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WEAKLY COMPATIBLE MAPS AND FIXED POINT RESULTS IN SUPER METRIC SPACE

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Abstract. The aim of this research paper is to extend the results of Karapinar and Khojasteh [13], Karapinar and Fulga [12] in super metric space by using weakly compatible mappings. Further, the analogue results of Jungck[10], Das and Naik [6], and Ciric [5] in quasi-contraction mappings are generalized in super metric space.

1. Introduction and Preliminaries

Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a mapping which satisfies

(d_1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
(d_2) $d(x, y) = d(y, x)$ for all $x, y, \in X$,
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. (triangular inequality)

Then, the pair (X, d) is called a Euclidean metric space or a metric space.
Considering Banach pioneering fixed point theorem, a large number of results have been observed. Normally, there are two main features which are used to extend and generalize the metric fixed point theory. Either the contractive conditions are weakened or abstract structure is changed.
The well-known Banach's contraction mapping principle states that if $f : X \to X$ is a contraction on X (i. e. $d(fx, fy) \leq qd(x, y)$ for some q < 1 and all x, y $\in$ X) and X is complete , then f has a unique fixed point.
A large number of generalizations of this result have appeared in the literature of fixed point theory. Ciric [5] introduced and studied quasi-contraction as one of the most general contractive type map.

Definition 1.1 ([5])**.** A mapping f : X $\to$X of a metric space X into itself is said to be a quasi- contraction if and only if there exists a number q, $0 \leq q < 1$, such that

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$$d(fx, fy) \leq q \max[d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)]$$

holds for every x, y ∈ X. The well-known Ciric [5] result is that quasi-contraction f possesses a unique fixed point.

Proposition 1.2. Let f be a quasi-contraction on a complete metric space X , then f has a unique fixed point.

Proposition 1.3 ([6])**.** Let (X, ρ) be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further, let f(X) is a subset of g(X) , and there exists a constant $\lambda \in$ (0,1) such that for every x, y ∈ X.

$$\rho(fx, fy) \leq \lambda M_\rho(x, y)$$

where

$$M_\rho(x, y) = \max[\rho(gx, gy), \rho(gx, fx), \rho(gx, fy), \rho(gy, fy), \rho(gy, fx)].$$

Then f and g have a unique fixed point.

The structure of metric space has been generalized and extended in various directions. We mention here only some famous generalized spaces such as: Cone metric space [9], S - metric space [15], G-metric space [14], Complex- valued metric space [2] , quasi- metric space [3]. Recently in 2022, Karapinar and Khojasteh [13] have introduced a new extension of metric space and named it as Super metric space.

Definition 1.4. Let X be a nonempty set and $m : X \times X \to [0, +\infty)$ be a mapping satisfying

- (m_1) if $m(x, y) = 0$, then $x = y$ for all $x, y \in X$,
- (m_2) $m(x, y) = m(y, x)$ for all $x, y \in X$,
- (m_3) there exists $s \geq 1$ such that for all $y \in X$, there exist distinct sequences $\{x_n\}, \{y_n\} \subset X$, with $m(x_n, y_n) \to 0$ when n tends to infinity, such that

$$\lim_{n\to\infty} \sup m(y_n, y) \leq s \lim_{n\to\infty} \sup m(x_n, y).$$

Then, the pair (X, m) is called a super metric space.

Definition 1.5. Let (X, m) be a super metric space and let $\{x_n\}$ be a sequence in X. We say

- (i) $\{x_n\}$ converges to x in X if and only if $m(x_n, x) \to 0$, as $n \to \infty$.
- (ii) $\{x_n\}$ is a Cauchy sequence in X if and only if $\lim_{n\to\infty} \sup\{m(x_n, x_m) : m > n\} = 0$.
- (iii) (X, m) is a complete super metric space if and only if every Cauchy sequence is convergent in X.

Proposition 1.6 ([12])**.** On a super metric space, the limit of a convergent sequence is unique.

Definition 1.7 ([1])**.** Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 1.8 ([11])**.** A pair (f, g) of self mappings of metric space (X, d) is said to be weakly compatible if the mappings commute at all of their coincidence points, that is, $fx = gx$ for some $x \in X$ implies $fgx = gfx$.

Proposition 1.9 ([1])**.** Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

In the present paper, we generalize the fixed point theorems of Karapinar and Khojasteh [13], Karapinar and Fluga [12], and Jungck [10]. Further, the results of Das and Naik [6], and Ciric [5] are proved in super metric space with a different approach.

2. Main Results

Theorem 2.1. *Let (X, m) be a complete super metric space and the mappings $f, g : X \to X$ satisfy*

$$m(fx, fy) \leq k\ max[m(gx, gy), m(gx, fx), \frac{m(gx, fy)}{2s}, m(gy, fy), m(gy, fx)] \quad (2.1)$$

for all $x, y \in X$ and $0 \leq k < 1$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point of X. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0$. In this way, we can construct two distinct sequences $\{fx_n\}$ and $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, we have $gx_n = gx_{n+1}$, then f and g have a point of coincidence. On the contrary, let $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$.

Thus, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} m(gx_n, gx_{n+1}) &= m(fx_{n-1}, fx_n) \\ &\leq k \max\Bigg[m(gx_{n-1}, gx_n), m(gx_{n-1}, fx_{n-1}), \\ &\qquad \frac{m(gx_{n-1}, fx_n)}{2s}, m(gx_n, fx_n), m(gx_n, fx_{n-1})\Bigg] \\ &= k \max\Bigg[m(gx_{n-1}, gx_n), m(gx_{n-1}, gx_n), \\ &\qquad \frac{m(gx_{n-1}, gx_{n+1})}{2s}, m(gx_n, gx_{n+1}), m(gx_n, gx_n)\Bigg] \\ &= k \max\left[m(gx_{n-1}, gx_n), \frac{m(gx_{n-1}, gx_{n+1})}{2s}, m(gx_n, gx_{n+1})\right]. \end{aligned}$$

Let us denote $A = \left[m(gx_{n-1}, gx_n), \frac{m(gx_{n-1}, gx_{n+1})}{2s}, m(gx_n, gx_{n+1})\right]$.
If $\max A = m(gx_n, gx_{n+1})$,then

$$m(gx_n, gx_{n+1}) \leq k\ m(gx_n, gx_{n+1}) < m(gx_n, gx_{n+1}),$$

which is a contradiction.
Further, if $\max A = \frac{m(gx_{n-1},gx_{n+1})}{2s}$, then

$$m(gx_n, gx_{n+1}) \leq k\left[\frac{m(gx_{n-1}, gx_{n+1})}{2s}\right] < \frac{m(gx_{n-1}, gx_{n+1})}{2s},$$

and using (m_3), we have

$$\begin{aligned}\lim_{n\to\infty} \sup m(gx_n, gx_{n+1}) &\leq \frac{1}{2s} \lim_{n\to\infty} \sup m(gx_{n-1}, gx_{n+1}) \\ &\leq \frac{s}{2s} \lim_{n\to\infty} \sup m(fx_{n-1}, gx_{n+1}) \\ &= \frac{1}{2} \lim_{n\to\infty} \sup m(gx_n, gx_{n+1}),\end{aligned}$$

which is again a contradiction. Therefore,

$$\begin{aligned}m(gx_n, gx_{n+1}) &= m(fx_{n-1}, fx_n) \\ &\leq k\ m(gx_{n-1}, gx_n) \\ &\leq k^2\ m(gx_{n-2}, gx_{n-1}) \\ &\ \vdots \\ &\leq k^n\ m(gx_0, gx_1). \end{aligned} \tag{2.2}$$

Our aim is to prove that $\{gx_n\}$ is Cauchy sequence. Let $\epsilon > 0$.
Since $\lim\limits_{n\to\infty} k^n\ m(gx_0, gx_1) = 0$, there exists $N \in \mathbb{N}$ such that

$$k^n\ [m(gx_0, gx_1)] < \epsilon \quad \text{for all } n \geq N.$$

Therefore, using (2.2) for all $n \geq$ N

$$m(gx_n, gx_{n+1}) < \epsilon. \tag{2.3}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$m(gx_n, gx_m) < \epsilon \quad \text{for all } m \geq n \geq N. \tag{2.4}$$

Now from (2.3), we get that the result is true for $m = n + 1$. If $x_n = x_m$, (2.4) is trivially true.

Without loss of generality, we can take $x_n \neq x_m$. Suppose (2.4) is true for $m = k$ i.e.

$$\lim_{n\to\infty} \sup m(gx_n, gx_k) = 0.$$

Therefore, by using (2.1) for $m = k+1$ we have

$$\begin{aligned} m(gx_n, gx_{k+1}) = m(fx_{n-1}, fx_k) &\le k \max\Bigg[m(gx_{n-1}, gx_k), m(gx_{n-1}, fx_{n-1}), \\ &\qquad \frac{m(gx_{n-1}, fx_k)}{2s}, m(gx_k, fx_k), m(gx_k, fx_{n-1}) \Bigg] \\ &= k \max\Bigg[m(gx_{n-1}, gx_k), m(gx_{n-1}, gx_n), \\ &\qquad \frac{m(gx_{n-1}, gx_{k+1})}{2s}, m(gx_k, gx_{k+1}), m(gx_k, gx_n) \Bigg]. \end{aligned}$$

Put $B = \left[m(gx_{n-1}, gx_k), m(gx_{n-1}, gx_n) \frac{m(gx_{n-1}, gx_{k+1})}{2s}, m(gx_k, gx_{k+1}), m(gx_k, gx_n) \right]$.

If max $B = m(gx_{n-1}, gx_k)$, then

$$\begin{aligned} m(gx_n, gx_{k+1}) &= m(fx_{n-1}, fx_k) \\ &\le k\ m(gx_{n-1}, gx_k). \end{aligned}$$

Taking $n \to \infty$

$$\lim_{n\to\infty} \sup m(gx_n, g_{k+1}) \le k \lim_{n\to\infty} \sup m(gx_{n-1}, gx_k).$$

Using (m_3), we get

$$\lim_{n\to\infty} \sup m(gx_n, gx_{k+1}) \le s\ k \lim_{n\to\infty} \sup m(fx_{n-1}, gx_k) \quad = s\ k \lim_{n\to\infty} \sup m(gx_n, gx_k) = 0.$$

If max $B = m(gx_{n-1}, gx_n)$, then

$$\begin{aligned} \lim_{n\to\infty} \sup m(gx_n, gx_{k+1}) &\le k \lim_{n\to\infty} \sup m(gx_{n-1}, gx_n) \\ &< \lim_{n\to\infty} \sup m(gx_{n-1}, gx_n) \\ &\le s \lim_{n\to\infty} \sup m(fx_{n-1}, gx_n)\ (by\ m_3) \\ &= s \lim_{n\to\infty} \sup m(gx_n, g_n) \\ &= 0. \end{aligned}$$

If max $B = \frac{m(gx_{n-1}, gx_{k+1})}{2s}$, then

$$\begin{aligned} \lim_{n\to\infty} \sup m(gx_n, gx_{k+1}) &\le \frac{1}{2s} k \lim_{n\to\infty} \sup m(gx_{n-1}, gx_{k+1}) \\ &< \frac{1}{2s} \lim_{n\to\infty} \sup m(gx_{n-1}, gx_{k+1}) \\ &\le \frac{s}{2s} \lim_{n\to\infty} \sup m(fx_{n-1}, g_{k+1}) (by\ m_3) \\ &= \frac{1}{2} \lim_{n\to\infty} \sup m(gx_n, g_{k+1}), \end{aligned}$$

which is a contradiction.
If $\max B = m(gx_k, gx_{k+1})$, then

$$\begin{aligned}\lim_{n\to\infty}\sup m(gx_n, gx_{k+1}) &\le k\lim_{n\to\infty}\sup m(gx_k, gx_{k+1})\\ &< \lim_{n\to\infty}\sup m(gx_k, g_{k+1})\\ &\le s\lim_{n\to\infty}\sup m(fx_k, g_{k+1})\ (by m_3)\\ &\le s\lim_{n\to\infty}\sup m(gx_{k+1}, g_{k+1})\\ &= 0.\end{aligned}$$

If $\max B = m(gx_n, gx_k)$, then the result is clear.
Hence, by induction $\lim_{n\to\infty}\sup m(gx_n, gx_{k+1}) = 0$.
This shows that $\{gx_n\}$ is a Cauchy sequence. By completeness of $g(X)$, we get that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$, such that $gp = q = \lim_{n\to\infty} gx_n$. We will show that $gp = fp$.
We have,

$$\begin{aligned}m(gp, fp) &= \lim_{n\to\infty} m(gx_n, fp)\\ &= \lim_{n\to\infty} m(fx_{n-1}, fp).\end{aligned}$$

Consider $m(fx_{n-1}, fp)$ and applying (2.1), we obtain

$$\begin{aligned}m(fx_{n-1}, fp) &\le k\max\Bigg[m(gx_{n-1}, gp), m(gx_{n-1}, fx_{n-1}),\\ &\qquad \frac{m(gx_{n-1}, fp)}{2s}), m(gp, fp), m(gp, fx_{n-1})\Bigg]\\ &< \max\Bigg[m(gx_{n-1}, gp), m(gx_{n-1}, fx_{n-1}),\\ &\qquad \frac{m(gx_{n-1}, fp)}{2s}), m(gp, fp), m(gp, fx_{n-1})\Bigg]\\ &= \max\Bigg[m(gx_{n-1}, gp), m(gx_{n-1}, gx_n),\\ &\qquad \frac{m(gx_{n-1}, fp)}{2s}), m(gp, fp), m(gp, gx_n)\Bigg].\end{aligned}$$

Taking $n \to \infty$, gives

$$m(gp, fp) < \max\Bigg[m(gp, gp), m(gp, gp), \tfrac{m(gp,fp)}{2s}), m(gp, fp), m(gp, gp)\Bigg] = m(gp, fp),$$

which is a contradiction.
Therefore $gp = fp$. We will now show that f and g have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in X$. By applying (2.1), it follows

that

$$\begin{aligned} m(gp, gq) &= m(fp, fq) \\ &\leq k\ max\left[m(gp, gq), m(gp, fp), \frac{m(gp, fq)}{2s}, m(gq, fq), m(gq, fp)\right] \\ &\leq k\ m(gp, gq) < m(gp, gq), \end{aligned}$$

which is a contradiction. Hence we have $gp = gq$.

This implies that f and g have a unique point of coincidence. By Proposition 1.9, we conclude that f and g have a unique common fixed point.
This complete the proof of theorem. □

Remark 2.2. Let $g = I_X$, be identity map on X in Theorem 2.1, we get a generalization and extension of Karapinar and Khojasteh result [13, Theorem 2.6.]
"Let (X, M) be a complete super metric space and let $T : X \to X$ be a mapping. Suppose that $0 < k < 1$ such that

$$m(Tx, Ty) \leq k\ m(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point in X".

Remark 2.3. An example [13, Example 2.7] is proposed to apply [13, Theorem 2.6.], where $X = [2, 3]$ and $T : X \to X$ is defined as

$$Tx = \begin{cases} 2, & x \neq 3, \\ \frac{3}{2}, & x = 3. \end{cases}$$

But the mapping T is not a valid mapping on $X = [2, 3]$. Thus, the main motto of the example is forfeited.

Remark 2.4. In Example 2.7 [13], there seems to be no typographical error in writing the set $X = [2, 3]$, since Theorem 2.6 [13] is verified for $2 \leq x < 3$.

Remark 2.5. [8] A rectification on Remark 2.3 is given and generalization of Karapinar and Khojasteh result [13, Theorem 2.6] is obtained.

Remark 2.6. If we take $g = I_X$, the Identity map on X in Theorem 2.1., we can deduce the analogue of Proposition 1.3 in super metric space.

Remark 2.7. Taking $g = I_X$, the Identity map on X in Theorem 2.1., one can deduce an extended analogue of the following result of Jungck [10] in complete super metric space:
"Let (X, ρ) be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further, let f and g satisfy $f(X) \subset g(X)$ and there exists a constant $\lambda \in (0, 1)$ such that for every $x, y \in X$

$$\rho(fx, fy) \leq \lambda\ \rho(gx, gy).$$

Then f and g have a unique common fixed point".

Example 2.8. *Let* $X = [1, 3]$ *and define*

$$m(x, y) = \begin{cases} xy, & x \neq y, \\ 0, & x = y. \end{cases}$$

It has been shown in [13] *that* (X, m) *is a super metric space. Further, let* $k = \frac{1}{2}$. *Now consider* $f, g : X \to X$ *as follows*

$$fx = \begin{cases} 2, & x \neq 3, \\ \frac{3}{2}, & x = 3 \end{cases} \quad \text{and} \quad gx = 4 - x.$$

Here $g(X) = [1, 3]$, $f(X) \subset g(X)$ *and* $g(X)$ *is complete space.*
We obtain that f *and* g *satisfy the contractive conditions of Theorem 2.1. Indeed for* $x \neq 3$, $y = 3$ *and* $s = 6$, *we obtain*

$$m(fx, fy) = m\left(2, \frac{3}{2}\right) = 2 \times \frac{3}{2} = 3.$$

We calculate the right hand side of Theorem 2.1 for this example as follows :

(i) $m(gx, gy) = m(gx, 1) = gx$, *where* $gx \in (1, 3]$.

(ii) $m(gx, fx) = [m(gx, 2)] = 2gx$, *where* $gx \in (1, 3]$.

(iii) $\frac{m(gx,fy)}{2s} = \frac{1}{2}\left[\frac{m(gx,\frac{3}{2})}{s}\right] = \frac{1}{2}(\frac{3gx}{2s}) \leq \frac{1}{2}(\frac{3}{2}gx)$, *where* $gx \in (1, 3]$.

(iv) $m(gy, fy) = m(1, \frac{3}{2}) = \frac{3}{2}$.

(v) $m(gy, fx) = m(1, 2) = 2$.

Therefore, maximum of right hand side of this example for Theorem 2.1 is 2 gx *where* $gx \in (1, 3]$.
The other cases are straightforward. Now for $x = 2$, $fx = gx$ *and* $fgx = gfx$. *So,* 2 *is the unique point of coincidence of* f *and* g. *Thus all the conditions of Theorem 2.1 are satisfied. Therefore,* 2 *is the unique common fixed point by Theorem 2.1.*

Theorem 2.9. *Let* (X, m) *be a complete super metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$m(fx, fy) \leq k\left[\max\left\{m(gx, gy), \frac{m(gx, fy) + m(gy, fx)}{2s}, \frac{m(gx, fx)m(gx, fy) + m(gy, fy)m(gy, fx)}{m(gx, fy) + m(gy, fx) + 1}\right\}\right] \tag{2.5}$$

for all $x, y \in X$. *If* $f(X) \subset g(X)$ *and* $g(X)$ *is a complete subspace of* X, *then* f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0$. Inductively, we can construct two distinct sequences $\{fx_n\}$ and $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Thus, we can suppose that

$gx_n \neq gx_{n+1}$, for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
m(gx_n, gx_{n+1}) &= m(fx_{n-1}, fx_n)\\
&\leq k\Bigg[\max\Bigg\{m(gx_{n-1}, gx_n), \frac{m(gx_{n-1}, fx_n) + m(gx_n, fx_{n-1})}{2s},\\
&\qquad \frac{m(gx_{n-1}, fx_{n-1})\ m(gx_{n-1}, fx_n) + m(gx_n, fx_n)\ m(gx_n, fx_{n-1})}{m(gx_{n-1}, fx_n) + m(gx_n, fx_{n-1}) + 1}\Bigg\}\Bigg]\\
&= k\Bigg[\max\Bigg\{m(gx_{n-1}, gx_n), \frac{m(gx_{n-1}, gx_{n+1}) + m(gx_n, gx_n)}{2s},\\
&\qquad \frac{m(gx_{n-1}, gx_n)\ m(gx_{n-1}, gx_{n+1}) + m(gx_n, gx_{n+1})\ m(gx_n, gx_n)}{m(gx_{n-1}, gx_{n+1}) + m(gx_n, gx_n) + 1}\Bigg\}\Bigg]\\
&\leq k\left[\max\left\{m(gx_{n-1}, gx_n), \frac{m(gx_{n-1}, gx_{n+1})}{2s}\right\}\right].
\end{aligned}$$

If $\max\left[m(gx_{n-1}, gx_n), \frac{m(gx_{n-1}, gx_{n+1})}{2s}\right] = \frac{m(gx_{n-1}, gx_{n+1})}{2s}$, then

$$m(gx_n, gx_{n+1}) \leq k\Big[\frac{m(gx_{n-1}, gx_{n+1})}{2s}\Big] < \frac{m(gx_{n-1}, gx_{n+1})}{2s}.$$

Taking limit as $n \to \infty$ on both sides implies that

$$\begin{aligned}
\lim_{n\to\infty} \sup m(gx_n, gx_{n+1}) &< \frac{1}{2s} \lim_{n\to\infty} \sup m(gx_{n-1}, gx_{n+1})\\
&\leq \frac{s}{2s} \lim_{n\to\infty} \sup m(fx_{n-1}, gx_{n+1})\ (by\ m_3)\\
&= \frac{1}{2} \lim_{n\to\infty} \sup m(gx_n, gx_{n+1}),
\end{aligned}$$

giving a contradiction. Therefore,

$$m(gx_n, gx_{n+1}) \leq k\ m(gx_{n-1}, gx_n).$$

That is, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
m(gx_n, gx_{n+1}) &= m(fx_{n-1}, fx_n)\\
&\leq k\ m(gx_{n-1}, gx_n)\\
&\leq k^2\ m(gx_{n-2}, gx_{n-1})\\
&\ \ \vdots\\
&\leq k^n\ m(gx_0, gx_1).
\end{aligned}$$

We will show that $\{gx_n\}$ is a Cauchy sequence.

Since $\lim\limits_{n\to\infty} k^n\ m(gx_0, gx_1) = 0$, there exists $N \in \mathbb{N}$, such that

$$k^n\ m(gx_0, gx_1) < \epsilon \quad \text{for all } n \geq \text{N}.$$

This implies that

$$m(gx_n, gx_{n+1}) < \epsilon \quad \text{for all } n \geq N. \tag{2.6}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$m(gx_n, gx_m) < \epsilon \quad \text{for all } m \geq n \geq N \tag{2.7}$$

by induction on m. From (2.6), the result is true for $m = n+1$. Suppose that (2.7) holds for $m = k$. Therefore, for $m = k+1$, we have

$$\begin{aligned} m(gx_n, gx_{k+1}) &= m(fx_{n-1}, fx_k) \\ &\leq k\Bigg[\max\Bigg\{m(gx_{n-1}, gx_k), \frac{m(gx_{n-1}, fx_k) + m(gx_k, fx_{n-1})}{2s}, \\ &\qquad \frac{m(gx_{n-1}, fx_{n-1})\ m(gx_{n-1}, fx_k) + m(gx_k, fx_k)\ m(gx_k, fx_{n-1})}{m(gx_{n-1}, fx_k) + m(gx_k, fx_{n-1}) + 1}\Bigg\}\Bigg]. \end{aligned}$$

Denote $A = \Big[m(gx_{n-1}, gx_k), \frac{m(gx_{n-1},fx_k)+m(gx_k,fx_{n-1})}{2s}, \frac{m(gx_{n-1},fx_{n-1})\ m(gx_{n-1},fx_k)+m(gx_k,fx_k)\ m(gx_k,fx_{n-1})}{m(gx_{n-1},fx_k)+m(gx_k,fx_{n-1})+1}\Big]$.

If $\max A = [m(gx_{n-1}, gx_k)]$, then

$$m(gx_n, gx_{k+1}) \leq k\ m(gx_{n-1}, gx_k) < m(gx_{n-1}, gx_k).$$

Using (m_3),

$$\begin{aligned} \lim_{n\to\infty} \sup m(gx_n, gx_{k+1}) &\leq s \lim_{n\to\infty} \sup m(fx_{n-1}, gx_k) \\ &= s \lim_{n\to\infty} \sup m(gx_n, gx_k) \\ &= 0. \end{aligned}$$

Hence

$$m(gx_n, gx_{k+1}) < \epsilon. \tag{2.8}$$

If $\max A = \frac{m(gx_{n-1},fx_k)+m(gx_k,fx_{n-1})}{2s}$, then

$$\begin{aligned} m(gx_n, gx_{k+1}) &\leq k\left[\frac{m(gx_{n-1}, fx_k) + m(gx_k, fx_{n-1})}{2s}\right] \\ &< \frac{m(gx_{n-1}, fx_k) + m(gx_k, fx_{n-1})}{2s} \\ &= \frac{m(gx_{n-1}, gx_{k+1}) + m(gx_k, gx_n)}{2s}. \end{aligned}$$

Taking $n \to \infty$ and using (m_3),

$$\begin{aligned} \lim_{n\to\infty} \sup m(gx_n, gx_{k+1}) &< \frac{1}{2s} \lim_{n\to\infty} \sup m(gx_{n-1}, gx_{k+1}) + \frac{1}{2s} \lim_{n\to\infty} \sup m(gx_n, gx_k) \\ &\leq \frac{s}{2s} \lim_{n\to\infty} \sup m(fx_{n-1}, gx_{k+1}) + \frac{s}{2s} \lim_{n\to\infty} \sup m(gx_n, fx_k) \\ &= \frac{1}{2} \lim_{n\to\infty} \sup m(gx_n, gx_{k+1} + \frac{1}{2} \lim_{n\to\infty} \sup m(gx_n, gx_{k+1}), \end{aligned}$$

which gives a contradiction.
If $\max A = \frac{m(gx_{n-1},fx_{n-1})\ m(gx_{n-1},fx_k)+m(gx_k,fx_k)\ m(gx_k,fx_{n-1})}{m(gx_{n-1},fx_k)+m(gx_k,fx_{n-1})+1}$, then

$$\begin{aligned}
m(gx_n, gx_{k+1}) &\leq k\left[\frac{m(gx_{n-1}, fx_{n-1})\ m(gx_{n-1}, fx_k) + m(gx_k, fx_k)\ m(gx_k, fx_{n-1})}{m(gx_{n-1}, fx_k) + m(gx_k, fx_{n-1}) + 1}\right]\\
&= k\left[\frac{m(gx_{n-1}, gx_n)\ m(gx_{n-1}, gx_{k+1}) + m(gx_k, gx_{k+1})\ m(gx_k, gx_n)}{m(gx_{n-1}, gx_{k+1}) + m(gx_k, gx_n) + 1}\right]\\
&= k\left[\frac{m(gx_{n-1}, gx_n)\ m(gx_{n-1}, gx_{k+1}) + m(gx_k, gx_{k+1})\ m(gx_k, gx_n)}{m(gx_{n-1}, gx_{k+1}) + m(gx_k, gx_n) + 1}\right]\\
&= k\left[\frac{m(gx_{n-1}, gx_n)\ m(gx_{n-1}, gx_{k+1})}{m(gx_{n-1}, gx_{k+1})\ m(gx_k, gx_n) + 1} + \frac{m(gx_k, gx_{k+1})\ m(gx_k, gx_n)}{m(gx_{n-1}, gx_{k+1})\ m(gx_k, gx_n) + 1}\right]\\
&\leq k\Big[m(gx_{n-1}, gx_n) + m(gx_k, gx_{k+1})\Big].
\end{aligned}$$

Taking $n \to \infty$ and using (m_3), we have

$$\begin{aligned}
\lim_{n\to\infty} \sup m(gx_n, gx_{k+1}) &\leq k \lim_{n\to\infty} \sup[m(gx_{n-1}, gx_n) + m(gx_k, gx_{k+1})]\\
&\leq s\ k \lim_{n\to\infty} \sup[m(fx_{n-1}, gx_n) + m(fx_k, gx_{k+1})]\\
&= s\ k \lim_{n\to\infty} \sup[m(gx_n, gx_n) + m(gx_{k+1}, gx_{k+1})\\
&= 0, \qquad \text{since } s \geq 1 \text{ is finite.}
\end{aligned}$$

Therefore,

$$m(gx_n, gx_{k+1}) < \epsilon. \tag{2.9}$$

Thus (2.8) holds for all $m \geq n \geq N$. It follows that $\{gx_n\}$ is a Cauchy sequence. By the completeness of $g(X)$, we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q = \lim_{n\to\infty} gx_n$. We will show that $gp = fp$. Suppose that $gp \neq fp$.
We have,

$$\begin{aligned}
m(gp, fp) &= \lim_{n\to\infty} m(gx_n, fp)\\
&= \lim_{n\to\infty} m(fx_{n-1}, fp).
\end{aligned}$$

Consider $m(fx_{n-1}, fp)$ and applying (2.5), we obtain

$$\begin{aligned}
m(fx_{n-1}, fp) &\leq k\Bigg[\max\Bigg\{m(gx_{n-1}, gp), \frac{m(gx_{n-1}, fp) + m(gp, fx_{n-1})}{2s},\\
&\qquad \frac{m(gx_{n-1}, fx_{n-1})\ m(gx_{n-1}, fp) + m(gp, fp)\ m(gp, fx_{n-1})}{m(gx_{n-1}, fp) + m(gp, fx_{n-1}) + 1}\Bigg\}\Bigg]\\
&= k\Bigg[\max\Bigg\{m(gx_{n-1}, gp), \frac{m(gx_{n-1}, fp) + m(gp, gx_n)}{2s},\\
&\qquad \frac{m(gx_{n-1}, gx_n)\ m(gx_{n-1}, fp) + m(gp, fp)\ m(gp, gx_n)}{m(gx_{n-1}, fp) + m(gp, gx_n) + 1}\Bigg\}\Bigg].
\end{aligned}$$

Taking limit as $n \to \infty$

$$m(gp, fp) \leq k\Bigg[\max\Bigg\{m(gp, gp), \frac{m(gp, fp) + m(gp, gp)}{2s}, \frac{m(gp, gp)\ m(gp, fp) + m(gp, fp)\ m(gp, gp)}{m(gp, fp) + m(gp, gp) + 1}\Bigg\}\Bigg]$$
$$= k\Big[\frac{m(gp, fp)}{2s}\Big] < \Big[\frac{m(gp, fp)}{2s}\Big],$$

giving a contradiction, since $s \geq 1$. So, $gp = fp$.
We now show that f and g have a unique point of coincidence. Let $fq = gq$ for some $q \in X$.
Assume that $gp \neq gq$. By applying (2.5), it follows that

$$\begin{aligned}
m(gp, gq) &= m(fp, fq) \\
&\leq k\Bigg[\max\Bigg\{m(gp, gq), \frac{m(gq, fq) + m(gq, fp)}{2s}, \frac{m(gp, fp)\ m(gp, fq) + m(gq, fq)\ m(gq, fp)}{m(gp, fq)\ m(gp, fp) + 1}\Bigg\}\Bigg] \\
&= k\Bigg[\max\Bigg\{m(gp, gq), \frac{m(gq, gq) + m(gq, gp)}{2s}, \frac{m(gp, gp)\ m(gp, gq) + m(gq, gq)\ m(gq, gp)}{m(gp, gq)\ m(gp, gp) + 1}\Bigg\}\Bigg] \\
&= k\Big[\max\Big\{m(gp, gq), \frac{m(gp, gq)}{s}\Big\}\Big] \\
&= k\ m(gp, gq).
\end{aligned}$$

Therefore, $m(gp, gq) \leq k\ m(gp, gq) < m(gp, gq)$ which leads to a contradiction. Hence $gp = gq$.

This implies that f and g have a unique point of coincidence. By Proposition 1.9, we can conclude that f and g have a unique common fixed point. □

Remark 2.10. If we take $g = I_X$, the identity map on X in Theorem 2.9., one can deduce the following result of Karapinar and Fulga [10] as a corollary.

Corollary 2.11. Let (X, m, s) be a complete super metric space and $T : X \to X$ be an asymptotically regular mapping. If there exists $k \in [0, 1)$, such that

$$m(Tx, Ty) \leq k \max\left[m(x, y), \frac{m(x, Ty) + m(y, Tx)}{2s}, \frac{m(x, Tx)\ m(x, Ty) + m(y, Ty)\ m(y, Tx)}{m(x, Ty) + m(y, Tx) + 1}\right].$$

Then T has a unique fixed point.

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