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VIETA-LUCAS POLYNOMIAL COMPUTATIONAL TECHNIQUE FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this study, we introduce a computational technique for tackling Volterra Integro-Differential Equations (VIDEs) using shifted Vieta-Lucas polynomials as the foundational basis functions. The approach involves adopting an approximative solution strategy through the utilization of Vieta-Lucas polynomials. These polynomials are then integrated into the pertinent VIDEs. Subsequently, the resulting equation is subjected to collocation at evenly spaced intervals, generating a system of linear algebraic equations with unspecified Vieta-Lucas coefficients. To solve this system, we employ a matrix inversion method to deduce the unknown constants. Once these constants are determined, they are incorporated into the earlier assumed approximate solution, thus yielding the sought-after approximated solution. To validate the accuracy and efficiency of this technique, we conducted numerical experiments. The obtained results underscore the outstanding performance of our method in comparison to outcomes found in existing literature. The precision and effectiveness of the approach are further illustrated through the utilization of tables.

1. Introduction

Integro-differential equations (IDEs) are types of mathematical equations that involve both derivatives and integrals. They arise in various fields of science and engineering, including physics, biology, economics, and finance, where systems exhibit memory or history-dependent behavior. Unlike Ordinary Differential Equations (ODEs) that involve only derivatives, integro-differential equations incorporate the influence of past values of the unknown function through the integration term. Integro-differential equations often appear in problems involving diffusion, propagation of waves, population dynamics, and control theory, among others. They provide a more realistic description of phenomena that exhibit memory effects or

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spatial interactions. Since most IDEs cannot be solved analytically, researchers have focused on developing numerical methods to obtain approximate solutions. Several authors have contributed to this area. For example, [1] employed the differential transform method, [2] used the Bernstein operational matrix approach, [3] applied the Chebyshev collocation method, [4] employed Lucas collocation method, and [5] introduced the reliable iterative method for Volterra-Fredholm IDEs. In [6], Euler polynomials with the least squares method are used to solve IDEs. For Fredholm-Volterra IDEs, various methods were utilized. [7]employed the projection method based on a Bernstein collocation approach; [8] used the Bernstein collocation method; [9] applied a fixed-point iterative algorithm; and [10] employed a collocation method based on Bernstein polynomials. In [11], a new numerical method was developed specifically for solving systems of Volterra IDEs. In [12], the Lucas polynomial is employed to solve nonlinear differential equations with variable delays. The use of third-kind Chebyshev polynomials for solving IDEs was examined in [13] and [14]. In [15] and [16], a Computational algorithm is used to find the solution of fractional Fredholm IDEs and Volterra-Fredholm IDEs. In [17], the homotopy perturbation approach is employed for Fractional Volterra and Fredholm IDEs. Other methods mentioned in this study include the quadrature-difference method [18], and Adomian's decomposition approach [19], which were used to solve Fredholm IDEs. Based on the works mentioned above, this study proposes a computational algorithm that utilizes shifted Legendre polynomials. This technique is inspired by previous research and aims to enhance the outcomes achieved by [14] and [6]. This work considers the Volterra integro-differential equation, which is represented in the following form:

$$\sum_{i=0}^{n} \rho_i(z) w^i(z) = f(z) + \lambda \int_0^z K(z,\nu) w(\nu) d\nu. \tag{1}$$

This equation is accompanied by initial conditions:

$$w^j(0) = w_j, j = 0, 1, 2, \ldots n-1. \tag{2}$$

In the above equations, j denotes the order of derivatives, and the functions K and $\rho_i(z)$ (where $i = 0, 1, 2, \ldots, n$) are known. It is important to note that $\rho_i(z)$ and λ are nonzero. Additionally, $f(z)$ is a known function, and $w^i(z)$ is the ith derivatives of the unknown function $w(z)$ that needs to be determined. Equation (1) is commonly referred to as a VIDE.

2. Material and Method

Definition 1

An integral equation is an equation that has an unknown function, $w(z)$, that appears under the integral sign. Standard integral equation have the following form:

$$w(z) = f(z) + \lambda \int_{h(z)}^{g(z)} K(z,t) w(t) dt.$$

where $K(z,t)$ is a function of two variables z and t known as the kernel or the nucleus of the integral equation, g(z) and $h(z)$ are the limits of integration, λ is a constant parameter.

Definition 2

The Vieta-Lucas polynomials exhibit orthogonality and are defined for $|\zeta| < 2$ as follows: $VL_n(\zeta) = 2\cos(n\theta)$, where $\theta = \cos^{-1}(\frac{\zeta}{2})$, $\theta \in [0, \pi]$

An explicit power series formula for $VL_n(\zeta)$ is given by:

$VL_n(\zeta) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil} (-1)^i \frac{n\Gamma(n-i)}{\Gamma(i+1)\Gamma(n+1-2i)} \zeta^{n-2i}$, $n \geq 2$ where $\lceil \frac{n}{2} \rceil$ denotes the ceiling function. The iterative formula for generating the Vieta-Lucas polynomials is: $VL_n(\zeta) = \zeta VL_{n-1}(\zeta) - \zeta VL_{n-2}(\zeta)$, $n \geq 2$, starting with $VL_0(\zeta) = 2$ and $VL_1(\zeta) = \zeta$.

Consequently, the initial Vieta-Lucas polynomials are as follows: $VL_0(\zeta) = 2$, $VL_1(\zeta) = \zeta$, $VL_2(\zeta) = \zeta^2 - 2, \ldots,$

These polynomials hold significance due to their orthogonality properties and find applications in various mathematical contexts.

Definition 3

The Shifted Vieta-Lucas polynomials of degree n on $[0, 1]$ can be obtained from $VL_n(\zeta)$ as demonstrated below: The Vieta-Lucas polynomials exhibit orthogonality and are defined for $|\zeta| < 2$ as follows:

$VL^*_n(\zeta) = VL_n(4\zeta - 2) = VL_{2n}(2\sqrt{\zeta})$

The expression for $VL^*_n(\zeta)$ can also be established using the explicit power series formula:

$VL^*_n(\zeta) = 2n \sum_{i=0}^{n} (-1)^i \frac{4^{n-i}\Gamma(2n-i)}{\Gamma(i+1)\Gamma(2n-2i+1)} \zeta^{n-i}$, $n \geq 2$

The iterative formula for generating the Shifted Vieta-Lucas polynomials is:

$VL^*_{n+1}(\zeta) = (4\zeta - 2)VL^*_n(\zeta) - VL^*_{n-1}(\zeta)$, $n \geq 1$, starting with $VL^*_0(\zeta) = 2$ and $VL^*_1(\zeta) = 4\zeta - 2$.

Hence, the initial Shifted Vieta-Lucas polynomials are as follows:

$VL^*_0(\zeta) = 2$, $VL^*_1(\zeta) = 4\zeta - 2$, $VL^*_2(\zeta) = 16\zeta^2 - 16\zeta + 2, \ldots,$

These polynomials exhibit distinct properties and applications, and they stem from the original Vieta-Lucas polynomials with appropriate shifts.

Definition 4

Collocation method: The collocation method is a numerical technique used for solving differential equations and other mathematical problems by transforming them into algebraic equations. In this method, a set of discrete points, known as collocation points, is chosen within the domain of the problem. The differential equation or mathematical problem is then evaluated at these collocation points, and the resulting equations are solved algebraically to approximate the solution.

Definition 5

Approximate solution: An approximate solution refers to an estimation or an educated guess of a value, quantity, or solution to a problem that is not obtained precisely but is close enough to the actual or true value to be useful for practical purposes. In various fields, including mathematics, science, engineering, and computing, it's common to encounter problems that are difficult or even impossible to solve exactly due to their complexity, nonlinearity, or lack of analytical solutions. In such cases, an approximate solution provides a practical way to gain insights, make predictions, or solve problems within an acceptable level of accuracy.

Definition 6
Exact solution: An exact solution refers to a precise and rigorous mathematical expression or representation that completely satisfies a given problem or equation. In various mathematical, scientific, and engineering contexts, finding an exact solution is highly valued because it provides an unambiguous and complete description of the problem at hand. An exact solution fully adheres to the principles and conditions of the problem, leaving no room for uncertainty or approximation.

Definition 7
Absolute Error(AE): We defined absolute error as follows in this study: Absolute Erro: $= |\mathbf{W(z)} - w(z)|$; $0 \leq z \leq 1$, where $\mathbf{W(z)}$is the exact solution and $w(z)$ is the Approximate Solution(AS).

3. Demonstration of the method

Proposed method
In the pursuit of obtaining a numerical approximation for the general class of problems addressed in this study, we introduced an approximate solution using shifted Vieta-Lucas polynomials expressed as:

$$w(z) = \sum_{i=0}^{r} VL_r^*(z)a_r. \tag{3}$$

Here $a_r, r = 0(1)n$ are the constants to be determined. With this in mind, we substituted Equation (3) into Equation (1), leading to:

$$\sum_{i=0}^{n} \rho_i(z) \sum_{i=0}^{r} VL_r^{*r}(z)a_r = f(z) + \int_0^z K(z,\nu) \sum_{i=0}^{r} VL_r^*(\nu)a_r d\nu. \tag{4}$$

Where $VL_r^{*r}(z)$ is the r^{th} derivative of $VL_r^*(z)$.

Defining $p(z) = \sum_{i=0}^{n} \rho_i(z) \sum_{i=0}^{r} VL_r^{*r}(z)a_r$, $q(z) = \int_0^z K(z,\nu) \sum_{i=0}^{r} VL_r^*(\nu)a_r d\nu$ Eq. (4) transform into:

$$p(z) - q(z) = f(z) \tag{5}$$

To formulate the linear algebraic system of equations for the $(n+1)$ unknown constants $a_r^{\prime}s$ we employ the collocation method at equidistant points $z_i = a + \frac{(b-a)i}{n}$, $(i = 0(1)(n))$. on Eq. (5). Furthermore, incorporating the initial conditions of Eq. (2), the problem takes on a matrix representation:

$$\begin{pmatrix} H_{11} & H_{12} & H_{13} & \cdots & \cdots & \cdots & H_{1n} \\ H_{21} & H_{22} & H_{23} & \cdots & \cdots & \cdots & H_{2r} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ H_{m1} & H_{m2} & H_{m3} & \cdots & \cdots & \cdots & H_{mn} \\ H_{11}^{0} & H_{12}^{0} & H_{13}^{0} & \cdots & \cdots & \cdots & H_{1r}^{0} \\ H_{21}^{1} & H_{22}^{1} & H_{23}^{1} & \cdots & \cdots & \cdots & H_{2r}^{1} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ H_{m1}^{n-1} & A_{m2}^{n-1} & H_{m3}^{n-1} & \cdots & \cdots & \cdots & H_{mn}^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} G_{11} \\ G_{22} \\ \vdots \\ \vdots \\ Gmn \\ G_{11}^{0} \\ G_{22}^{1} \vdots \\ \vdots \\ G_{mn}^{n-1} \end{pmatrix} \tag{6}$$

where $H_i's$ and $H_i^{0's}$ are the coefficients of $a_i's$ and $G_i's$ are values of $f(z_i)$. The matrix inversion approach is then used to solve the system of equations in order to obtain the unknown constants.

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} & \cdots & \cdots & \cdots & H_{1n} \\ H_{21} & H_{22} & H_{23} & \cdots & \cdots & \cdots & H_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ H_{m1} & H_{m2} & H_{m3} & \cdots & \cdots & \cdots & H_{mr} \\ H_{11}^{0} & H_{12}^{0} & H_{13}^{0} & \cdots & \cdots & \cdots & H_{1n}^{0} \\ H_{21}^{1} & H_{22}^{1} & H_{23}^{1} & \cdots & \cdots & \cdots & H_{2n}^{1} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ H_{m1}^{n-1} & H_{m2}^{n-1} & H_{m3}^{n-1} & \cdots & \cdots & \cdots & A_{mn}^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} G_{11} \\ G_{22} \\ \vdots \\ \vdots \\ Gmn \\ G_{11}^{0} \\ G_{22}^{1} \vdots \\ \vdots \\ G_{mn}^{r-1} \end{pmatrix} \tag{7}$$

Ultimately, the sought-after approximate solution is obtained by solving Eq. (7) and substituting the determined constant values into the assumed approximate solution.

4. Numerical Applications

Example 1 [14]: Consider fourth-order Volterra integro- differential equation

$$w^{iv}(z) = -1 + w(z) + \int_0^w (z-\nu)w(\nu)d\nu$$

Subject to the conditions $w(0) = -1$, $w^{'}(0) = 1$, $w^{''}(0) = 1$, $w^{'''}(0) = -1$. The exact solution is $w(z) = \sin z - \cos z$.

Example 2 [14] Consider the Volterra Integro-differential equation of second order

$$w^{''}(z) = 2 - 2z\sin z - \int_0^w (z-\nu)w(\nu)d\nu$$

Subject to the conditions $w(0) = 0$, $w^{'}(0) = 0$. The exact solution is $w(z) = z \sin z$

Example 3 [6] Consider the FVIDE of first order

$$w^{''}(z) = z - \int_0^w (z - \nu) w(\nu) d\nu$$

$w(0) = 0$,$w^{'}(0) = 1$ $-1 \leq s \leq 1$.

5. Numerical Results

Table 1. Shows comparison of the Absolute Error (AE) for example 1

w_i	ES	AS	AE the proposed Method at n=10	AE of [14] at n=10
0.0	-1.00000000000000	-0.99999999970000	$3.000E-10$	$6.00E-09$
0.2	-0.78139724700000	-0.78139724670000	$2.500E-10$	$2.10E-09$
0.4	-0.53164265170000	-0.53164265150000	$3.500E-10$	$6.20E-09$
0.6	-0.26069314150000	-0.26069314110000	$4.000E-10$	$6.80E-09$
0.8	0.02064938160000	0.02064938220000	$6.000E-10$	$4.77E-09$
1.0	0.30116867890000	0.30116867970000	$8.000E-10$	$9.55E-07$

Table 2. Shows comparison of the Absolute Error (AE) for example 2

w_i	ES	AS	AE the proposed Method at n=10	AE of [14] at n=10
0.0	0.00000000000000	-0.00000000013468	$1.347E-10$	$1.13E-10$
0.2	0.03973386616000	0.03973386603000	$1.100E-10$	$2.56E-07$
0.4	0.15576733690000	0.15576733680000	$0.000E+00$	$2.22E-07$
0.6	0.33878548400000	0.33878548390000	$3.000E-10$	$1.68E-07$
0.8	0.57388487270000	0.57388487260000	$1.000E-10$	$5.38E-07$
1.0	0.84147098480000	0.84147098460000	$2.000E-10$	$9.55E-07$

Table 3. Shows comparison of the Absolute Error (AE) for example 2

w_i	ES	AS	AE the proposed Method at n=5	AE of [6] at n=5
0.0	0.00000	0.00000002382332	$-2.3823E-08$	$0.2500E-4$
0.1	0.1001668	0.10016637490000	$4.2510E-07$	$0.2364E-4$
0.2	0.2013360	0.20132872620000	$7.2738E-06$	$0.2148E-4$
0.3	0.3045203	0.30447456400000	$4.57360E-05$	$0.1932E-4$
0.4	0.4107523	0.41057233410000	$1.7996E-04$	$0.1702E-4$
0.5	0.5210953	0.52056007870000	$5.352213E-04$	$0.1408E-4$

6. Discussion of results

This study investigated three instances. Examples 1 and 2 were resolved through the approach presented in [14], utilizing Chebyshev third-kind polynomials as the underlying basis. A comparison of the outcomes in tables 1 and 2 reveals the superiority of the suggested method employing Vieta Lucas polynomials over the results presented in [14]. Furthermore, example 3 was tackled by [6] employing Euler polynomials combined with the least squares technique. A comparison of the outcomes in tables 1 and 2 demonstrates the enhanced accuracy of the proposed method compared to the findings of [6].

7. Conclusion

In this study, the suggested approach has been adeptly employed to yield numerical solutions for VIDEs employing Vieta- Lucas polynomials. Through the utilization of tables, three distinct numerical illustrations have been employed to showcase the accuracy and effectiveness of the technique. The comparison across Table 1–3 clearly reveals the heightened precision of the employed approach, as evidenced by smaller error values in contrast to those reported in [14] and [6]. Hence, researchers have a foundation to extend this methodology to address a range of other VIDEs based on the insights gained from this study.

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