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# CONVERGENCE, STABILITY, AND DATA DEPENDENCE RESULTS FOR A NEW ITERATION METHOD IN BANACH SPACE

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Abstract. In this paper, we introduce a new iterative process and show that our iteration scheme is faster than other existing iteration schemes with the help of numerical examples. Next, we have established convergence, stability, and data dependency results for the approximation of fixed points of the Contractive- like mapping in the framework of uniformly convex Banach space.

## 1. Introduction

Fixed point problems posses either existence result or approximate solution. Iterative methods have become one of the most interesting topics for numerical analysis to approximate fixed points and study their convergence results.The iterative method has two main aspects, the number of iterations and stability. When the number of iterations is small and stable, the method is considered successful, effective, and better than its counterpart in approximation. Over the years, researchers have developed many iterative schemes to obtain approximate solutions of fixed point problems for different mappings or operators over different spaces.

In 1953, Mann [14], introduced an iterative scheme. Later in 1974, Ishikawa [10], introduced an iterative scheme which was two step iterative schemes. In 2000, Noor[15], introduced three step iterative scheme for approximating fixed point problems. Later several researchers modified Mann, Ishikawa, and Noor iterations etc. The data-dependence result concerning Mann-Ishikawa iteration [19] where the data dependence of Ishikawa iteration was proved for contractions mappings. Soltuz et al.[20]proved data-dependence results for Ishikawa iteration for the contractive-like

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operators.
In 2014 Gursoy et al.[7] introduced a new iteration process:

$$\begin{cases} x_0 \in K \\ x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)Tx_n + \alpha_n Tz_n \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n \end{cases} \tag{1}$$

where $\alpha_n, \beta_n \in (0,1)$. This iteration process also called Picard-S iteration and used to approximate the fixed point of contraction mappings.They proved a data dependence result for fixed point of contraction mappings with the help of the new iteration method.
In 2018 Dogan et al. [5] introduced the following three step iteration process:

$$\begin{cases} x_0 \in K \\ x_{n+1} = (1-\alpha_n)Tz_n + \alpha_n Ty_n \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n \\ z_n = Tx_n, \end{cases} \tag{2}$$

where $\alpha_n, \beta_n \in (0,1)$. We compared the rate of convergence with the iterative scheme and Picard-S due to Gursoy. They proved that strong convergence result for this iteration method and also shows that this result of its data dependency.
Motivated by the above work done in this direction, we introduce a new iteration process for approximating fixed point of Contractive like mapping in Uniformly Banach Space. Let $X$ be a uniformly Banach space and $K$ be any nonempty closed convex subset of $X$. Let $T : K \to K$ be a mapping and for each $x_0 \in K$, the sequence $\{x_n\}$ in $K$ is defined by

$$\begin{cases} x_0 \in K \\ x_{n+1} = T((1-\alpha_n)Tz_n + \alpha_n y_n) \\ y_n = T((1-\beta_n)Tx_n + \beta_n Tz_n) \\ z_n = Tx_n, \end{cases} \tag{3}$$

where $\alpha_n, \beta_n \in (0,1)$. We also prove convergence and stability results, data dependency result and rate of convergence with the help of numerical examples.

## 2. Preliminaries

In this section, we recall some definitions and results to be used in establishing the main results.

**Definition 1.** [6] A Banach space $X$ is said to be uniformly convex if for each $\epsilon \in (0,2]$ there is a $\delta > 0$ such that $x, y \in X$

$$||(x+y)/2|| \leq 1 - \delta \textit{ whenever } \quad ||x-y|| \geq \epsilon \textit{ and } ||x|| = ||y|| = 1.$$

**Definition 2.** [16]A Banach space $X$ is said to satisfy Opial's property if for each sequence $\{x_n\}$ in $X$ converging weakly to $x \in X$, we have

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||,$$

for all $y \in X$ s.t. $x \neq y$.

**Definition 3.** [9] Let $X$ be a Banach space and let $T : X \to X$ be a self map. The mapping $T$ is called contractive like mapping if there exist a constant $\delta \in [0, 1)$ and a strictly increasing and continuous function $\xi : [0, \infty) \to [0, \infty)$ with $\xi(0) = 0$ such that for all $x, y \in X$,

$$||Tx - Ty|| \leq \delta||x - y|| + \xi(||x - Tx||). \tag{4}$$

.

**Definition 4.** [3] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequence of positive numbers such that converge to $a$ and $b$ respectively. Assume that there exists

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

(i) If $l = 0$, then the sequence $\{a_n\}$ converges faster than the sequence $\{b_n\}$.
(ii) If $0 < l < \infty$, then we say that the sequence $\{a_n\}$ and $\{b_n\}$ have the same rate of converges.
(iii) If $l = \infty$, then the sequence $\{b_n\}$ converges faster than sequence $\{a_n\}$.

**Definition 5.** [8] Let $\{t_n\}$ be any arbitrary sequence in $X$. Then an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point $x^*$, is said to $T$-stable, if for $\epsilon_n = ||t_{n+1} - f(T, t_n)||$, $\forall n \in N$, we have $\lim\limits_{n\to\infty} \epsilon_n = 0$ if and only if $\lim\limits_{n\to\infty} t_n = x^*$.

**Lemma 1.** [3] *Suppose that for two fixed point iteration processes $\{u_n\}$ and $\{v_n\}$ both converging to the same fixed point $x^*$, the error estimates*

$$||u_n - x^*|| \leq a_n \quad n \geq 1,$$
$$||v_n - x^*|| \leq b_n \quad n \geq 1.$$

*are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to $x^*$.*

**Lemma 2.** [1] *If $\lambda$ is a real number such that $0 \leq \lambda < 1$ and $\{\epsilon_n\}$ is the sequence of positive numbers such that*

$$\lim_{n\to\infty} \epsilon_n = 0$$

*then for an sequence of positive numbers $v_n$ satisfying*

$$v_{n+1} \leq \lambda v_n + \epsilon_n, \quad for\ n = 1, 2, ...,$$

*we have*

$$\lim_{n\to\infty} v_n = 0.$$

**Lemma 3.** [18]*Let $X$ be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ $\forall n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of $X$ s.t. $\limsup_{n\to\infty} ||x_n|| \leq a$, $\limsup_{n\to\infty} ||y_n|| \leq a$ and $\limsup_{n\to\infty} ||t_n x_n + (1 - t_n)y_n|| = a$ hold for some $a \geq 0$. Then $\lim\limits_{n\to\infty} ||x_n - y_n|| = 0$.*

## 3. Convergence and Stability Results

**Theorem 1.** *Let $K \neq \phi$ be a closed convex subset of a uniformly convex Banach space $X$ and $T : K \to K$ be a map with a fixed point $x^*$ satisfying the condition 4. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by iteration process (3), where $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$ and $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$. Then the iteration scheme $\{x_n\}_{n=0}^{\infty}$ convergence to the fixed point of $T$.*

*Proof.* Using iteration process (3) and (4), we have

$$\begin{aligned}||x_{n+1}-x^*|| &= ||T((1-\alpha_n)Tz_n+\alpha_n y_n)-x^*|| \\ &\leq \delta||(1-\alpha_n)Tz_n+\alpha_n y_n-x^*||+\xi(||x^*-Tx^*||) \\ &= \delta(1-\alpha_n)||Tz_n-x^*||+\delta\alpha_n||y_n-x^*|| \\ &\leq \delta^2(1-\alpha_n)||z_n-x^*||+\delta\alpha_n||y_n-x^*|| \end{aligned} \tag{5}$$

$$\begin{aligned}||y_n-x^*|| &= ||T((1-\beta_n)Tx_n+\beta_n Tz_n)-x^*|| \\ &\leq \delta||(1-\beta_n)Tx_n+\beta_n Tz_n-x^*||+\xi(||x^*-Tx^*||) \\ &= \delta(1-\beta_n)||Tx_n-x^*||+\delta\beta_n||Tz_n-x^*|| \\ &\leq \delta^2(1-\beta_n)||x_n-x^*||+\delta^2\beta_n||z_n-x^*|| \end{aligned} \tag{6}$$

$$\begin{aligned}||z_n-x^*|| &= ||Tx_n-x^*|| \\ &\leq \delta||x_n-x^*|| \end{aligned} \tag{7}$$

From equation (5) and (6), we have

$$\begin{aligned}||y_n-x^*|| &\leq \delta^2(1-\beta_n)||x_n-x^*||+\delta^3\beta_n||x_n-x^*|| \\ &= \delta^2(1-(1-\delta)\beta_n)||x_n-x^*|| \end{aligned} \tag{8}$$

From equation (5),(7) and (8), we get

$$\begin{aligned}||x_{n+1}-x^*|| &\leq \delta^3(1-\alpha_n)||x_n-x^*||+\alpha_n\delta^3(1-(1-\delta)\beta_n)||x_n-x^*|| \\ &= \delta^3(1-\alpha_n\beta_n(1-\delta))||x_n-x^*|| \\ &\leq \delta^{3+3}\prod_{k=n-1}^{n}(1-\alpha_k\beta_k(1-\delta))||x_{n-1}-x^*|| \\ &\vdots \\ &\leq \delta^{3(n+1)}\prod_{k=0}^{n}(1-\alpha_k\beta_k(1-\delta))||x_0-x^*|| \end{aligned} \tag{9}$$

where $\delta \in (0,1)$ and also $\alpha_k, \beta_k \in (0,1)$. Since $1-x \leq e^{-x}$, $x \in [0,1]$. So from equation (9), we get

$$||x_{n+1}-x^*|| \leq \frac{||x_0-x^*||\delta^{3(n+1)}}{e^{(1-\delta)\sum_{k=0}^{\infty}\alpha_k\beta_k}}$$

Since $\sum_{k=0}^{\infty}\alpha_k\beta_k=\infty$. So taking the limit on both sides

$$\lim_{n\to\infty}||x_{n+1}-x^*||=0.$$

□

**Theorem 2.** *Let $K \neq \phi$ be a closed convex subset of a uniformly convex Banach space $X$ and $T : K \to K$ be a map with a fixed point $x^*$ satisfying the condition* (4)*. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by iteration process* (3)*, where $\{\alpha_n\}, \{\beta_n\} \in (0,1)$ . Then the iteration process* (3) *is $T-stable$.*

*Proof.* Let $\{t_n\}$ be an arbitrary sequence in $K$ and the sequence generated by (3) $x_{n+1} = f(T, x_n)$ converges to a unique fixed point $x^*$ and $\epsilon = ||t_{n+1} - f(T, t_n)||$. We shall prove that $\lim_{n\to\infty} \epsilon = 0$ if and only if $\lim_{n\to\infty} t_n = x^*$. Suppose $\lim_{n\to\infty} \epsilon = 0$ and

$$\begin{aligned}
||t_{n+1} - x^*|| &= ||t_{n+1} - f(T, t_n) + f(T, t_n) - x^*|| \\
&\leq ||t_{n+1} - f(T, t_n)|| + ||f(T, t_n) - x^*|| \\
&= \epsilon + ||T((1 - \alpha_n)Tr_n + \alpha_n s_n) - x^*|| \\
&\leq \epsilon + \delta(1 - \alpha_n)||Tr_n - x^*|| + \delta\alpha_n||s_n - x^*|| \\
&\leq \epsilon + \delta^2(1 - \alpha_n)||r_n - x^*|| + \delta\alpha_n||s_n - x^*|| \\
&= \epsilon + \delta^2(1 - \alpha_n)||Tt_n - x^*|| + \delta\alpha_n||T((1 - \beta_n)Tt_n + \beta_n Tr_n) - x^*|| \\
&\leq \epsilon + \delta^3(1 - \alpha_n)||t_n - x^*|| + \delta^2\alpha_n((1 - \beta_n)||Tt_n - x^*|| + \beta_n||Tr_n - x^*||) \\
&\leq \epsilon + \delta^3(1 - \alpha_n)||t_n - x^*|| + \delta^3\alpha_n((1 - \beta_n)||t_n - x^*|| + \beta_n||r_n - x^*||) \\
&\leq \epsilon + \delta^3(1 - \alpha_n)||t_n - x^*|| + \delta^3\alpha_n((1 - \beta_n)||t_n - x^*|| + \beta_n\delta||t_n - x^*||) \\
&= \epsilon + \delta^3(1 - \alpha_n)||t_n - x^*|| + \delta^3\alpha_n(1 - (1 - \delta)\beta_n)||t_n - x^*|| \\
&= \epsilon + \delta^3(1 - \alpha_n\beta_n(1 - \delta))||t_n - x^*||
\end{aligned}$$

Since $\delta \in [0, 1)$ and $\alpha_n$ , $\beta_n \in (0, 1)$, so $\delta^3(1 - \alpha_n\beta_n(1 - \delta)) < 1$. Using Lemma 2, we have

$$\lim_{n\to\infty} ||t_n - x^*|| = 0$$
$$\lim_{n\to\infty} t_n = x^*.$$

Converse part: Suppose $\lim_{n\to\infty} t_n = x^*$ and

$$\begin{aligned}
\epsilon &= ||t_{n+1} - f(T, t_n)|| \\
&\leq ||t_{n+1} - x^*|| + ||x^* - f(T, t_n)|| \\
&= ||t_{n+1} - x^*|| + ||T((1 - \alpha_n)Tr_n + \alpha_n s_n) - x^*|| \\
&\leq ||t_{n+1} - x^*|| + \delta^2(1 - \alpha_n)||r_n - x^*|| + \delta\alpha_n||s_n - x^*|| \\
&= ||t_{n+1} - x^*|| + \delta^2(1 - \alpha_n)||Tt_n - x^*|| + \delta\alpha_n||T((1 - \beta_n)Tt_n + \beta_n Tr_n) - x^*|| \\
&\leq ||t_{n+1} - x^*|| + \delta^3(1 - \alpha_n)||t_n - x^*|| + \delta^2\alpha_n((1 - \beta_n)||Tt_n - x^*|| + \beta_n||Tr_n - x^*||) \\
&\leq ||t_{n+1} - x^*|| + \delta^3(1 - \alpha_n)||t_n - x^*|| + \delta^3\alpha_n((1 - \beta_n)||t_n - x^*|| + \beta_n||r_n - x^*||) \\
&\leq ||t_{n+1} - x^*|| + \delta^3(1 - \alpha_n)||t_n - x^*|| + \delta^3\alpha_n((1 - \beta_n)||t_n - x^*|| + \beta_n\delta||t_n - x^*||) \\
&= ||t_{n+1} - x^*|| + \delta^3(1 - \alpha_n)||t_n - x^*|| + \delta^3\alpha_n(1 - (1 - \delta)\beta_n)||t_n - x^*|| \\
&= ||t_{n+1} - x^*|| + \delta^3(1 - \alpha_n\beta_n(1 - \delta))||t_n - x^*||
\end{aligned}$$

Since $\lim_{n\to\infty} t_n = x^*$, we have

$$\lim_{n\to\infty} \epsilon = 0.$$

Hence iteration process (3) is $T - stable$. □

## 4. Data Dependency Results

We consider the new iteration process (3) for the operator $S$ as follows:

$$\begin{cases} u_0 \in K \\ u_{n+1} = S((1-\alpha_n)Sw_n + \alpha_n v_n) \\ v_n = S((1-\beta_n)Su_n + \beta_n Sw_n) \\ w_n = Su_n, \end{cases} \tag{10}$$

where $\alpha_n, \beta_n \in (0,1)$.

**Definition 6.** [4] Let $T, S : K \to K$ be two operators. $S$ is said to be approximate operator for $T$ if for some $\epsilon > 0$ we have $||Tx - Sx|| \leq \epsilon$, for all $x \in K$.

**Lemma 4.** [20] *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers for which there exists $n_0 \in N$ such that for all $n \geq n_0$ satisfying the relation*

$$a_n \leq (1-b_n)a_n + b_n c_n,$$

*where $b_n \in (0,1)$ for all $n \in N$, $\sum_{n=0}^{\infty} b_n = \infty$ and $c_n \geq 0$ for all $n \in N$. Then the following holds:*

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} c_n.$$

**Theorem 3.** *Let $T$ and $S$ be defined on a nonempty subset $K$ s.t. $T$ be a contractive like operator with fixed point $x^*$ and $S$ is approximate operator for $T$ with $Sy^* = y^*$. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (3) and iterative sequence $\{u_n\}_{n=0}^{\infty}$ is generated by (10) with the assumption $(1-\alpha_n\beta_n) < \alpha_n\beta_n$ and $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$. If $\lim_{n\to\infty} u_n = y^*$, then*

$$||x^* - y^*|| \leq \frac{8\epsilon}{1-\delta}$$

*where $\epsilon > 0$ a fixed number.*

*Proof.* By using iteration (3) and (10), we obtain

$$\begin{aligned} ||z_n - w_n|| &= ||Tx_n - Su_n|| \\ &\leq ||Tx_n - Tu_n|| + ||Tu_n - Su_n|| \\ &\leq \delta||x_n - u_n|| + \xi(||Tx_n - x_n||) + \epsilon \end{aligned} \tag{11}$$

$$\begin{aligned} ||y_n - v_n|| &= ||T((1-\beta_n)Tx_n + \beta_n Tz_n) - S((1-\beta_n)Su_n + \beta_n Sw_n)|| \\ &\leq ||T((1-\beta_n)Tx_n + \beta_n Tz_n) - T((1-\beta_n)Su_n + \beta_n Sw_n)|| \\ &+ ||T((1-\beta_n)Su_n + \beta_n Sw_n) - S((1-\beta_n)Su_n + \beta_n Sw_n)|| \\ &\leq \delta[||(1-\beta_n)Tx_n + \beta_n Tz_n - (1-\beta_n)Su_n + \beta_n Sw_n)||] \\ &+ \xi(||T((1-\beta_n)Tx_n + \beta_n Tz_n) - (1-\beta_n)Tx_n + \beta_n Tz_n||) + \epsilon \\ &= \delta[(1-\beta_n)||Tx_n - Su_n|| + \beta_n||Tz_n - Sw_n||] \\ &+ \xi(||T((1-\beta_n)Tx_n + \beta_n Tz_n) - (1-\beta_n)Tx_n + \beta_n Tz_n||) + \epsilon \\ &= \delta[(1-\beta_n)(||Tx_n - Tu_n|| + ||Tu_n - Su_n||) + \beta_n(||Tz_n - Tw_n|| + ||Tw_n - Sw_n||)] \\ &+ \xi(||T((1-\beta_n)Tx_n + \beta_n Tz_n) - (1-\beta_n)Tx_n + \beta_n Tz_n||) + \epsilon \\ &\leq \delta[(1-\beta_n)(\delta||x_n - u_n|| + \xi(||Tx_n - x_n||) + \epsilon) + \beta_n(\delta||z_n - w_n|| + \xi(||Tz_n - z_n||) + \epsilon)] \\ &+ \xi(||T((1-\beta_n)Tx_n + \beta_n Tz_n) - (1-\beta_n)Tx_n + \beta_n Tz_n||) + \epsilon \end{aligned} \tag{12}$$

From equation (11) and (12), we get

$$\begin{aligned}
||y_n - v_n|| &\leq \delta[(1-\beta_n)(\delta||x_n - u_n|| + \xi(||Tx_n - x_n||) + \epsilon) \\
&+ \beta_n(\delta(\delta||x_n - u_n|| + \xi(||Tx_n - x_n||) + \epsilon) + \xi(||Tz_n - z_n||) + \epsilon) \\
&+ \xi(||T((1-\beta_n)Tx_n + \beta_n Tz_n) - (1-\beta_n)Tx_n + \beta_n Tz_n||)] + \epsilon \\
&= \delta[\delta(1-(1-\delta)\beta_n)||x_n - u_n|| + (1-(1-\delta)\beta_n)\xi(||Tx_n - x_n||) + \beta_n\xi(||Tz_n - z_n||) \\
&+ (1+\beta_n\delta)\epsilon] + \xi(||T((1-\beta_n)Tx_n + \beta_n Tz_n) - (1-\beta_n)Tx_n + \beta_n Tz_n||) + \epsilon
\end{aligned} \tag{13}$$

$$\begin{aligned}
||x_{n+1} - u_{n+1}|| &= ||T((1-\alpha_n)Tz_n + \alpha_n y_n) - S((1-\alpha_n)Sw_n + \alpha_n v_n)|| \\
&\leq ||T((1-\alpha_n)Tz_n + \alpha_n y_n) - T((1-\alpha_n)Sw_n + \alpha_n v_n)|| \\
&+ ||T((1-\alpha_n)Sw_n + \alpha_n v_n) - S((1-\alpha_n)Sw_n + \alpha_n v_n)|| \\
&\leq \delta[||(1-\alpha_n)Tz_n + \alpha_n y_n) - ((1-\alpha_n)Sw_n + \alpha_n v_n||] \\
&+ \xi(||T((1-\alpha_n)Tz_n + \alpha_n y_n) - (1-\alpha_n)Tz_n + \alpha_n y_n||) + \epsilon \\
&= \delta[(1-\alpha_n)||Tz_n - Sw_n|| + \alpha_n||y_n - v_n||] \\
&+ \xi(||T((1-\alpha_n)Tz_n + \alpha_n y_n) - (1-\alpha_n)Tz_n + \alpha_n y_n||) + \epsilon \\
&\leq \delta[(1-\alpha_n)(||Tz_n - Tw_n|| + ||Tw_n - Sw_n||) + \alpha_n||y_n - v_n||] \\
&+ \xi(||T((1-\alpha_n)Tz_n + \alpha_n y_n) - (1-\alpha_n)Tz_n + \alpha_n y_n||) + \epsilon \\
&\leq \delta[(1-\alpha_n)(\delta||z_n - w_n|| + \xi(||Tz_n - z_n||) + \epsilon) + \alpha_n||y_n - v_n||] \\
&+ \xi(||T((1-\alpha_n)Tz_n + \alpha_n y_n) - (1-\alpha_n)Tz_n + \alpha_n y_n||) + \epsilon
\end{aligned} \tag{14}$$

From equation (12), (13) and (14), we get

$$\begin{aligned}
||x_{n+1} - u_{n+1}|| &\leq \delta[(1-\alpha_n)(\delta(\delta||x_n - u_n|| + \xi(||Tx_n - x_n||) + \epsilon) + \xi(||Tz_n - z_n||) + \epsilon) \\
&+ \alpha_n(\delta[\delta(1-(1-\delta)\beta_n)||x_n - u_n|| + (1-(1-\delta)\beta_n)\xi(||Tx_n - x_n||) + \beta_n\xi(||Tz_n - z_n||) \\
&+ (1+\beta_n\delta)\epsilon] + \xi(||T((1-\beta_n)Tx_n + \beta_n Tz_n) - (1-\beta_n)Tx_n + \beta_n Tz_n||) + \epsilon)] \\
&+ \xi(||T((1-\alpha_n)Tz_n + \alpha_n y_n) - (1-\alpha_n)Tz_n + \alpha_n y_n||) + \epsilon \\
&= \delta[\delta^2(1-(1-\delta)\alpha_n\beta_n)||x_n - u_n|| + \delta(1-(1-\delta)\alpha_n\beta_n)\xi(||Tx_n - x_n||) \\
&+ (1-\alpha_n(1-\beta_n)\delta)\xi(||Tz_n - z_n||) + (\delta + 1 - \alpha_n + \alpha_n\delta + \alpha_n\beta_n\delta^2)\epsilon \\
&+ \alpha_n\xi(||T((1-\beta_n)Tx_n + \beta_n Tz_n) - (1-\beta_n)Tx_n + \beta_n Tz_n||)] \\
&+ \xi(||T((1-\alpha_n)Tz_n + \alpha_n y_n) - (1-\alpha_n)Tz_n + \alpha_n y_n||) + \epsilon
\end{aligned} \tag{15}$$

Since $\alpha_n, \beta_n \in (0,1)$ and $\delta \in (0,1)$ it implies that $\delta^2(1-\alpha_n(1-\beta_n)), \delta^2(1-(1-\delta)\alpha_n\beta_n), \delta^2, \delta^3, \alpha_n\delta^2, \alpha_n\beta_n\delta^3 < 1$. Thus

$$\begin{aligned}
||x_{n+1} - u_{n+1}|| &\leq (1-(1-\delta)\alpha_n\beta_n)||x_n - u_n|| + \xi(||Tx_n - x_n||) + \xi(||Tz_n - z_n||) \\
&+ \xi(||T((1-\beta_n)Tx_n + \beta_n Tz_n) - (1-\beta_n)Tx_n + \beta_n Tz_n||) \\
&+ \xi(||T((1-\alpha_n)Tz_n + \alpha_n y_n) - (1-\alpha_n)Tz_n + \alpha_n y_n||) + +4\epsilon
\end{aligned} \tag{16}$$

By assumption $(1-\alpha_n\beta_n) < \alpha_n\beta_n$ i.e. $1 < 2\alpha_n\beta_n$ and taking $(1-\alpha_n)Tz_n+\alpha_n y_n = l_n$ , $(1-\beta_n)Tx_n + \beta_n Tz_n = t_n$, we obtain

$$\begin{aligned}
||x_{n+1} - u_{n+1}|| &\leq (1-(1-\delta)\alpha_n\beta_n)||x_n - u_n|| + \xi(||Tx_n - x_n||) + \xi(||Tz_n - z_n||) \\
&+ \xi(||Tt_n - t_n||) + \xi(||Tl_n - l_n||) + +4\epsilon \\
&\leq (1-(1-\delta)\alpha_n\beta_n)||x_n - u_n|| + 2\alpha_n\beta_n\xi(||Tx_n - x_n||) + 2\alpha_n\beta_n\xi(||Tz_n - z_n||) \\
&+ 2\alpha_n\beta_n\xi(||Tt_n - t_n||) + 2\alpha_n\beta_n\xi(||Tl_n - l_n||) + 4.2\alpha_n\beta_n\epsilon \\
&\leq (1-(1-\delta)\alpha_n\beta_n)||x_n - u_n|| + \alpha_n\beta_n(1-\delta)\{\frac{2\xi(||Tx_n - x_n||) + 2\xi(||Tz_n - z_n||)}{(1-\delta)} \\
&+ \frac{2\xi(||Tt_n - t_n||) + 2\xi(||Tl_n - l_n||) + 8\epsilon}{(1-\delta)}\}
\end{aligned}$$

Take $a_n = ||x_n - u_n||, b_n = \alpha_n\beta_n(1-\delta)$ and

$$c_n = \frac{2\xi(||Tx_n - x_n||) + 2\xi(||Tz_n - z_n||)}{(1-\delta)} + \frac{2\xi(||Tt_n - t_n||) + 2\xi(||Tl_n - l_n||) + 8\epsilon}{(1-\delta)}$$

Since $\xi$ is a continuous strictly increasing function and $\{x_n\}, \{z_n\}, \{t_n\}, \{l_n\}$ are sequences converge to the fixed point of $T$, then

$$\lim_{n\to\infty} \xi(||Tx_n - x_n||) = \lim_{n\to\infty} \xi(||Tz_n - z_n||) = \lim_{n\to\infty} \xi(||Tt_n - t_n||) = \lim_{n\to\infty} \xi(||Tl_n - l_n||) = 0.$$

By Using Lemma 4, we obtain

$$0 \leq \limsup_{n\to\infty} ||x_n - u_n|| \leq \limsup_{n\to\infty} \frac{8\epsilon}{(1-\delta)} \tag{17}$$

Using our hypothesis that $\lim_{n\to\infty} u_n = y^*$ , (17) and Theorem 1, we conclude that

$$||x^* - y^*|| \leq \frac{8\epsilon}{(1-\delta)}$$

□

## 5. Rate of Convergence

**Theorem 4.** *Let $K \neq \phi$ be a closed convex subset of a uniformly convex Banach space $X$ and $T : K \to K$ be a map with a fixed point $x^*$ satisfying the condition* (4)*. Let $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by iteration process* (2) *and* (3)*, where $\{\alpha_n\}, \{\beta_n\} \in (0,1)$.Then $\{x_n\}_{n=0}^{\infty}$ convergences faster than $\{u_n\}_{n=0}^{\infty}$, i.e. our process* (3) *converges faster than* (2)*.*

*Proof.* From iteration (2) and equation (4), we have

$$\begin{aligned}
||u_{n+1} - x^*|| &= ||(1-\alpha_n)Tw_n + \alpha_n Tv_n - x^*|| \\
&\leq (1-\alpha_n)||Tw_n - x^*|| + \alpha_n||Tv_n - x^*|| \\
&\leq (1-\alpha_n)\delta||w_n - x^*|| + \alpha_n\delta||v_n - x^*|| 
\end{aligned} \tag{18}$$

$$\begin{aligned}
||v_n - x^*|| &= ||(1-\beta_n)Tu_n + \beta_n Tw_n - x^*|| \\
&\leq (1-\beta_n)||Tu_n - x^*|| + \beta_n||Tw_n - x^*|| \\
&\leq \delta(1-\beta_n)||u_n - x^*|| + \beta_n\delta||w_n - x^*||
\end{aligned} \tag{19}$$

$$\begin{aligned}
||w_n - x^*|| &= ||Tu_n - x^*|| \\
&\leq \delta||u_n - x^*||
\end{aligned} \tag{20}$$

From equation (18), (19) and (20), we get

$$\begin{aligned}
||u_{n+1} - x^*|| &\leq \delta^2(1 - \alpha_n\beta_n(1-\delta))||u_n - x^*|| \\
&\leq \delta^{2+2}\prod_{k=n-1}^{n}(1 - \alpha_k\beta_k(1-\delta))||u_{n-1} - x^*|| \\
&\vdots \\
&\leq \delta^{2(n+1)}\prod_{k=0}^{n}(1 - \alpha_k\beta_k(1-\delta))||u_0 - x^*|| \qquad (21)
\end{aligned}$$

Now using equation (9) in Theorem 1, we have

$$||x_{n+1} - x^*|| \leq \delta^{3(n+1)}\prod_{k=0}^{n}(1 - \alpha_k\beta_k(1-\delta))||x_0 - x^*||$$

So

$$\lim_{n\to\infty}\frac{||x_{n+1} - x^*||}{||u_{n+1} - x^*||} \leq \frac{\delta^{3(n+1)}||x_0 - x^*||}{\delta^{2(n+1)}||u_0 - x^*||}$$

Since $\delta \in [0, 1)$, we get

$$\lim_{n\to\infty}\frac{||x_{n+1} - x^*||}{||u_{n+1} - x^*||} = 0.$$

Hence by Definition (4) and Lemma 1 , we have $\{x_n\}$ convergence faster than $\{u_n\}$. So our iteration process (3) converges faster than (2). □

## 6. Numerical Example

**Example 1.** *Let* $X = R$ *and* $K \subset X$*, where* $K = [0, 100]$*. Let* $T : K \to K$ *be mapping defined by* $T(x) = \sqrt{x^2 - 9x + 90}$*, for all* $x \in K$*. Clearly, the unique fixed point of* $T$ *is* 10*. We take* $\alpha_n = \beta_n = \frac{2}{3}$ *and initial value* $x_0 = 20$*.Table* 1 *shows that the iteration process* (3) *converges to* $x^* = 10$ *faster than iteration process* (2) *and Gursoy* (1)*. The convergence behaviour of the iteration process are represented in the figure* 1*.*

| *Iteration* | *New (3)* | *Dogan(2)* | *Gursoy(1)* |
|---|---|---|---|
| 1 | 20.00000000 | 20.00000000 | 20.00000000 |
| 2 | 13.24500116 | 14.78008054 | 17.60681686 |
| 3 | 10.61434213 | 11.73001069 | 15.54151371 |
| 4 | 10.08782611 | 10.49400837 | 13.84431381 |
| 4 | 10.01181134 | 10.12564948 | 12.53260550 |
| 5 | 10.00157428 | 10.03080054 | 11.58761197 |
| 6 | 10.00020958 | 10.00747735 | 10.95373194 |
| 7 | 10.00002790 | 10.00181091 | 10.55465092 |
| 8 | 10.00000371 | 10.00043832 | 10.31546401 |
| 9 | 10.00000049 | 10.00010608 | 10.17691612 |
| 10 | 10.00000007 | 10.00002567 | 10.09838485 |
| 11 | 10.00000001 | 10.00000621 | 10.05444742 |
| 12 | 10.00000000 | 10.00000150 | 10.03004916 |
| 13 | 10.00000000 | 10.00000036 | 10.01655848 |
| 14 | 10.00000000 | 10.00000009 | 10.00911672 |
| 15 | 10.00000000 | 10.00000002 | 10.00501709 |
| 16 | 10.00000000 | 10.00000001 | 10.00276028 |
| 17 | 10.00000000 | 10.00000000 | 10.00151842 |
| 18 | 10.00000000 | 10.00000000 | 10.00083521 |
| 19 | 10.00000000 | 10.00000000 | 10.00045939 |
| 20 | 10.00000000 | 10.00000000 | 10.00025267 |

Table 1. Comparison Table

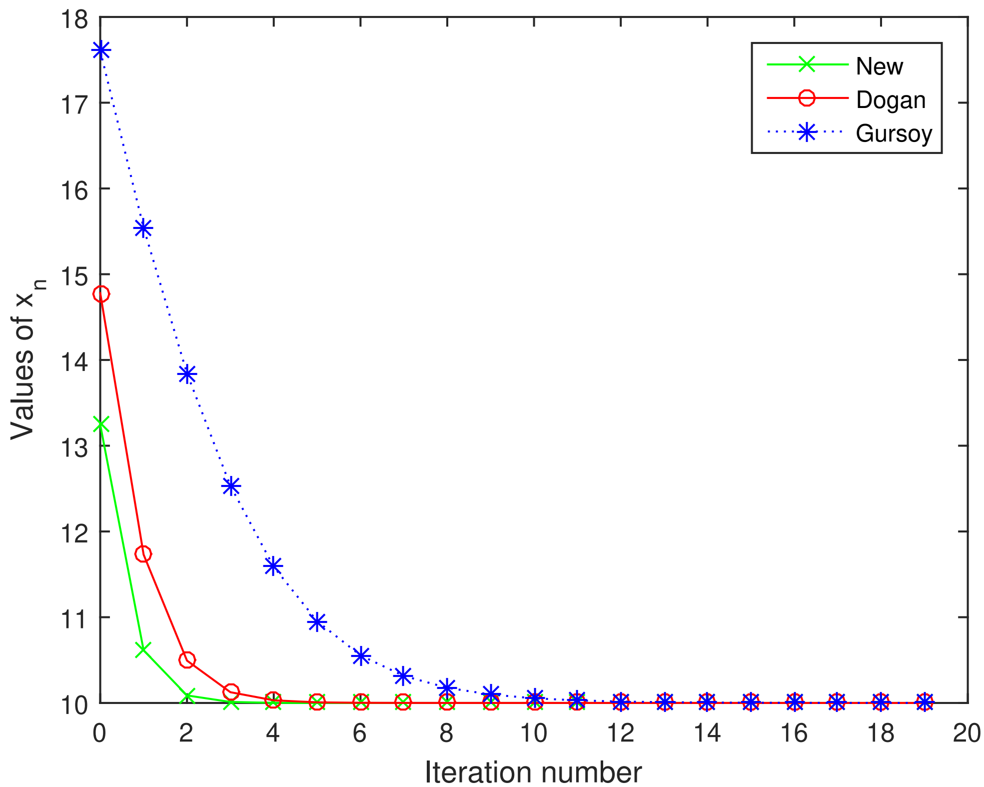


Figure 1. Comparison Plot

## 7. Conflict of interest

The authors declare that they have no conflict of interest.

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