*-points have multiplicities at least* t *and* s *respectively and* $\alpha(\not\equiv 0) \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_{\mathcal{F}}(\mathbb{C})$*. Suppose* P *satisfies one of the following conditions:*
(1) $n > max\Big\{k_2+k_3+2,\ \Big(1-\delta(\infty, f)\Big)\left(\frac{11}{t}\right)+\left(\frac{7}{s}\right)+max(0, 7-k_2)+max(0, 11-k_3)\Big\}$
when $w = 0$*;*
(2) $n > max\Big\{k_2+k_3+2,\ \Big(1-\delta(\infty, f)\Big)\left(\frac{6}{t}\right)+\left(\frac{9}{2s}\right)+max(0, \frac{9}{2}-k_2)+max(0, 6-k_3)\Big\}$
when $w = 1$*;*
(3) $n > max\Big\{k_2+k_3+2,\ \Big(1-\delta(\infty, f)\Big)\left(\frac{5}{t}\right)+\left(\frac{4}{s}\right)+max(0, 4-k_2)+max(0, 5-k_3)\Big\}$
when $w \geq 2$*,*
then if $f'P'(f)$ *and* $\mathcal{F}'P'(\mathcal{F})$ *share* (α, w)*, then* $f = \mathcal{F}$*.*

By taking all k_i's equal to 1, a better formulation can be obtained as below.

Theorem 1.2. *Let* f *be a non-constant meromorphic function,* $\mathcal{F}$ *be a L-function and* P *be a polynomial of uniqueness for* $\mathcal{M}(\mathbb{C})$ *such that* P' *is of the form* $b(x-a_1)^n\prod_{i=2}^{l}(x-a_i)$ *where* $b \in \mathbb{C}\backslash\{0\}$*, and* $n > 0$*,* $l \geq 3$ *are integers. Let* s, t *be positive integers. Suppose that* f*,* $\mathcal{F} \in \mathcal{M}(\mathbb{C})$ *whose poles and* a_1*-points have multiplicities*

at least t and s respectively and $\alpha(\not\equiv 0) \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_{\mathcal{F}}(\mathbb{C})$, then if $f'P'(f)$ and $\mathcal{F}'P'(\mathcal{F})$ share (α, w), where $w \in \mathbb{N} \cup \{0\}$ and one of the following three conditions is satisfied:
(1) $n > \Big(1 - \delta(\infty, f)\Big)\left(\frac{11}{t}\right) + \left(\frac{7}{s}\right) + 4l$ *when* $w = 0$;
(2) $n > \Big(1 - \delta(\infty, f)\Big)\left(\frac{6}{t}\right) + \left(\frac{9}{2s}\right) + \frac{3}{2}l$ *when* $w = 1$;
(3) $n > \Big(1 - \delta(\infty, f)\Big)\left(\frac{5}{t}\right) + \left(\frac{4}{s}\right) + l$ *when* $w \geq 2$;
then if $f'P'(f)$ and $\mathcal{F}'P'(\mathcal{F})$ share (α, w), then $f = \mathcal{F}$.

Theorem 1.3. *Let f be a non-constant meromorphic function, $\mathcal{F}$ be a L-function, t, s be positive integers and $a \in \mathbb{C}\backslash\{0\}$. Suppose that f, $\mathcal{F} \in \mathcal{M}(\mathbb{C})$ whose poles and a_1-points have multiplicities at least t and s respectively and $\alpha(\not\equiv 0) \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_{\mathcal{F}}(\mathbb{C})$, then if $f'f^n(f-a)$ and $\mathcal{F}'\mathcal{F}^n(\mathcal{F}-a)$ share (α, w), where $w \in \mathbb{N} \cup \{0\}$ and one of the following conditions is satisfied:*
(1) $n > \Big(1 - \delta(\infty, f)\Big)\left(\frac{11}{t}\right) + \left(\frac{7}{s}\right) + 8$ *when* $w = 0$;
(2) $n > \Big(1 - \delta(\infty, f)\Big)\left(\frac{6}{t}\right) + \left(\frac{9}{2s}\right) + 3$ *when* $w = 1$;
(3) $n > \Big(1 - \delta(\infty, f)\Big)\left(\frac{5}{t}\right) + \left(\frac{4}{s}\right) + 2$ *when* $w \geq 2$,
then either $f = \mathcal{F}$ or there exists some $h \in \mathcal{M}(\mathbb{C})$ satisfying $f = \frac{a(n+2)}{n+1}\frac{(1-h^{n+1})}{(1-h^{n+2})}h$, $\mathcal{F} = \frac{a(n+2)}{n+1}\frac{(1-h^{n+1})}{(1-h^{n+2})}$.

Theorem 1.4. *Let f be a non-constant meromorphic function, $\mathcal{F}$ be a L-function, $t, s, m(\geq 2)$ be positive integers and $a \in \mathbb{C}\backslash\{0\}$. Suppose that f, $\mathcal{F} \in \mathcal{M}(\mathbb{C})$ whose poles and a_1-points have multiplicities at least t and s respectively and $\alpha(\not\equiv 0) \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_{\mathcal{F}}(\mathbb{C})$, then if $f'f^n(f-a)^m$ and $\mathcal{F}'\mathcal{F}^n(\mathcal{F}-a)^m$ share (α, w), where $w \in \mathbb{N} \cup \{0\}$ and one of the following conditions is satisfied:*
(1) $n > max\Big\{k^*, \Big(1 - \delta(\infty, f)\Big)\left(\frac{11}{t}\right) + \left(\frac{7}{s}\right) + max(0, 11-m)\Big\}$ *when* $w = 0$;
(2) $n > max\Big\{k^*, \Big(1 - \delta(\infty, f)\Big)\left(\frac{6}{t}\right) + \left(\frac{9}{2s}\right) + max(0, 6-m)\Big\}$ *when* $w = 1$;
(3) $n > max\Big\{k^*, \Big(1 - \delta(\infty, f)\Big)\left(\frac{5}{t}\right) + \left(\frac{4}{s}\right) + max(0, 5-m)\Big\}$ *when* $w \geq 2$,
where $k^* = \frac{2m(t+1)}{t} - (m-1)$, then one of the following two cases holds:
(I) $f = d\mathcal{F}$ for a constant d such that $d^u = 1$, where $u = gcd(n+m+1, n+m, ..., n+1)$,
(II) f and $\mathcal{F}$ satisfy the algebraic equation $R(f, \mathcal{F}) = 0$, where

$$R(\omega_1, \omega_2) =$$
$$\omega_1^{n+1}\left(\frac{\omega_1^m}{m+n+1} - ma\frac{\omega_1^{m-1}}{m+n} + \frac{m(m-1)a^2}{2}\frac{\omega_1^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m a^m}{n+1}\right)$$
$$- \omega_2^{n+1}\left(\frac{\omega_2^m}{m+n+1} - ma\frac{\omega_2^{m-1}}{m+n} + \frac{m(m-1)a^2}{2}\frac{\omega_2^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m a^m}{n+1}\right).$$

In particular, if $m = 2$ and $a = 1$, then $f = \mathcal{F}$.

2. **Lemmas**

In order to prove our results, we need the following lemmas.

Lemma 2.1. [28] *Let $\mathcal{F}$ be a L-function with degree d. Then*

$$T(r,\mathcal{F}) = \frac{d}{\pi} r log r + O(r).$$

Lemma 2.2. [17] *Let $\mathcal{F}$ be a L-function. Then $N(r,\infty;\mathcal{F}) = N(r,\mathcal{F}) = S(r,\mathcal{F}) = O(log r)$.*

Lemma 2.3. [32] *Let $f \in \mathcal{M}(\mathbb{C})$ and $P(f) = b_n f^n + b_{n-1} f^{n-1} + \cdots + b_0$, where $b_j \in \mathcal{M}_f(\mathbb{C})$, $(j = 0, 1, \cdots, n)$, $b_n \not\equiv 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.4. [32] *Suppose that $f \in \mathcal{M}(\mathbb{C})$ and k is a positive integer. Then*

$$Z(r, f^{(k)}) \leq Z(r, f) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.5. [3] *Let $f \in \mathcal{M}(\mathbb{C})$ be non-constant meromorphic function. Then*

$$T(r, f) - Z(r, f) \leq T(r, f') - Z(r, f') + S(r, f).$$

Lemma 2.6. [3] *Let $P'(x) = b(x-a_1)^n \prod_{i=2}^{l} (x-a_i)^{k_i} \in \mathbb{C}[x]$, $(a_i \neq a_j,\ for\, i \neq j)$ with $b \in \mathbb{C}\backslash\{0\}$, $l \geq 2$ and let $k = \sum_{i=2}^{l} k_i$. Suppose that $f,\ g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\theta = P'(f)f'P'(g)g'$. If $\theta \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$, then we have the following:*
if $l = 2$, then $n \in \{k, k+1, 2k, 2k+1, 3k+1\}$;
if $l = 3$, then $n \in \{\frac{k}{2}, k+1, 2k+1, 3k_2 - k, 3k_3 - k\}$;
if $l \geq 4$, then $n = k+1$.
In particular, if $f, g \in \mathcal{A}(\mathbb{C})$, then θ does not belong to $\mathcal{A}_f(\mathbb{C})$.

Lemma 2.7. *Let f be a non-constant meromorphic function, $\mathcal{F}$ be a L-function such that 0 is not a Picard's exeptional value of f and $\mathcal{F}$. Let n, m be positive integers and $P_1(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ be a polynomial of degree m with $a_0(\neq 0), a_1, \cdots, a_m(\neq 0)$ being complex constants and $\gamma(\leq m)$ be the number of distinct roots of the equation $P_1(z) = 0$. Suppose that $f,\ \mathcal{F} \in \mathcal{M}(\mathbb{C})$ whose poles have multiplicities at least t and $\alpha(\not\equiv 0) \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_{\mathcal{F}}(\mathbb{C})$, such that $n > k_1^*$, where $k_1^* = \frac{2m(t+1)}{t\gamma} - (m-1)$ then $f' f^n P_1(f) \mathcal{F}' \mathcal{F}^n P_1(\mathcal{F}) \not\equiv \alpha^2$.*

Proof. Suppose that

$$f' f^n P_1(f) \mathcal{F}' \mathcal{F}^n P_1(\mathcal{F}) \equiv \alpha^2. \tag{2.1}$$

Let v_i be the distinct zeros of $P_1(z) = 0$ with multiplicity p_i, where $i = 1, 2, \cdots, \gamma$, $1 \leq \gamma \leq m$ and $\sum_{i=1}^{\gamma} p_i = m$.
Now by the Second Fundamental Theorem for f and $\mathcal{F}$ we get respectively

$$\gamma T(r, f) \leq \overline{Z}(r, f) + \overline{N}(r, f) + \sum_{i=1}^{\gamma} \overline{Z}(r, v_i; f) - \overline{Z}_0(r, f') + S(r, f) \tag{2.2}$$

and

$$\gamma T(r, \mathcal{F}) \leq \overline{Z}(r, \mathcal{F}) + \sum_{i=1}^{\gamma} \overline{Z}(r, v_i; \mathcal{F}) - \overline{Z}_0(r, \mathcal{F}') + S(r, \mathcal{F}), \tag{2.3}$$

where $\overline{Z}(r, v_i; f)$ denotes the reduced counting function of zeros of $f - v_i$, $(i = 1, 2, \cdots, \gamma)$ and $\overline{Z}_0(r, f')$ denotes the reduced counting function of those zeros of f' which are not the zeros of f and $f - v_i$, $(i = 1, 2, \cdots, \gamma)$ and $\overline{Z}(r, v_i; \mathcal{F})$, $\overline{Z}_0(r, \mathcal{F}')$ are similarly defined.

Let z_0 be a zero of f with multiplicity p such that $\alpha(z_0) \neq 0$ or ∞. Clearly z_0 must be a pole of $\mathcal{F}$ with multiplicity q. Then from (2.1), we get $np+p-1 = nq+mq+q+1$. This gives

$$(n+1)(p-q) = mq+2. \tag{2.4}$$

From (2.4), we get $p-q \geq 1$ and hence $q \geq \frac{n-1}{m}$. Now, $np+p-1 = nq+mq+q+1$ gives $p \geq \frac{n+m-1}{m}$.

Thus, we have

$$\overline{Z}(r,f) \leq \frac{m}{n+m-1} Z(r,f) \leq \frac{m}{n+m-1} T(r,f). \tag{2.5}$$

Let z_1 be a zero of $f - v_i$ with multiplicity q_i $(i = 1, 2, \cdots, \gamma)$ such that $\alpha(z_1) \neq 0$ or ∞, then obviously z_1 must be a pole of $\mathcal{F}$ with multiplicity $r(\geq t)$. Then from (2.1), we get $q_i p_i + q_i - 1 = (n+m+1)r + 1 \geq (n+m+1)t + 1$.

This gives $q_i \geq \left[\frac{(n+m+1)t+2}{p_i+1}\right]$ for $i = 1, 2, \cdots, \gamma$ and hence, we get

$$\overline{Z}(r, v_i; f) \leq \frac{p_i+1}{(n+m+1)t+2} Z(r, v_i; f) \leq \frac{p_i+1}{(n+m+1)t+2} T(r,f).$$

Clearly,

$$\sum_{i=1}^{\gamma} \overline{Z}(r, v_i; f) \leq \frac{m+\gamma}{(n+m+1)t+2} T(r,f). \tag{2.6}$$

Similarly, we have

$$\overline{Z}(r,\mathcal{F}) \leq \frac{m}{n+m-1} T(r,\mathcal{F}) \tag{2.7}$$

and

$$\sum_{i=1}^{\gamma} \overline{Z}(r, v_i; \mathcal{F}) \leq \frac{m+\gamma}{(n+m+1)t+2} T(r,\mathcal{F}). \tag{2.8}$$

Also, it is clear from (2.1), (2.7) and (2.8) that

$$\begin{aligned} \overline{N}(r,f) &\leq \overline{Z}(r,\mathcal{F}) + \sum_{i=1}^{\gamma} \overline{Z}(r, v_i; \mathcal{F}) + \overline{Z}_0(r,\mathcal{F}') + S(r,f) + S(r,\mathcal{F}) \\ &\leq \left(\frac{m}{n+m-1} + \frac{m+\gamma}{(n+m+1)t+2}\right) T(r,\mathcal{F}) + \overline{Z}_0(r,\mathcal{F}') + S(r,f) + S(r,\mathcal{F}). \end{aligned} \tag{2.9}$$

Then from (2.2), (2.5), (2.6) and (2.9), we get

$$\begin{aligned} \gamma T(r,f) \leq &\left(\frac{m}{n+m-1} + \frac{m+\gamma}{(n+m+1)t+2}\right) \{T(r,f) + T(r,\mathcal{F})\} \\ &+ \overline{Z}_0(r,\mathcal{F}') - \overline{Z}_0(r,f') + S(r,f) + S(r,\mathcal{F}). \end{aligned} \tag{2.10}$$

Similarly from (2.3), (2.7) and (2.8), we get

$$\gamma T(r,\mathcal{F}) \leq \left(\frac{m}{n+m-1} + \frac{m+\gamma}{(n+m+1)t+2}\right) T(r,\mathcal{F}) - \overline{Z}_0(r,\mathcal{F}') + S(r,f) + S(r,\mathcal{F}). \tag{2.11}$$

By adding (2.10) and (2.11), we get

$$\begin{aligned}\gamma\{T(r,f)+T(r,\mathcal{F})\} \le & \left(\frac{m}{n+m-1}+\frac{m+\gamma}{(n+m+1)t+2}\right)T(r,f)-\overline{Z}_0(r,f')\\ & +2\left(\frac{m}{n+m-1}+\frac{m+\gamma}{(n+m+1)t+2}\right)T(r,\mathcal{F})\\ & +S(r,f)+S(r,\mathcal{F}). \qquad (2.12)\end{aligned}$$

Let $A=\left(\frac{m}{n+m-1}+\frac{m+\gamma}{(n+m+1)t+2}\right)$. Then (2.12) becomes,

$$\gamma\{T(r,f)+T(r,\mathcal{F})\} \le A\Big(T(r,f)\Big)+2A\Big(T(r,\mathcal{F})\Big)+S(r,f)+S(r,\mathcal{F}).$$

This implies

$$\gamma\{T(r,f)+T(r,\mathcal{F})\} < 2A\Big(T(r,f)\Big)+2A\Big(T(r,\mathcal{F})\Big)+S(r,f)+S(r,\mathcal{F}).$$

This implies

$$(\gamma-2A) < \frac{S(r,f)+S(r,\mathcal{F})}{T(r,f)+T(r,\mathcal{F})}. \qquad (2.13)$$

We note that when $n+m-1>\left[\frac{2m(t+1)}{t\gamma}\right]$, implies $(n+m+1)t+2>\left[\frac{2(m+\gamma)(t+1)}{\gamma}\right]$, *i.e.*, when $n>k_1^*$, where $k_1^*=\left[\frac{2m(t+1)}{t\gamma}\right]-(m-1)$, then clearly $(\gamma-2A)>0$. Thus, (2.13) leads to a contradiction.
This completes the proof of Lemma 2.7. □

Now, by changing the notation $N(r,0;f)$ by $Z(r,f)$ for zeros of f, $N(r,\infty;f)$ by $N(r,f)$ for poles of f (similarly for g) and replacing the meromorphic function g by $\mathcal{F}$, we can obtain the following lemmas by a similar argument as in the proof of Theorem 1 in [13] and Lemmas 2.14 and 2.15 in [2] respectively.

Lemma 2.8. *Let f, $\mathcal{F}\in\mathcal{M}(\mathbb{C})$ and let $a\in\mathcal{M}_f(\mathbb{C})\cap\mathcal{M}_{\mathcal{F}}(\mathbb{C})$ be such that $a(z)\not\equiv 0,\infty$. If f and $\mathcal{F}$ share $(a,2)$ then one of the following conditions holds:*

(1) $T(r,f)\le Z_{[2]}(r,f)+Z_{[2]}(r,\mathcal{F})+N_{[2]}(r,f)+N_{[2]}(r,\mathcal{F})+S(r,f)+S(r,\mathcal{F})$ *and a similar inequality holds for $\mathcal{F}$;*
(2) $f\mathcal{F}=a^2$;
(3) $f=\mathcal{F}$.

Lemma 2.9. *Let f, $\mathcal{F}\in\mathcal{M}(\mathbb{C})$ and let $a\in\mathcal{M}_f(\mathbb{C})\cap\mathcal{M}_{\mathcal{F}}(\mathbb{C})$ be such that $a(z)\not\equiv 0,\infty$. If f and $\mathcal{F}$ share $(a,1)$ then one of the following conditions holds:*

(1) $T(r,f)\le Z_{[2]}(r,f)+Z_{[2]}(r,\mathcal{F})+N_{[2]}(r,f)+N_{[2]}(r,\mathcal{F})+\frac{1}{2}\overline{Z}(r,f)+\frac{1}{2}\overline{N}(r,f)+S(r,f)+S(r,\mathcal{F})$ *and a similar inequality holds for $\mathcal{F}$;*
(2) $f\mathcal{F}=a^2$;
(3) $f=\mathcal{F}$.

Lemma 2.10. *Let f, $\mathcal{F}\in\mathcal{M}(\mathbb{C})$ and let $a\in\mathcal{M}_f(\mathbb{C})\cap\mathcal{M}_{\mathcal{F}}(\mathbb{C})$ be such that $a(z)\not\equiv 0,\infty$. If f and $\mathcal{F}$ share $(a,0)$ then one of the following conditions holds:*

(1) $T(r,f)\le Z_{[2]}(r,f)+Z_{[2]}(r,\mathcal{F})+N_{[2]}(r,f)+N_{[2]}(r,\mathcal{F})+2\overline{Z}(r,f)+\overline{Z}(r,\mathcal{F})+2\overline{N}(r,f)+\overline{N}(r,\mathcal{F})+S(r,f)+S(r,\mathcal{F})$ *and a similar inequality holds for $\mathcal{F}$;*
(2) $f\mathcal{F}=a^2$;
(3) $f=\mathcal{F}$.

By following the proof of Lemma 6 in [7], we can get the following Lemma.

Lemma 2.11. *Let f, $\mathcal{F} \in \mathcal{M}(\mathbb{C})$ be non-constant and let $n \geq 6$ be an integer. If*

$$\frac{(n-1)(n-2)}{2} f^n - n(n-2) f^{n-1} + \frac{n(n-1)}{2} f^{n-2} = \frac{(n-1)(n-2)}{2} \mathcal{F}^n - n(n-2) \mathcal{F}^{n-1} + \frac{n(n-1)}{2} \mathcal{F}^{n-2},$$

then $f = \mathcal{F}$.

3. Proof of Theorems

3.1. Proof of Theorem 1.1.

Proof. Let $\Phi = P(f)$ and $\Psi = P(\mathcal{F})$. We may assume without loss of generality that $a_1 = 0$. Then clearly Φ' and Ψ' share (α, w). From the definitions of Φ and Ψ it follows that

$$\begin{aligned} Z_{[2]}(r, \Phi') &+ Z_{[2]}(r, \Psi') + N_{[2]}(r, \Phi') + N_{[2]}(r, \Psi') \\ &\leq 2\overline{Z}(r, f) + 2\sum_{i=2}^{l} \overline{Z}(r, f - a_i) + Z(r, f') + 2\overline{Z}(r, \mathcal{F}) + 2\sum_{i=2}^{l} \overline{Z}(r, \mathcal{F} - a_i) \\ &\quad + Z(r, \mathcal{F}') + 2\overline{N}(r, f) + 2\overline{N}(r, \mathcal{F}) + S(r, f) + S(r, \mathcal{F}) \\ &\leq \frac{2}{s} Z(r, f) + 2\sum_{i=2}^{l} \overline{Z}(r, f - a_i) + Z(r, f') + \frac{2}{s}(r, \mathcal{F}) + 2\sum_{i=2}^{l} \overline{Z}(r, \mathcal{F} - a_i) \\ &\quad + Z(r, \mathcal{F}') + \frac{2}{t} N(r, f) + S(r, f) + S(r, \mathcal{F}). \end{aligned} \tag{3.1}$$

Noting the proof of Theorem 3 in [3] for the fact that $P(x) = x^{n+1}Q(x)$, where $Q(x)$ is some polynomial of degree k and using Lemma 2.5, we get

$$\begin{aligned} T(r, \Phi) &\leq T(r, \Phi') + Z(r, f^{n+1}Q(f)) - Z(r, f'P'(f)) + S(r, f) \\ &\leq T(r, \Phi') + (n+1)Z(r, f) + Z(r, Q(f)) - nZ(r, f) - \sum_{i=2}^{l} k_i Z(r, f - a_i) \\ &\quad - Z(r, f') + S(r, f) \\ &\leq T(r, \Phi') + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^{l} k_i Z(r, f - a_i) - Z(r, f') + S(r, f). \end{aligned} \tag{3.2}$$

Similarly, we get

$$T(r, \Psi) \leq T(r, \Psi') + Z(r, \mathcal{F}) + Z(r, Q(\mathcal{F})) - \sum_{i=2}^{l} k_i Z(r, \mathcal{F} - a_i) - Z(r, \mathcal{F}') + S(r, \mathcal{F}). \tag{3.3}$$

Now the following three cases can be distinguished:

Case 1. Let $w = 0$. Now the following three subcases needs to be considered:

Subcase 1.1. Suppose if (1) of Lemma 2.10 holds. Then

$$\begin{aligned} T(r, \Phi') \leq Z_{[2]}(r, \Phi') &+ Z_{[2]}(r, \Psi') + N_{[2]}(r, \Phi') + N_{[2]}(r, \Psi') + 2\overline{Z}(r, \Phi') \\ &+ \overline{Z}(r, \Psi') + 2\overline{N}(r, \Phi') + \overline{N}(r, \Psi') + S(r, \Phi') + S(r, \Psi'), \end{aligned} \tag{3.4}$$

and using Lemma 2.2, we have

$$T(r,\Psi') \leq Z_{[2]}(r,\Psi') + Z_{[2]}(r,\Phi') + N_{[2]}(r,\Psi') + N_{[2]}(r,\Phi') + 2\overline{Z}(r,\Psi') + \overline{Z}(r,\Phi') + \overline{N}(r,\Phi') + S(r,\Phi') + S(r,\Psi'). \tag{3.5}$$

From Lemma 2.4, we have

$$Z(r,f')+Z(r,\mathcal{F}') \leq Z(r,f-a_2)+\frac{1}{t}N(r,f)+Z(r,\mathcal{F}-a_2)+S(r,f)+S(r,\mathcal{F}). \tag{3.6}$$

Also, we have

$$Z(r,Q(f)) \leq kT(r,f) + S(r,f) \quad \text{and} \quad Z(r,Q(\mathcal{F})) \leq kT(r,\mathcal{F}) + S(r,\mathcal{F}). \tag{3.7}$$

Using (3.1), (3.2) and (3.4), we obtain

$$T(r,\Phi) \leq \left(1+\frac{4}{s}\right) Z(r,f) + \frac{3}{s}Z(r,\mathcal{F}) + \sum_{i=2}^{l}(4-k_i)Z(r,f-a_i) + 3\sum_{i=2}^{l} Z(r,\mathcal{F}-a_i) + 2Z(r,\mathcal{F}') + \frac{4}{t}N(r,f) + 2Z(r,f') + Z(r,Q(f)) + S(r,f) + S(r,\mathcal{F}). \tag{3.8}$$

Similarly, using (3.1), (3.3) and (3.5), we have

$$T(r,\Psi) \leq \left(1+\frac{4}{s}\right) Z(r,\mathcal{F}) + \frac{3}{s}Z(r,f) + \sum_{i=2}^{l}(4-k_i)Z(r,\mathcal{F}-a_i) + 3\sum_{i=2}^{l} Z(r,f-a_i) + 2Z(r,f') + \frac{3}{t}N(r,f) + 2Z(r,\mathcal{F}') + Z(r,Q(\mathcal{F})) + S(r,f) + S(r,\mathcal{F}). \tag{3.9}$$

Both Φ and Ψ are polynomials of degree $n+k+1$ in f and $\mathcal{F}$ respectively. Thus, by adding (3.8) and (3.9) and using (3.6), (3.7), Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned}
&(n+k+1)\{T(r,f)+T(r,\mathcal{F})\} \\
&\leq \left(1+\frac{7}{s}\right)\Big\{Z(r,f)+Z(r,\mathcal{F})\Big\} + \sum_{i=3}^{l}(7-k_i)\Big\{Z(r,f-a_i)+Z(r,\mathcal{F}-a_i)\Big\} \\
&\quad + \frac{7}{t}N(r,f) + 4\Big\{Z(r,f-a_2)+\overline{N}(r,f)+Z(r,\mathcal{F}-a_2)\Big\} + k\Big\{T(r,f)+T(r,\mathcal{F})\Big\} \\
&\quad + (7-k_2)\Big\{Z(r,f-a_2)+Z(r,\mathcal{F}-a_2)\Big\} + S(r,f)+S(r,\mathcal{F}).
\end{aligned}$$

This implies

$$\left[n-\frac{7}{s}-\sum_{i=3}^{l} max(0,7-k_i) - max(0,11-k_2)\right]\Big(T(r,f)+T(r,\mathcal{F})\Big) \leq S(r,f)+S(r,\mathcal{F})+\frac{11}{t}N(r,f),$$

since $\frac{N(r,f)}{T(r,f)+T(r,\mathcal{F})} \leq \frac{N(r,f)}{T(r,f)}$, we have

$$\left[n-\frac{7}{s}-\sum_{i=3}^{l} max(0,7-k_i) - max(0,11-k_2)\right] \leq \frac{S(r,f)+S(r,\mathcal{F})}{T(r,f)+T(r,\mathcal{F})} + \frac{11}{t}\Big(1-\delta(\infty,f)\Big),$$

which contradicts,

$$n > \Big[\Big(1-\delta(\infty,f)\Big)\left(\frac{11}{t}\right)+\left(\frac{7}{s}\right)+\sum_{i=3}^{l} max(0,7-k_i)+max(0,11-k_2)\Big].$$

Subcase 1.2. Suppose if (2) of Lemma 2.10 holds. Then $\Phi'\Psi' = \alpha^2$, a contradiction by Lemma 2.6.
Subcase 1.3. Suppose if (3) of Lemma 2.10 holds. Then $\Psi' = \Phi'$. From this we deduce that $\Psi = \Phi + c_1$, for some constant c_1 and hence $T(r,\mathcal{F}) = T(r,f) + S(r,\mathcal{F})$. We claim that $c_1 = 0$. If not, then we have $\Phi = \Psi - c_1$. By Nevanlinna's Second Fundamental Theorem and (3.7), we get

$$\begin{aligned} T(r,\Psi) &\leq \overline{N}(r,\Psi) + \overline{Z}(r,\Psi) + \overline{Z}(r,\Psi - c_1) + S(r,\mathcal{F}) \\ &\leq \overline{Z}(r,\mathcal{F}) + \overline{Z}(r,Q(\mathcal{F})) + \overline{Z}(r,f) + \overline{Z}(r,Q(f)) + S(r,\mathcal{F}) \\ &\leq \left(2k + \frac{2}{s}\right) T(r,\mathcal{F}) + S(r,\mathcal{F}), \end{aligned}$$

i.e., $\left[n - \left(k - 1 + \frac{2}{s}\right)\right] T(r,\mathcal{F}) \leq S(r,\mathcal{F})$, a contradiction as $n > k+2$ and $k+2 > k-1+\frac{2}{q}$. This proves our claim and so $P(f) = P(\mathcal{F})$. Hence $P(z)$ being a polynomial of uniqueness gives $f = \mathcal{F}$.
Case 2. Let $w = 1$. By using the Lemma 2.9 and considering the three subcases just like Case 1 and following the proof in a similar manner we can arrive at the required conclusion.
Case 3. Let $w \geq 2$. By using the Lemma 2.8 and considering the three subcases just like Case 1 and following the proof in a similar manner we can arrive at the required conclusion.
This completes the proof of Theorem 1.1. □

3.2. **Proof of Theorem 1.2.** Since all $k_i's$ are equal to 1, the proof follows exactly in a similar manner to the proof of Theorem 1.1 by taking $Z_{[2]}(r,\Phi') \leq 2\overline{Z}(r,f) + \sum_{i=2}^{l} Z(r,f-a_i) + Z(r,f')$ (Similarly for $Z_{[2]}(r,\Psi')$) and by taking $Z(r,f') \leq Z(r,f) + \overline{N}(r,f)$.

3.3. **Proof of Theorem 1.3.**

Proof. As in the proof of Theorem 5 in [3], suppose $P(x) \in \mathbb{C}[x]$ is a polynomial such that $P'(x) = x^n(x-a)$. Let $\Phi = P(f)$ and $\Psi = P(\mathcal{F})$. Then Φ' and Ψ' share (α, w). Then by a similar argument as in Theorem 1.1, we have

$$\begin{aligned} &Z_{[2]}(r,\Phi') + Z_{[2]}(r,\Psi') + N_{[2]}(r,\Phi') + N_{[2]}(r,\Psi') \\ &\qquad \leq 2\overline{Z}(r,f) + \overline{Z}(r,f-a) + Z(r,f') + 2\overline{Z}(r,\mathcal{F}) + \overline{Z}(r,\mathcal{F}-a) \\ &\qquad\quad + Z(r,\mathcal{F}') + 2\overline{N}(r,f) + 2\overline{N}(r,\mathcal{F}) + S(r,f) + S(r,\mathcal{F}) \\ &\qquad \leq \frac{2}{s} Z(r,f) + \overline{Z}(r,f-a) + Z(r,f') + \frac{2}{s}(r,\mathcal{F}) + \overline{Z}(r,\mathcal{F}-a) \\ &\qquad\quad + Z(r,\mathcal{F}') + \frac{2}{t} N(r,f) + S(r,f) + S(r,\mathcal{F}). \end{aligned} \tag{3.10}$$

Using Lemma 2.5 and noting that $P(x) = x^{n+1}Q^*(x)$ for some polynomial $Q^*(x) \in \mathbb{C}[x]$ with $deg(Q^*) = 1$, we get

$$\begin{aligned} T(r,\Phi) &\leq T(r,\Phi') + Z(r, f^{n+1}Q^*(f)) - Z(r, f'P'(f)) + S(r,f) \\ &\leq T(r,\Phi') + (n+1)Z(r,f) + Z(r,Q^*(f)) - nZ(r,f) - Z(r,f-a) \\ &\quad - Z(r,f') + S(r,f) \\ &\leq T(r,\Phi') + Z(r,f) + Z(r,Q^*(f)) - Z(r,f-a) - Z(r,f') + S(r,f). \end{aligned} \tag{3.11}$$

Similarly, we get

$$T(r,\Psi) \leq T(r,\Psi') + Z(r,\mathcal{F}) + Z(r,Q^*(\mathcal{F})) - Z(r,\mathcal{F}-a) - Z(r,\mathcal{F}') + S(r,\mathcal{F}). \tag{3.12}$$

We now distinguish the following three cases:
Case 4. Let $w = 0$. Now we consider the following three subcases:
Subcase 4.1. Suppose that (1) of Lemma 2.10 holds. Then

$$\begin{aligned} T(r,\Phi') \leq\ & Z_{[2]}(r,\Phi') + Z_{[2]}(r,\Psi') + N_{[2]}(r,\Phi') + N_{[2]}(r,\Psi') + 2\overline{Z}(r,\Phi') \\ &+ \overline{Z}(r,\Psi') + 2\overline{N}(r,\Phi') + \overline{N}(r,\Psi') + S(r,\Phi') + S(r,\Psi'), \end{aligned} \tag{3.13}$$

and using Lemma 2.2, we have

$$\begin{aligned} T(r,\Psi') \leq\ & Z_{[2]}(r,\Psi') + Z_{[2]}(r,\Phi') + N_{[2]}(r,\Psi') + N_{[2]}(r,\Phi') + 2\overline{Z}(r,\Psi') \\ &+ \overline{Z}(r,\Phi') + \overline{N}(r,\Phi') + S(r,\Phi') + S(r,\Psi'). \end{aligned} \tag{3.14}$$

From Lemma 2.4, we have

$$Z(r,f') + Z(r,\mathcal{F}') \leq Z(r,f-a) + \frac{1}{t}N(r,f) + Z(r,\mathcal{F}-a) + S(r,f) + S(r,\mathcal{F}). \tag{3.15}$$

Also, we have

$$Z(r,Q^*(f)) \leq T(r,f) + S(r,f) \quad \text{and} \quad Z(r,Q^*(\mathcal{F})) \leq T(r,\mathcal{F}) + S(r,\mathcal{F}). \tag{3.16}$$

Using (3.10), (3.11) and (3.13), we obtain

$$\begin{aligned} T(r,\Phi) \leq\ & \left(1 + \frac{4}{s}\right) Z(r,f) + \frac{3}{s}Z(r,\mathcal{F}) + 2Z(r,f-a) + 2Z(r,\mathcal{F}-a) \\ &+ 2Z(r,\mathcal{F}') + 2Z(r,f') + \frac{4}{t}N(r,f) + Z(r,Q^*(f)) + S(r,f) + S(r,\mathcal{F}). \end{aligned} \tag{3.17}$$

Similarly, using (3.10), (3.12) and (3.14), we have

$$\begin{aligned} T(r,\Psi) \leq\ & \left(1 + \frac{4}{s}\right) Z(r,\mathcal{F}) + \frac{3}{s}Z(r,f) + 2Z(r,\mathcal{F}-a) + 2Z(r,f-a) \\ &+ 2Z(r,f') + 2Z(r,\mathcal{F}') + \frac{3}{t}N(r,f) + Z(r,Q^*(\mathcal{F})) + S(r,f) + S(r,\mathcal{F}). \end{aligned} \tag{3.18}$$

Since Φ and Ψ are polynomials in f and $\mathcal{F}$ respectively and both of them are of degree $n+2$, by adding (3.17), (3.18) and using (3.15), (3.16), Lemma 2.2 and

Lemma 2.3 we get

$$\begin{aligned}(n+2)\Big\{T(r,f)+T(r,\mathcal{F})\Big\} \leq \left(1+\frac{7}{s}\right)\Big\{Z(r,f)+Z(r,\mathcal{F})\Big\}+\Big\{T(r,f)+T(r,\mathcal{F})\Big\}\\ +\frac{7}{t}N(r,f)+4\Big\{Z(r,f-a)+\overline{N}(r,f)+Z(r,\mathcal{F}-a)\Big\}\\ +4\Big\{Z(r,f-a)+Z(r,\mathcal{F}-a)\Big\}+S(r,f)+S(r,\mathcal{F}).\end{aligned}$$

This implies

$$\left[n-\frac{7}{s}-8\right]\Big(T(r,f)+T(r,\mathcal{F})\Big) \leq S(r,f)+S(r,\mathcal{F})+\frac{11}{t}N(r,f),$$

since $\frac{N(r,f)}{T(r,f)+T(r,\mathcal{F})} \leq \frac{N(r,f)}{T(r,f)}$, thus we have

$$\left[n-\frac{7}{s}-8\right] \leq \frac{S(r,f)+S(r,\mathcal{F})}{T(r,f)+T(r,\mathcal{F})}+\frac{11}{t}\Big(1-\delta(\infty,f)\Big),$$

which contradicts $n > \left[8+(1-\delta(\infty,f))\left(\frac{11}{t}\right)+\left(\frac{7}{s}\right)\right]$.

Subcase 4.2. Next we assume that (2) of Lemma 2.10 holds. Then $\Phi'\Psi' = \alpha^2$, *i.e.*, $f^n(f-a)f'\mathcal{F}^n(\mathcal{F}-a)\mathcal{F}' = \alpha^2$. Then by Lemma 2.7 we arrive at a contradiction as $n > \left[8+\Big(1-\delta(\infty,f)\Big)\left(\frac{11}{t}\right)+\left(\frac{7}{s}\right)\right]$.

Subcase 4.3. Suppose (3) of Lemma 2.10 holds. Then $\Psi' = \Phi'$. From this we deduce that $\Psi = \Phi + c_2$, for some constant c_2 and hence $T(r,\mathcal{F}) = T(r,f)+S(r,\mathcal{F})$. We claim that $c_2 = 0$. If not, then we have $\Phi = \Psi - c_2$. By Nevanlinna's Second Fundamental Theorem and (3.16), we get

$$\begin{aligned}T(r,\Psi) &\leq \overline{N}(r,\Psi)+\overline{Z}(r,\Psi)+\overline{Z}(r,\Psi-c_2)+S(r,\mathcal{F})\\ &\leq \overline{Z}(r,\mathcal{F})+\overline{Z}(r,Q^*(\mathcal{F}))+\overline{Z}(r,f)+\overline{Z}(r,Q^*(f))+S(r,\mathcal{F})\\ &\leq \left(2+\frac{2}{s}\right)T(r,\mathcal{F})+S(r,\mathcal{F}),\end{aligned}$$

i.e., $\left[n-\frac{2}{s}\right]T(r,\mathcal{F}) \leq S(r,\mathcal{F})$, a contradiction as $n > \left[8+(1-\delta(\infty,f))\left(\frac{11}{t}\right)+\left(\frac{7}{s}\right)\right]$. This proves our claim and so we have $P(f) = P(\mathcal{F})$. Let $f = h\mathcal{F}$. If $h = 1$, we have $f = \mathcal{F}$. Otherwise, $P(f) = P(\mathcal{F})$ gives

$$\begin{aligned}\frac{f^{n+2}}{n+2}-a\frac{f^{n+1}}{n+1} &= \frac{\mathcal{F}^{n+2}}{n+2}-a\frac{\mathcal{F}^{n+1}}{n+1},\\ \textit{i.e.,}\quad f^{n+1}\left(f-a\frac{n+2}{n+1}\right) &= \mathcal{F}^{n+1}\left(\mathcal{F}-a\frac{n+2}{n+1}\right).\end{aligned}$$

From the above relation it follows that

$$f = \frac{a(n+2)}{n+1}\frac{(1-h^{n+1})}{(1-h^{n+2})}h \quad \text{and} \quad \mathcal{F} = \frac{a(n+2)}{n+1}\frac{(1-h^{n+1})}{(1-h^{n+2})}.$$

Case 5. Let $w = 1$. By using the Lemma 2.9 and considering the three subcases just like Case 4 and following the proof in a similar manner we can arrive at the required conclusion.

Case 6. Let $w \geq 2$. By using the Lemma 2.8 and considering the three subcases just like Case 4 and following the proof in a similar manner we can arrive at the required conclusion.

This completes the proof of Theorem 1.3. □

3.4. **Proof of Theorem 1.4.**

Proof. Let

$$P(x)=\frac{x^{m+n+1}}{m+n+1}-ma\frac{x^{m+n}}{m+n}+\frac{m(m-1)a^2}{2}\frac{x^{m+n-1}}{m+n-1}-\cdots+\frac{(-1)^m a^m x^{n+1}}{n+1}, \tag{3.19}$$

such that $P'(x)=x^n(x-a)^m$, where $m\geq 2$. Let $\Phi=P(f)$ and $\Psi=P(\mathcal{F})$. Then Φ' and Ψ' share (α,w). Note that $P(x)=x^{n+1}Q_1(x)$ of degree m. Then we consider the following three cases.

Case 7. Let $w=0$. Then we consider the following three subcases as in Theorem 1.1.

Subcase 7.1. Suppose that (1) of Lemma 2.10 holds. Then

$$\begin{aligned} T(r,\Phi') \leq\ & Z_{[2]}(r,\Phi')+Z_{[2]}(r,\Psi')+N_{[2]}(r,\Phi')+N_{[2]}(r,\Psi')+2\overline{Z}(r,\Phi') \\ & +\overline{Z}(r,\Psi')+2\overline{N}(r,\Phi')+\overline{N}(r,\Psi')+S(r,\Phi')+S(r,\Psi'), \end{aligned} \tag{3.20}$$

and using Lemma 2.2, we have

$$\begin{aligned} T(r,\Psi') \leq\ & Z_{[2]}(r,\Psi')+Z_{[2]}(r,\Phi')+N_{[2]}(r,\Psi')+N_{[2]}(r,\Phi')+2\overline{Z}(r,\Psi') \\ & +\overline{Z}(r,\Phi')+\overline{N}(r,\Phi')+S(r,\Phi')+S(r,\Psi'). \end{aligned} \tag{3.21}$$

From Lemma 2.4, we have

$$Z(r,f')+Z(r,\mathcal{F}')\leq Z(r,f-a)+\frac{1}{t}N(r,f)+Z(r,\mathcal{F}-a)+S(r,f)+S(r,\mathcal{F}). \tag{3.22}$$

Also we have

$$Z(r,Q_1(f))\leq mT(r,f)+S(r,f) \quad \text{and} \quad Z(r,Q_1(\mathcal{F}))\leq mT(r,\mathcal{F})+S(r,\mathcal{F}). \tag{3.23}$$

Then by a similar argument as in Subcase 1.1 of Case 1 of Theorem 1.1, we get a contradiction as $n>\left[(1-\delta(\infty,f))\left(\frac{11}{t}\right)+\left(\frac{7}{s}\right)+max(0,11-m)\right]$.

Subcase 7.2. Next we assume that (2) of Lemma 2.10 holds. Then $\Phi'\Psi'=\alpha^2$. By Lemma 2.7 we arrive at a contradiction as $n>\left[\frac{2m(t+1)}{t}\right]-(m-1)$.

Subcase 7.3. Suppose (3) of Lemma 2.10 holds. Then $\Psi'=\Phi'$. From this we deduce that $\Psi=\Phi+c_3$. We claim that $c_3=0$, if not then proceeding like Subcase 1.3 of Case 1 of Theorem 1.1,, we arrive at a contradiction as $n>\left[\frac{2m(t+1)}{t}\right]-(m-1)$ and $\left[\frac{2m(t+1)}{t}\right]-(m-1)>\left(\frac{2}{s}\right)+(m-1)$. Let $h=\frac{f}{\mathcal{F}}$. If h is constant, then using (3.19) we get

$$b_m g^m(h^{m+n+1}-1)+b_{m-1}g^{m-1}(h^{m+n}-1)+\cdots+b_0(h^{n+1}-1)\equiv 0,$$

where $b_m=\frac{1}{m+n+1}$, $b_{m-1}=\frac{-ma}{m+n},...,b_0=\frac{(-1)^m a^m}{n+1}$ with $a\neq 0$, which implies that $h^u=1$, where $u=gcd(m+n+1,m+n,...,n+1)$. Thus $f=d\mathcal{F}$, for a constant d such that $d^u=1$. If h is not a constant, then f and $\mathcal{F}$ satisfy the algebraic equation $R(f,\mathcal{F})=0$, where

$$\begin{aligned} R(\omega_1,\omega_2)= & \\ & \omega_1^{n+1}\left(\frac{\omega_1^m}{m+n+1}-ma\frac{\omega_1^{m-1}}{m+n}+\frac{m(m-1)a^2}{2}\frac{\omega_1^{m-2}}{m+n-1}-\cdots+\frac{(-1)^m a^m}{n+1}\right) \\ & -\omega_2^{n+1}\left(\frac{\omega_2^m}{m+n+1}-ma\frac{\omega_2^{m-1}}{m+n}+\frac{m(m-1)a^2}{2}\frac{\omega_2^{m-2}}{m+n-1}-\cdots+\frac{(-1)^m a^m}{n+1}\right). \end{aligned}$$

Moreover, by Lemma 2.11, we get $f = \mathcal{F}$, for $m = 2$ and $a = 1$.
Case 8. Let $w = 1$. By using the Lemma 2.9 and considering the three subcases just like Case 7 and following the proof in a similar manner we can arrive at the required conclusion.
Case 9. Let $w \geq 2$. By using the Lemma 2.8 and considering the three subcases just like Case 7 and following the proof in a similar manner we can arrive at the required conclusion.
This completes the proof of Theorem 1.4. □

4. Conclusion

L-functions are Selberg class functions with Riemann zeta function as the prototype and since L-functions are analytically continued as meromorphic functions, it is easy to study the value distribution and uniqueness problems on L-functions and arbitrary meromorphic functions. Also since L-functions neither have a pole nor zero at the origin, but is having only possible pole at $s = 1$ allows us to build upon and extend the previous results obtained by Boussaf et al. [3] and enables us to obtain uniqueness results on a L-function $\mathcal{F}$ and an arbitrary meromorphic function f, in the form $f'P'(f)$ and $\mathcal{F}'P'(\mathcal{F})$. Consideration of the concept of weighted sharing and least multiplicity further generalizes the results.

Acknowledgments

Authors are indebt to the editor and refrees for their careful reading and valuable suggestions which helped to improve the manuscript.

References

[1] T. T. H. An, J. T. Y. Wang, P. M. Wong. Strong uniqueness polynomials: the complex case. *Complex Var. Theory Appl.* 49(1) (2004), 25–54.
[2] A. Banerjee. Meromorphic functions sharing one value. *Int. J. Math. Math. Sci.* 22 (2005), 3587–3598.
[3] K. Boussaf, A. Escassut, J. Ojeda. Complex meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function. *Indag. Math. (N.S.)* 24(1) (2013), 15–41.
[4] A. Boutabaa, A. Escassut. Urs and ursims for p-adic meromorphic functions inside a disc. *Proc. Edinb. Math. Soc (2)* 44(3) (2001), 485–504.
[5] A. Escassut,J. Ojeda, C. C. Yang. Functional equations in a p-adic context. *J. Math. Anal. Appl.* 351(1) (2009), 350–359.
[6] M. L. Fang, X. H. Hua. Entire functions that share one value. *Nanjing Daxue Xuebao Shuxue Bannian Kan* 13(1) (1996), 44–48.
[7] M. L. Fang, H. Guo. On meromorphic functions sharing two values. *Analysis* 17(4) (1997), 355–366.
[8] M. L. Fang, W. Hong. A unicity theorem for entire functions concerning differential polynomials. *Indian J. Pure Appl. Math.* 32(9) (2001), 1343–1348.
[9] W.J. Hao, J. F. Chen. Uniqueness theorems for L-functions in the extended Selberg class. *Open Mathematics* 16(1) (2018), 1291-1299.
[10] W.K. Hayman. *Meromorphic functions.* Clarendon Press, Oxford, 1964.
[11] A. Kundu, A. Banerjee. Uniqueness of L function with special class of meromorphic function in the light of two shared sets. *Rend. Circ. Mat. Palermo (2)* 70(3) (2021), 1227–1244.
[12] I. Lahiri. Weighted sharing and uniqueness of meromorphic functions. *Nagoya Math. J.* 161 (2001), 193–206.
[13] I. Lahiri. Weighted value sharing and uniqueness of meromorphic functions. *Complex Variables Theory Appl.* 46(3) (2001), 241–253.
[14] I. Lahiri, N. Mandal. Uniqueness of nonlinear differential polynomials sharing simple and double 1-points. *Int. J. Math. Math. Sci.* 12 (2005), 1933–1942.

[15] P. Li, C. C. Yang. Some further results on the unique range sets of meromorphic functions. *Kodai Math. J.* 18(3) (1995), 437–450.
[16] W. Lin, H. X. Yi. Uniqueness theorems for meromorphic functions concerning fixed-points. *Complex Var. Theory Appl.* 49(11) (2004), 793–806.
[17] N. Mandal, N. K. Dattar. Uniqueness of L-function and its certain differential monomial concerning small functions. *J. Math. Comput. Sci.* 10(5) (2020), 2155-2163.
[18] N. Mandal, N. K. Datta. Polynomial Sharing and Uniqueness of Differential-Difference Polynomials of L-functions. *Advances in Dynamical Systems and Applications (ADSA)* 15(2) (2020), 171-185.
[19] N. T. Hoa. On the functional equation $P(f) = Q(g)$ in non-Archimedean field. *Acta Math. Vietnam.* 31(2) (2006), 167–180.
[20] J. Ojeda. Zeros of ultrametric meromorphic functions $f'f^n(f-a)^k - \alpha$. *Asian-Eur. J. Math.* 1(3) (2008), 415–429.
[21] J. Ojeda. Applications of the p-adic Nevanlinna theory to problems of uniqueness, advances in p-adic and non-Archimedean analysis, *Contemp. Math.* 508 (2010), 161–179.
[22] J. Ojeda. Uniqueness for ultrametric analytic functions. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* 54(102) (2011), no. 2, 153–165.
[23] P. N. Raj, H. P. Waghamore. Uniqueness results on L-functions and certain difference-differential polynomials. *J. Classical Anal.* 21(2) (2023), 153–171. doi:10.7153/jca-2023-21-09
[24] B. Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse. *Ges. Math. Werke und Wissenschaftlicher Nachlaß* 2(2) (1859), 145-155.
[25] P. Sahoo, S. Halder. Uniqueness results related to L-functions and certain differential polynomials. *Tbilisi Mathematical Journal* 11(4) (2018), 67-78.
[26] P. Sahoo, S. Halder. Results on L-functions and certain uniqueness questions of Gross. *Lithuanian Mathematical Journal* 60(1) (2020), 80-91.
[27] A. Selberg. Old and new conjectures and results about a class of Dirichlet series. *Proceedings of the Amalfi Conference on Analytic Number Theory* (Maiori, 1989), Univ. Salerno, Salerno, 1992, 367–385.
[28] J. Steuding. *Value-distribution of L-functions.* Lecture Notes in Mathematics, 1877, Springer, Berlin, 2007.
[29] J. T. Y. Wang. Uniqueness polynomials and bi-unique range sets for rational functions and non-Archimedean meromorphic functions. *Acta Arith.* 104(2) (2002), 183–200.
[30] C. C. Yang, X. H. Hua. Uniqueness and value-sharing of meromorphic functions. *Ann. Acad. Sci. Fenn. Math.* 22(2) (1997), 395–406.
[31] C. C. Yang, X. H. Hua. Unique polynomials of entire and meromorphic functions. *Mat. Fiz. Anal. Geom.* 4(3) (1997), 391–398.
[32] C. C. Yang, H. X. Yi. *Uniqueness theory of meromorphic functions.* Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.
[33] L. Yang. *Value distribution theory. Translated and revised from the 1982 Chinese original.* Springer-Verlag, Berlin; Science Press Beijing, Beijing, 1993.

Preetham N. Raj
Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore, Karnataka, India - 560 056
Email address: preethamnraj@gmail.com, preethamnraj@bub.ernet.in

Harina P. Waghamore
Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore, Karnataka, India - 560 056
Email address: harinapw@gmail.com