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AVERAGING PRINCIPLE FOR BSDES DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH NON LIPSCHITZ COEFFICIENTS

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Abstract. Stochastic averaging for a class of backward stochastic differential equations with fractional Brownian motion, of the Hurst parameter H in the interval $\left(\frac{1}{2},1\right)$, is investigated under the non-Lipschitz condition. An averaged fractional BSDEs for the original fractional BSDEs is proposed, and their solutions are quantitatively compared. Under some appropriate assumptions, the solutions to original systems can be approximated by the solutions to averaged stochastic systems, both in the sense of mean square and also in probability. The stochastic integral used throughout the paper is the divergence-type integral.

1. Introduction

The backward stochastic differential equations (BSDEs in short) were first studied by Pardoux and Peng [14] and have the following type:

$$Y_t = \xi + \int_t^T g\left(r, Y_r, Z_r\right) dr - \int_t^T Z_r dW_r, \quad t \in [0,T], \tag{1}$$

where $\{W_t : 0 \leq t \leq T\}$ is a d-dimensionnal Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$, the terminal value ξ is square integrable and g is mapping from $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R}$. They proved that equation (1) has a unique, adapted, and square-integrable solution when g is globally Lipschitz. This pioneering work was extensively used in many fields, like the stochastic interpretation of solutions to PDEs and financial mathematics. Since

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then, several authors have investigated BSDEs (see, among others, [3, 9, 12]). In all the above works, one notices that the coefficients of SDEs are usually assumed to satisfy the Lipschitz condition. However, many practical models of SDEs do not satisfy the Lipschitz condition. In view of the pressing need, the importance, and the impact on many diverse applications, it is necessary and also significant to consider some weaker conditions than the Lipschitz one. Fortunately, Mao [11] and Wang [15] have given much weaker conditions, which are regarded as the so-called non Lipschitz conditions.

In the present paper, we study a stochastic averaging technique for a class of the following FrBSDEs:

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_s)ds - \int_t^T Z_s dB_t^H, \quad t \in [0,T], \tag{2}$$

where $\left(B_t^H\right)_{t\geq 0}$ is the fractional Brownian motion, and $\{\eta_t\}_{0\leq t\leq T}$ is the solution of a stochastic differential equation driven by fractional Brownian motion. It's important to know that the stochastic averaging principle, which is usually used to approximate dynamical systems under random fluctuations, has a long and rich story in multiscale problems (see [13]).
Recently the averaging principle for BSDEs driven by two mutually independent fractional Brownian motions was studied by S. Aidara, Y. Sagna and I.Faye. We present an averaging principle and prove that the original FrBSDEs can be approximated by averaged FrBSDEs in the sense of mean square convergence and convergence in probability when a scaling parameter tends to zero.

The rest of the paper is arranged as follows: In Section 2, we recall some definitions and results about fractional stochastic integrals and the related Itô formula. In Section 3, we investigate the averaging principle for the fractional BSDEs under some proper conditions.

2. Fractional stochastic calculus

In this section, we shall recall some important definitions and results concerning the Malliavin calculus, the stochastic integral with respect to a fractional Brownian motion, and Itô's formula.

Let Ω be a non-empty set, $\mathcal{F}$ a $\sigma-$algebra of sets of Ω and $\mathbf{P}$ a probability measure defined on $\mathcal{F}$. The triplet $(\Omega, \mathcal{F}, \mathbf{P})$ defines a probability space and $\mathbf{E}$ the mathematical expectation with respect to the probability measure $\mathbf{P}$.

Let us recall that, for $H \in (0,1)$, a fractional Brownian motion $\left(B^H(t)\right)_{t\geq 0}$ with Hurst parameter H is a continuous and centered Gaussian process with covariance

$$\mathbf{E}[B^H(t)B^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \geq 0.$$

Denote $\phi(x) = H(2H-1)|x|^{2H-2},\ x \in \mathbf{R}$.
Let ξ and η be measurable functions on $[0,T]$. Define

$$\langle \eta, \xi \rangle_t = \int_0^t \int_0^t \phi(u-v)\xi(u)\eta(v)dudv \quad \text{and} \quad \|\xi\|_t^2 = \langle \xi, \xi \rangle_t.$$

Note that, for any $t \in [0,T]$, $\langle \eta, \xi \rangle_t$ is a Hilbert scalar product.
Let H be the completion of the set of measurable functions such that

$$\|\xi\|_t^2 < +\infty.$$

For a polynomial function of k variables f, let P_T be the set of all polynomials of fractional Brownian motion of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^H(\omega), \int_0^T \xi_2(t)dB_t^H(\omega), \ldots, \int_0^T \xi_k(t)dB_t^H(\omega)\right)$$

where $(\xi_n)_{n\in\mathbf{N}}$ be a sequence in H such that $\langle \xi_i, \xi_j\rangle_T = \delta_{ij}$.

The Malliavin derivative of $F \in P_T$ is given by

$$D_s^H F = \sum_{i=1}^k \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \ldots, \int_0^T \xi_k(t)dB_t^H\right)\xi_i(s), \quad 0 \le s \le T.$$

It is well known that the divergence operator $D^H : L^2(\Omega, F, \mathbf{P}) \to (\Omega, F, H)$ is closable. Hence we can consider the space $\mathbb{D}_{1,2}$ is the completion of P_T with the norm

$$||F||_{1,2}^2 = \mathbf{E}|F|^2 + \mathbf{E}||D_s^H F||_T^2.$$

We define $\mathbb{D}_s^H F = \int_0^T \phi(s-r)D_r^H F dr$ and denote by $\mathbb{L}_H^{1,2}$ the space of all stochastic processes $F : (\Omega, \mathcal{F}, \mathbf{P}) \longrightarrow H$ such that

$$\mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 ds dt\right) < +\infty.$$

We have the following (see [5, Proposition 6.25])

Theorem 2.1. *If $F \in \mathbb{L}_H^{1,2}$, then the Itô-Skorohod type stochastic integral $\int_0^T F_s dB_s^H$ exists in $L^2(\Omega, F, \mathbf{P})$ and satisfies*

$$\mathbf{E}\left(\int_0^T F_s dB_s^H\right) = 0 \quad \textit{and} \quad \mathbf{E}\left(\int_0^T F_s dB_s^H\right)^2 = \mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_t^H F_s \mathbb{D}_s^H F_t ds dt\right).$$

The following integration by parts given in [5, Theorem 11.1] will be useful in the sequel

Theorem 2.2. *Let $f_i(s), g_i(s)$, $i = 1, 2$, are in $\mathbb{D}_{1,2}$ and*

$$\mathbf{E}\int_0^T (|f_i(s)| + |g_i(s)|)\, ds < +\infty.$$

Assume that $\mathbb{D}_t^H f_1(s)$ and $\mathbb{D}_t^H f_2(s)$ are continuously differentiable w.r.t $(s,t) \in [0,T] \times [0,T]$ for almost all $\omega \in \Omega$. Suppose that

$$\mathbf{E}\int_0^T \int_0^T \left|\mathbb{D}_t^H f_i(s)\right|^2 ds dt < +\infty$$

and denote $X_i(t) = \int_0^t g_i(s)ds + \int_0^t f_i(s)dB_s^H, \quad i = 1, 2.$
Then we have for $0 \le t \le T$,

$$\begin{aligned} X_1(t)X_2(t) = &\int_0^t X_1(s)g_2(s)ds + \int_0^t X_1(s)f_2(s)dB_s^H + \int_0^t X_2(s)g_1(s)ds \\ &+ \int_0^t X_2(s)f_1(s)dB_s^H + \int_0^t \mathbb{D}_s^H X_1(s)f_2(s)ds + \int_0^t \mathbb{D}_s^H X_2(s)f_1(s)ds. \end{aligned}$$

Let us finish this section by giving an Itô formula for the divergence type integral (see [5, Theorem 10.3]).

Theorem 2.3. *Let f and $g : [0,T] \to \mathbf{R}$ be deterministic continuous functions. If*

$$X_t = X_0 + \int_0^t g_s ds + \int_0^t f_s dB_s^H, \quad 0 \le t \le T,$$

where X_0 is a constant and if $\varphi \in C^{1,2}([0,T] \times \mathbf{R}, \mathbf{R})$, then for $0 \le t \le T$

$$\varphi(t, X_t) = \varphi(0, X_0) + \int_0^t \frac{\partial \varphi}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial \varphi}{\partial x}(s, X_s) dX_s + \int_0^t \frac{\partial^2 \varphi}{\partial x^2}(s, X_s) \Big[\frac{d}{ds}||f||_s^2\Big] ds. \tag{3}$$

where

$$\begin{aligned}\frac{d}{ds}||f||_s^2 &= \frac{d}{ds}\int_0^s \int_0^s \phi(u-v) f_u\, f_v\, du\, dv \\ &= 2 f_s \int_0^s \phi(u-s)\, f_u\, du.\end{aligned}$$

In order to present a stochastic averaging principle, we need the following [16, Lemma 1]

Lemma 2.1. *Let B_t^H be a fractional Brownian motion with $\frac{1}{2} < H < 1$, and $u(s)$ be a stochastic process in $\mathbb{L}_H^{1,2}$. For every $T < +\infty$, there exists a constant $C_0(H,T) = HT^{2H-1}$ such that*

$$\mathbf{E}\left[\left(\int_0^T |u(s)|\, dB_s^H\right)^2\right] \le C_0(H,T)\mathbf{E}\left[\int_0^T |u(s)|^2\, ds\right] + C_0 T^2.$$

3. Averaging Principle for FrBSDEs

3.1. **Fractional BSDEs.** Let us consider the following process

$$\eta_t = \eta_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dB_s^H \quad 0 \le t \le T,$$

where η_0, b and σ satisfy:

- **•:** $\eta_0 \in \mathbf{R}$ is a given constant;
- **•:** $b, \sigma : [0,T] \to \mathbf{R}$ are continuous deterministic, σ is differentiable and $\sigma(t) \neq 0, t \in [0,T]$.

Note that, since

$$\|\sigma\|_t^2 = H(2H-1)\int_0^t \int_0^t |u-v|^{2H-2}\sigma(u)\sigma(v) du dv,$$

we have

$$\frac{d}{dt}\left(\|\sigma\|_t^2\right) = \sigma(t)\hat{\sigma}(t) > 0 \;\text{ where }\; \hat{\sigma}(t) = \int_0^t \phi(t-v)\sigma(v) dv, \quad 0 \le t \le T.$$

Given ξ a measurable real-valued random variable and the function

$$f : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R},$$

we consider the BSDEs driven by fractional Brownian motion

$$Y_t = \xi + \int_t^T f(r, \eta_r, Y_r, Z_r)\, dr - \int_t^T Z_r dB_r^H, \quad 0 \le t \le T \tag{4}$$

We introduce the following sets (where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}$) :

- **•:** $C^{1,2}_{\text{pol}}([0,T]\times\mathbf{R})$ is the space of all $C^{1,2}$-functions over $[0,T]\times\mathbf{R}$, which together with their derivatives are of polynomial growth,
- **•:** $V_{[0,T]}=\left\{Y(\cdot)=\psi(\cdot,\eta_\cdot);\quad \psi\in C^{1,2}_{\text{pol}}([0,T]\times\mathbf{R}),\ \dfrac{\partial\psi}{\partial t}\text{ is bounded},\ t\in[0,T]\right\}$,
- **•:** $\widetilde{V}^H_T$ the completion of $V_{[0,T]}$ under the following norm

$$\left\|Y\right\|_{\widetilde{V}^H_T}=\left(\int_0^T t^{2H-1}\mathbf{E}[|Y(t)|^2]dt\right)^{1/2}=\left(\int_0^T t^{2H-1}\mathbf{E}[|\psi(t,\eta_t)|^2]dt\right)^{1/2}.$$

- **•:** $B^2([0,T],\mathbf{R})=\widetilde{V}^{1/2}_{[0,T]}\times\widetilde{V}^H_{[0,T]}$ is a Banach space with the norm

$$\left\|(Y,Z)\right\|^2_{B^2}=\left\|Y\right\|^2_{\widetilde{V}^{\frac{1}{2}}_T}+\left\|Z\right\|^2_{\widetilde{V}^H_T}$$

- **•:** $\mathbb{Q}$ be the set of all nondecreasing, continuous and concave functions: $\rho(.):\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\rho(0)=0$, $\rho(s)>0$ for $s>0$ and

$$\int_0^{+\infty}\frac{du}{\rho(u)}=+\infty.$$

  For any $\rho\in\mathbb{Q}$, we can find a pair of positive constants a and b such that $\rho(v)\le a+bv$ for all $v\ge 0$.

Definition 3.1. *A pair of processes $(Y_t,Z_t)_{0\le t\le T}$ is called a solution to the FrBSDEs (4), if it satisfies eq.(4) with $(Y,Z)\in B^2([0,T],\mathbf{R})$.*

Let us recall the following result given in [10, Proposition 17] : Assume that there exists $\widetilde{K}>0$ such that f and g are $\widetilde{K}-$Lipschitz functions. Then eq. (4) has a unique solution $(Y_t,Z_t)_{0\le t\le T}\in\widetilde{V}^{1/2}_{[0,T]}\times\widetilde{V}^H_{[0,T]}$.

The next result (we omit the proof since it is an adaptation of Lemma 4.2 in [6]) is required in the proof of Proposition 3.1.

Lemma 3.2. *Assume that h_1 and $h_2\in C^{0,1}_{pol}([0,T]\times\mathbf{R})$ such that*

$$\int_0^t h_1(s,\eta_s)ds+\int_0^t h_2(s,\eta_s)dB^H_s=0,\quad 0\le t\le T.$$

Then we have

$$h_1(s,x)=h_2(s,x)=0.\quad 0\le s\le T,\ \ x\in\mathbf{R}.$$

Thanks to this result, we have the representation given in [**?**, Proposition 25] and in the proof of [1, Proposition 3.6]. Let $(Y_t,Z_t)_{0\le t\le T}$ be a solution of the FrBSDEs (4). Then:
(i) We have the stochastic representation

$$\mathbb{D}Y_t=\frac{\hat\sigma(t)}{\sigma(t)}Z_t,\quad 0\le t\le T,$$

(ii) There exists a constant $M>0$ such that

$$\frac{t^{2H-1}}{M}\le\frac{\hat\sigma(t)}{\sigma(t)}\le Mt^{2H-1},\quad 0\le t\le T.$$

Thanks to these preliminary results, we are now in a position to investigate our main subject.

3.2. **An averaging principle.** In this section, we will investigate the averaging principle for the FrBSDEs under non Lipschitz coefficients. Let us consider the standard form of the equation (4):

$$Y_t^\varepsilon = \xi + \varepsilon^{2H} \int_t^T f\left(r, \eta_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon\right) dr - \varepsilon^H \int_t^T Z_r^\varepsilon dB_r^H, \quad t \in [0,T]; \tag{5}$$

where $\eta_t^\varepsilon = \eta_0 + \varepsilon^{2H} \int_0^t b(s) ds + \varepsilon^H \int_0^t \sigma(s) dB_s^H, \quad t \in [0,T].$

According to the second part, equation (5) also has an adapted unique and square-integrable solution. We will examine whether the solution Y_t^ε can be approximated to the solution process $\overline{Y}_t$ of the simplified equation:

$$\overline{Y}_t = \xi + \varepsilon^{2H} \int_t^T \overline{f}\left(\eta_r^\varepsilon, \overline{Y}_r, \overline{Z}_r\right) dr - \varepsilon^H \int_t^T \overline{Z}_r dB_r^H, \quad t \in [0,T]; \tag{6}$$

where $\left(\overline{Y}_t, \overline{Z}_t\right)$ has the same properties as $(Y_t^\varepsilon, Z_t^\varepsilon)$.

We assume that the coefficients f and $\overline{f}$ are continuous functions and satisfy the following assumption:

- **(A1)** For all $(t, x, y_i, z_i) \in [0,T] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, $i = 1, 2$, we have
$$|f(t, x, y_1, z_1) - f(t, x, y_2, z_2)|^2 \leq \rho\left(|y_1 - y_2|^2\right) + \rho\left(|z_1 - z_2|^2\right)$$
- **(A2)** For any $t \in [0, T_1] \subset [0,T]$ and for all $(x, y, z) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, we have
$$\frac{1}{T_1 - t} \int_t^{T_1} \left|f(s, x, y, z) - \overline{f}(x, y, z)\right|^2 ds \leq \phi(T_1 - t)\left(1 + \rho\left(|y|^2\right) + \rho\left(|z|^2\right)\right)$$
where $\phi(T_1)$ is a bounded function.

In what follows, we establish the result which will be useful in the sequel.

Lemma 3.3. *Suppose that the original FrBSDEs* (5) *and the averaged FrBSDEs* (6) *both satisfy the assumptions* (**A1**) *and* (**A2**). *For a given arbitrarily small number* $u \in [0, t] \subset [0, T]$, *there exist* $L_1 > 0$ *and* $C_2 > 0$ *such that*

$$\mathbf{E}\left[\int_u^T s^{2H-1} \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds\right] \leq L_1 \mathbf{E}\left[\int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds\right] + C_2 (T - u). \tag{7}$$

Proof. Let us define $\overline{\Delta\delta}_s^\varepsilon = \delta_s^\varepsilon - \overline{\delta}_s$ for a process $\delta_s \in \{Y_s, Z_s\}$.

It is easily seen that the pair of processes $\left(\overline{\Delta Y}_t^\varepsilon, \overline{\Delta Z}_t^\varepsilon\right)_{0 \leq t \leq T}$ solves the FrBSDEs

$$\overline{\Delta Y}_t^\varepsilon = \varepsilon^{2H} \int_t^T \left(f(s, \eta_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon) - \overline{f}(\eta_s^\varepsilon, \overline{Y}_s, \overline{Z}_s)\right) ds - \varepsilon^H \int_t^T \overline{\Delta Z}_s^\varepsilon dB_s^H, \quad t \in [0,T].$$

Applying Itô's formula to $\left|\overline{\Delta Y}_t^\varepsilon\right|^2$, we obtain

$$\begin{aligned}\left|\overline{\Delta Y}_t^\varepsilon\right|^2 + \varepsilon^H \int_u^T \mathbb{D}_s^H \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_s^\varepsilon ds =& 2\varepsilon^{2H} \int_u^T \overline{\Delta Y}_s^\varepsilon \left(f(s, \eta_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon) - \overline{f}(\eta_s^\varepsilon, \overline{Y}_s, \overline{Z}_s)\right) ds \\ & - 2\varepsilon^H \int_u^T \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_s^\varepsilon dB_s^H \end{aligned} \tag{8}$$

Using the fact that $\left(\overline{\Delta Y}^{\varepsilon}_{s},\overline{\Delta Z}^{\varepsilon}_{s}\right)_{t\le s\le T}\in \widetilde{V}^{1/2}_{[0,T]}\times \widetilde{V}^{H}_{[0,T]}$ and $V_{[0,T]}\subset \mathbb{L}^{1,2}_{H}$ (see Lemma 8 in [**?**]) which implies in fact $F_s=\overline{\Delta Y}^{\varepsilon}_{s}\overline{\Delta Z}^{\varepsilon}_{s}\in \mathbb{L}^{1,2}_{H}$. Then by Theorem 2.1, we have

$$\mathbf{E}\left[\int_0^T \overline{\Delta Y}^{\varepsilon}_{s}\overline{\Delta Z}^{\varepsilon}_{s}dB^H_s\right]=0$$

Hence we deduce from (8)

$$\begin{aligned}\mathbf{E}\left[\left|\overline{\Delta Y}^{\varepsilon}_{t}\right|^2+\varepsilon^H\int_u^T \mathbb{D}^H_s\overline{\Delta Y}^{\varepsilon}_{s}\overline{\Delta Z}^{\varepsilon}_{s}ds\right]&=2\varepsilon^{2H}\mathbf{E}\left[\int_u^T \overline{\Delta Y}^{\varepsilon}_{s}\left(f(s,\eta^{\varepsilon}_s,Y^{\varepsilon}_s,Z^{\varepsilon}_s)-\overline{f}(\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_s)\right)ds\right]\\&\le 2\varepsilon^{2H}\mathbf{E}\left[\int_u^T \overline{\Delta Y}^{\varepsilon}_{s}\left(f(s,\eta^{\varepsilon}_s,Y^{\varepsilon}_s,Z^{\varepsilon}_s)-f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_s)\right)ds\right]\\&\quad+2\varepsilon^{2H}\mathbf{E}\left[\int_u^T \overline{\Delta Y}^{\varepsilon}_{s}\left(f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_s)-\overline{f}(\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_s)\right)ds\right] \quad (9)\\&=E_1+E_2.\end{aligned}$$

For E_1, by using the condition (**A1**) and Holder's inequality, for any $\alpha>0$, $2xy\le \alpha x^2+y^2/\alpha$, we deduce that

$$\begin{aligned}E_1\le&\alpha\varepsilon^{2H}\mathbf{E}\left[\int_u^T\left|\overline{\Delta Y}^{\varepsilon}_{s}\right|^2ds\right]+\frac{\varepsilon^{2H}}{\alpha}\mathbf{E}\left[\int_u^T\left|f(s,\eta^{\varepsilon}_s,Y^{\varepsilon}_s,Z^{\varepsilon}_s)-f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_s)\right|^2ds\right]\\\le&\alpha\varepsilon^{2H}\mathbf{E}\left[\int_u^T\left|\overline{\Delta Y}^{\varepsilon}_{s}\right|^2ds\right]+\frac{\varepsilon^{2H}}{\alpha}\mathbf{E}\left[\int_u^T\rho\left(\left|\overline{\Delta Y}^{\varepsilon}_{s}\right|^2\right)ds\right]+\frac{\varepsilon^{2H}}{\alpha}\mathbf{E}\left[\int_u^T\rho\left(\left|\overline{\Delta Z}^{\varepsilon}_{s}\right|^2\right)ds\right]\\\le&\varepsilon^{2H}\left(\alpha+\frac{b}{\alpha}\right)\mathbf{E}\left[\int_u^T\left|\overline{\Delta Y}^{\varepsilon}_{s}\right|^2ds\right]+\frac{b\varepsilon^{2H}}{\alpha}\mathbf{E}\left[\int_u^T\left|\overline{\Delta Z}^{\varepsilon}_{s}\right|^2ds\right]+\frac{2a\varepsilon^{2H}}{\alpha}(T-u)\end{aligned} \quad (10)$$

For E_2, by using assumption (**A2**), Holder's inequality and Young's inequality, we have

$$\begin{aligned}E_2&\le 2\varepsilon^{2H}\mathbf{E}\left[\left(\int_u^T\left|\overline{\Delta Y}^{\varepsilon}_{s}\right|^2ds\right)^{\frac12}\left(\int_t^T\left|f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_s)-\overline{f}(\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_s)\right|^2ds\right)^{\frac12}\right]\\&\le 2\varepsilon^{2H}\mathbf{E}\left[\left((T-u)\int_u^T\left|\overline{\Delta Y}^{\varepsilon}_{s}\right|^2ds\right)^{\frac12}\left(\frac{1}{T-u}\int_u^T\left|f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_s)-\overline{f}(\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_s)\right|^2ds\right)^{\frac12}\right]\\&\le 2\varepsilon^{2H}C_1\mathbf{E}\left[\left(\int_u^T\left|\overline{\Delta Y}^{\varepsilon}_{s}\right|^2ds\right)^{\frac12}\right]\\&\le 2\varepsilon^{2H}C_1\mathbf{E}\left[\int_u^T\left|\overline{\Delta Y}^{\varepsilon}_{s}\right|^2ds+T-u\right]\\&\le 2\varepsilon^{2H}C_1\mathbf{E}\left[\int_u^T\left|\overline{\Delta Y}^{\varepsilon}_{s}\right|^2ds\right]+2\varepsilon^{2H}C_1\left(T-u\right);\end{aligned} \quad (11)$$

where $C_1 = \sqrt{(T-u)\sup\limits_{u\leq s\leq T}\phi(s-u)\left[1 + b\sup\limits_{u\leq s\leq T}\mathbf{E}(\left|\overline{Y}_s\right|^2) + b\sup\limits_{u\leq s\leq T}\mathbf{E}(\left|\overline{Z}_s\right|^2) + 2a\right]}$.

By the stochastic representation given in Proposition 3.1, we have

$$\mathbf{E}\left[\int_u^T \mathbb{D}_s^H \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_s^\varepsilon ds\right] \geq \mathbf{E}\left[\frac{1}{M}\int_u^T s^{2H-1}\left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right] \tag{12}$$

Putting pieces together, we deduce from (9) that

$$\mathbf{E}\left[\left|\overline{\Delta Y}_t^\varepsilon\right|^2 + \frac{\varepsilon^H}{M}\int_u^T s^{2H-1}\left|\overline{\Delta Z}_s^\varepsilon\right|_s^2 ds\right] \leq \varepsilon^{2H}\left(\alpha + \frac{b}{\alpha} + 2C_1\right)\mathbf{E}\left[\int_u^T \left|\overline{\Delta Y}_s^\varepsilon\right|^2 ds\right] \tag{13}$$
$$+ \frac{b\varepsilon^{2H}}{\alpha}\mathbf{E}\left[\int_u^T \left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right] + \left(\frac{2a}{\alpha} + 2C_1\right)\varepsilon^{2H}(T-u)$$

Now we can compute

$$\begin{aligned}\frac{b\varepsilon^{2H}}{\alpha}\mathbf{E}\left[\int_u^T \left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right] &= \frac{b\varepsilon^{2H}}{\alpha}\mathbf{E}\left[\int_u^T \frac{1}{s^{2H-1}} \times s^{2H-1}\left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right] \\ &\leq \frac{b\varepsilon^{2H}}{\alpha} \times \frac{T^{2-2H} - u^{2-2H}}{2-2H}\mathbf{E}\left[\int_u^T s^{2H-1}\left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right] \\ &\leq \frac{b\varepsilon^{2H}T^{2-2H}}{\alpha(2-2H)}\mathbf{E}\left[\int_u^T s^{2H-1}\left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right] \end{aligned} \tag{14}$$

Therefore, we can write

$$\mathbf{E}\left[\left|\overline{\Delta Y}_t^\varepsilon\right|^2 + \frac{\varepsilon^H}{M}\int_u^T s^{2H-1}\left|\overline{\Delta Z}_s^\varepsilon\right|_s^2 ds\right] \leq \varepsilon^{2H}\left(\alpha + \frac{b}{\alpha} + 2C_1\right)\mathbf{E}\left[\int_u^T \left|\overline{\Delta Y}_s^\varepsilon\right|^2 ds\right] \tag{15}$$
$$+ \frac{b\varepsilon^{2H}T^{2-2H}}{\alpha(2-2H)}\mathbf{E}\left[\int_u^T s^{2H-1}\left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right]$$
$$+ \left(\frac{2a}{\alpha} + 2C_1\right)\varepsilon^{2H}(T-u)$$

Hence if we choose $\alpha = \alpha_0$ satisfying $\dfrac{\alpha_0(2-2H)\varepsilon^H - Mb\varepsilon^{2H}T^{2-2H}}{\alpha_0 M(2-2H)} = \varepsilon^{2H}$, then

$$\varepsilon^{2H}\mathbf{E}\left[\int_u^T s^{2H-1}\left|\overline{\Delta Z}_s^\varepsilon\right|_s^2 ds\right] \leq \varepsilon^{2H}\left(\alpha_0 + \frac{b}{\alpha_0} + 2C_1\right)\mathbf{E}\left[\int_u^T \left|\overline{\Delta Y}_s^\varepsilon\right|^2 ds\right] + \left(\frac{2a}{\alpha_0} + C_1\right)\varepsilon^{2H}(T-u)$$

Thus,

$$\mathbf{E}\int_u^T s^{2H-1}\left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds \leq L_1\mathbf{E}\int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + C_2(T-u), \tag{16}$$

where $L_1 = \alpha_0 + \dfrac{b}{\alpha_0} + 2C_1$ and $C_2 = \left(\frac{2a}{\alpha} + 2C_1\right)$. This completes the proof. □

Now, we claim the main theorem showing the relationship between solution processes Y_t^ε to the original (5) and $\overline{Y}_t$ to the averaged (6). It shows that the solution of the averaged (6) converges to that of the original (5) in the mean square sense.

Theorem 3.4. *Under the assumption of Lemma 3.3 are satisfied. For a given arbitrarily small number $\delta_1 > 0$, there exists $\varepsilon_1 \in [0, \varepsilon_0]$ and $\beta \in [0,1]$ such that for all $\varepsilon \in [0, \varepsilon_1]$ having*

$$\sup_{T\varepsilon^{1-\beta}\leq t\leq T} \mathbf{E}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2 \leq \delta_1.$$

Proof. With the help of Lemma 3.3, now we can prove the Theorem 3.4. Using the elementary inequelity and the isometry property, we derive that

$$\begin{aligned}
\mathbf{E}\left[\left|\overline{\Delta Y}_s^\varepsilon\right|^2\right] =&\mathbf{E}\left[\left|\varepsilon^{2H}\int_u^T \left[f(s,\eta_s^\varepsilon,Y_s^\varepsilon,Z_s^\varepsilon) - \overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)\right]ds - \varepsilon^H\int_u^T \overline{\Delta Z}_s^\varepsilon dB_s^H\right|^2\right] \\
\leq& 2\varepsilon^{4H}\mathbf{E}\left[\left|\int_u^T \left[f(s,\eta_s^\varepsilon,Y_s^\varepsilon,Z_s^\varepsilon) - \overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)\right]ds\right|^2\right] + 2\varepsilon^{2H}\mathbf{E}\left[\left|\int_u^T \overline{\Delta Z}_s^\varepsilon dB_s^H\right|^2\right] \\
\leq& 4\varepsilon^{4H}\mathbf{E}\left[\left|\int_u^T \left[f(s,\eta_s^\varepsilon,Y_s^\varepsilon,Z_s^\varepsilon) - f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)\right]ds\right|^2\right] \\
&+ 4\varepsilon^{4H}\mathbf{E}\left[\left|\int_u^T \left[f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s) - \overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)\right]ds\right|^2\right] + 2\varepsilon^{2H}\mathbf{E}\left[\left|\int_u^T \overline{\Delta Z}_s^\varepsilon dB_s^H\right|^2\right] \\
=& D_1 + D_2 + D_3; \qquad (17)
\end{aligned}$$

where $D_1 = 4\varepsilon^{4H}\mathbf{E}\left[\left|\int_u^T \left[f(s,\eta_s^\varepsilon,Y_s^\varepsilon,Z_s^\varepsilon) - f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)\right]ds\right|^2\right]$,

$D_2 = 4\varepsilon^{4H}\mathbf{E}\left[\left|\int_u^T \left[f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s) - \overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)\right]ds\right|^2\right]$ and $D_3 = 2\varepsilon^{2H}\mathbf{E}\left[\left|\int_u^T \overline{\Delta Z}_s^\varepsilon dB_s^H\right|^2\right]$.

Applying Holder's inequality and the assumption (**A1**), we obtain

$$\begin{aligned}
D_1 \leq& 4(T-u)\varepsilon^{4H}\mathbf{E}\left[\int_u^T \left|f(s,\eta_s^\varepsilon,Y_s^\varepsilon,Z_s^\varepsilon) - f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)\right|^2 ds\right] \\
\leq& 4(T-u)\varepsilon^{4H}\mathbf{E}\left[\int_u^T \rho\left(\left[\left|\overline{\Delta Y}_s^\varepsilon\right|^2\right) + \rho\left(\left|\overline{\Delta Z}_s^\varepsilon\right|^2\right)\right]ds\right] \\
\leq& 4(T-u)b\varepsilon^{4H}\mathbf{E}\left[\int_u^T \left|\overline{\Delta Y}_s^\varepsilon\right|^2 ds\right] + 4(T-u)b\varepsilon^{4H}\frac{T^{2-2H}}{2-2H}\mathbf{E}\left[\int_u^T s^{2H-2}\left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right] \\
&+ 8(T-u)^2 a\varepsilon^{4H} \qquad (18)
\end{aligned}$$

Then, together with Holder's inequality and the assumption (**A2**), we get

$$\begin{aligned}
D_2 \leq& 4(T-u)\varepsilon^{4H}\mathbf{E}\left[\int_u^T \left|f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)-\overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)\right|^2 ds\right] \\
\leq& 4(T-u)^2\varepsilon^{4H}\mathbf{E}\left[\frac{1}{T-u}\int_u^T \left|f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)-\overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_s)\right|^2 ds\right] \\
\leq& 4(T-u)^2\varepsilon^{4H}\sup_{u\leq s\leq T}[\phi(s-u)]\left(1+\sup_{u\leq s\leq T}\rho\left(\mathbf{E}\left(\left|\overline{Y}_s\right|^2\right)\right)+\sup_{u\leq s\leq T}\rho\left(\mathbf{E}\left(\left|\overline{Z}_s\right|^2\right)\right)\right) \\
=& C_3(T-u)^2\varepsilon^{4H}, \qquad (19)
\end{aligned}$$

where $C_3 = 4\sup_{u\leq s\leq T}[\phi(s-u)]\left(1+b\sup_{u\leq s\leq T}\mathbf{E}\left(\left|\overline{Y}_s\right|^2\right)+b\sup_{u\leq s\leq T}\mathbf{E}\left(\left|\overline{Z}_s\right|^2\right)+2a\right)$.

By the Lemma 2.1, we obtain

$$\begin{aligned}
D_3 &\leq 2\varepsilon^{2H}HT^{2H-1}\mathbf{E}\left[\int_u^T \left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right]+2\varepsilon^{2H}C_0T^2 \\
&\leq \frac{2\varepsilon^{2H}HT}{2-2H}\mathbf{E}\left[\int_u^T s^{2H-1}\left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right]+2\varepsilon^{2H}C_0T^2 \qquad (20)
\end{aligned}$$

Using above inequalities, from (17) we deduce

$$\begin{aligned}
\sup_{u\leq t\leq T}\mathbf{E}\left[\left|\overline{\Delta Y}_t^\varepsilon\right|^2\right] \leq& \left(4(T-u)b\varepsilon^{4H}\frac{T^{2-2H}}{2-2H}+\frac{2\varepsilon^{2H}HT}{2-2H}\right)\sup_{u\leq t\leq T}\mathbf{E}\left[\int_u^T \left|\overline{\Delta Z}_s^\varepsilon\right|^2 ds\right] \\
&+4(T-u)b\varepsilon^{4H}\sup_{u\leq t\leq T}\mathbf{E}\int_u^T \left|\overline{\Delta Y}_s^\varepsilon\right|^2 ds + C_3(T-u)^2\varepsilon^{4H}+2\varepsilon^{2H}C_0T^2+8(T-u)^2a\varepsilon^{4H}
\end{aligned}$$

Applying Lemma 3.3 to the above inequality we get

$$\begin{aligned}
\sup_{u\leq t\leq T}\mathbf{E}\left[\left|\overline{\Delta Y}_t^\varepsilon\right|^2\right] \leq& \left[4(T-u)b\varepsilon^{4H}\left(1+L_1\frac{T^{2-2H}}{2-2H}\right)+L_1\frac{2\varepsilon^{2H}HT}{2-2H}\right]\int_u^T \sup_{u\leq s_1\leq s}\mathbf{E}\left|\overline{\Delta Y}_{s_1}^\varepsilon\right|^2 ds \\
&+\varepsilon^{2H}\left[\frac{2C_2(T-u)}{2-2H}\left(2(T-u)b\varepsilon^{2H}T^{2-2H}+HT\right)+C_3(T-u)^2\varepsilon^{2H}+2C_0T^2+8(T-u)^2a\varepsilon^{2H}\right]. \qquad (21)
\end{aligned}$$

Thanks to Gronwall's inequality, we obtain

$$\begin{aligned}
\sup_{u\leq t\leq T}\mathbf{E}\left|\overline{\Delta Y}_t^\varepsilon\right|^2 \leq& \varepsilon^{2H}\left[\frac{2C_1(T-u)}{2-2H}\left(2(T-u)b\varepsilon^{2H}T^{2-2H}+HT\right)+C_3(T-u)^2\varepsilon^{2H}\right. \\
&\left.+2C_0T^2+8(T-u)^2a\varepsilon^{2H}\right]e^{(T-u)\left[4(T-u)b\varepsilon^{4H}\left(1+L_1\frac{T^{2-2H}}{2-2H}\right)+L_1\frac{2\varepsilon^{2H}HT}{2-2H}\right]}.
\end{aligned}$$

Obviously, the above estimate implies that there exist $\beta\in[0,1]$ and $K>0$ such that for every $t\in(0,K\varepsilon^{-2H\beta}]\subseteq[0,T]$,

$$\sup_{K\varepsilon^{-2H\beta}\leq t\leq T}\mathbf{E}\left|Y_t^\varepsilon-\overline{Y}_t\right|^2 \leq C_4\varepsilon^{1-2H\beta}, \qquad (22)$$

in which

$$C_4 = \varepsilon^{2H(1+\beta)-1}\left[\frac{2C(T-K\varepsilon^{-2H\beta})}{2-2H}\left(2(T-K\varepsilon^{-2H\beta})b\varepsilon^{2H}T^{2-2H}+HT\right)+C_3(T-K\varepsilon^{-2H\beta})^2\varepsilon^{2H}\right.$$
$$\left.+2C_0T^2+8(T-K\varepsilon^{-2H\beta})^2a\varepsilon^{2H}\right]e^{(T-K\varepsilon^{-2H\beta})\left[4(T-K\varepsilon^{-2H\beta})b\varepsilon^{4H}\left(1+L_1\frac{T^{2-2H}}{2-2H}\right)+L_1\frac{2\varepsilon^{2H}HT}{2-2H}\right]}$$

is constant.

Consequently, for any number $\delta_1 > 0$, we can choose $\varepsilon_1 \in [0, \varepsilon_0]$ such that for every $\varepsilon_1 \in [0, \varepsilon_0]$ and for each $t \in (0, K\varepsilon^{-2H\beta}]$

$$\sup_{K\varepsilon^{-2H\beta}\le t\le T} \mathbf{E}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2 \le \delta_1. \tag{23}$$

This completes the proof. □

With Theorem 3.4, it is easy to show the convergence in probability between solution processes Y_t^ε to the original (5) and $\overline{Y}_t$ to the averaged (6).

Corollary 3.0. *Suppose that the original FrBSDEs* (5) *and the averaged FrBSDEs* (6) *both satisfy the assumptions* (**A1**) *and* (**A2**). *For a given arbitrary small number* $\delta_2 > 0$, *there exists* $\varepsilon_2 \in [0, \varepsilon_0]$ *such that for all* $\varepsilon \in (0, \varepsilon_2]$, *we have*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{K\varepsilon^{-2H\beta}\le t\le T}\left|Y_t^\varepsilon - \overline{Y}_t\right| > \delta_2\right) = 0 \tag{24}$$

where β *is defined by Theorem 3.4.*

Proof. By Theorem 3.4 and the Chebyshev inequality, for any given number $\delta_2 > 0$, we can obtain

$$\mathbb{P}\left(\sup_{K\varepsilon^{-2H\beta}\le t\le T}\left|Y_t^\varepsilon - \overline{Y}_t\right| > \delta_2\right) \le \frac{1}{\delta_2^2}\mathbf{E}\left(\sup_{K\varepsilon^{-2H\beta}\le t\le T}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2\right) \le \frac{C_4\varepsilon^{1-2H\beta}}{\delta_2^2}.$$

Let $\varepsilon \to 0$ and the required result follows. □

Remark 1. *Corollary 3.0 means the convergence in probability between the original solution* $(Y_t^\varepsilon, Z_t^\varepsilon)$ *and the averaged solution* $\left(\overline{Y}_t, \overline{Z}_t\right)$.

4. Conclusion

The averaging principle is an effective method for simplifying deterministic systems as well as stochastic systems. In this paper, we consider the averaging Principle for BSDEs driven by fractional Brownian motion with non Lipschitz coefficients. Our contribution is to prove that the original FrBSDEs can be approximated by averaged FrBSDEs in the sense of mean square convergence and convergence in probability when a scaling parameter tends to zero.

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