*Electronic Journal of Mathematical Analysis and Applications*
Vol. 12(2) July 2024, No.1.
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

# CONVERGENCE OF THAKUR ITERATION SCHEME FOR MEAN NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

OMPRAKASH SAHU AND AMITABH BANERJEE

ABSTRACT. The purpose of this paper, we modify the Thakur iteration process into hyperbolic metric spaces where the symmetry condition is satisfied and establish strong and $\Delta$- convergence theorems for mean nonexpansive mappings in uniformly convex hyperbolic spaces. We provide an example of mean nonexpansive mapping which is not nonexpansive mapping. Using this example and some numerical texts, we infer empirically that the Thakur iteration process converges faster than the Abbas, Agarwal, Noor, Ishikawa and Mann iteration process.

## 1. INTRODUCTION

Let $(X, d)$ be a metric space and $C$ be a nonempty closed and convex subset of $X$. A mapping $T : C \to C$ is said to be

(i) nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \forall x, y \in C.$$

(ii) mean nonexpansive mapping if

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Ty), \forall x, y \in C.$$

where $a, b \geq 0$ and $a + b \leq 1$.

The class of mean nonexpansive mapping was first introduced by Zhang [27], who proved that a mean nonexpansive mapping has a fixed point in a weakly compact convex subset $C$ (with normal structure) of a Banach space. Since then several authors have studied mean nonexpansive mappings in Banach Spaces. For examples, Wu and Zhang[25] investigated some properties of mean nonexpansive mappings and they proved that if $a + b < 1$, then mean nonexpansive mapping

2010 *Mathematics Subject Classification.* 47H09, 47H10 47H20.

*Key words and phrases.* Hyperbolic Space, Thakur iteration process, Mean nonexpansive mapping, Strong and $\Delta$-convergence theorem.

Submitted Feb 6, 2024, Revised March 14, 2024.

has a unique fixed point. In the same year, Zhao[28] proved convergence of Picard and Mann iteration for mean nonexpansive mappings. In 2007, Gu and Li [10] proved strong convergence of Ishikawa iteration scheme for mean nonexpansive mapping in the framework of uniformly convex Banach space. In 2012, Ouahab et al.[20] studied fixed point result for mean nonexpansive mapping in Hilbert space. Ouahab et al.[20] introduced modulus of the convexity and Chebyshev radius. In 2014, Zuo [30] proved that mean nonexpansive mapping has approximate fixed-point sequence, and, under some suitable conditions, we get some existence and uniqueness theorems of fixed point. In 2015, mean nonexpansive mappings was introduced and studied in CAT(0) space by Zhou and Cui [29] using the following Ishikawa iteration for $x_1 \in C$, $\{t_n\}$, $\{s_n\} \subset [0, 1]$ define iteratively by

$$\begin{cases} y_n = (1 - s_n)x_n \oplus s_n T x_n \\ x_{n+1} = (1 - t_n)x_n \oplus t_n T y_n \end{cases} \tag{1}$$

They proved both strong and $\Delta$-convergence theorems for the sequence $\{x_n\}$ generated by the above iteration.

In 2017, Chen et al. [5] introduced the concept of a mean nonexpansive set-valued mapping in Banach spaces, and extended Nadler's fixed point theorem and Lim's fixed point theorem to the case of mean nonexpansive set-valued mappings. In 2018, Akbar et al. [4] introduced a new iterative scheme to approximate the fixed point of mean nonexpansive mapping in CAT(0) spaces. They proved convergence results for mean nonexpansive mapping in CAT(0) space. In 2021, Ahmad et al. [3], established weak and strong convergence theorems for mean nonexpansive maps in Banach spaces under the Picard–Mann hybrid iteration process.

In 2022 Ezeora et al. [8] studied some fixed points properties and demiclosedness principle for mean nonexpansive mappings in uniformly convex hyperbolic space and established both strong and $\Delta$-convergence theorems for approximating a common fixed point of two mean nonexpansive mappings using the following iterative scheme introduced by Abbas and Nazir [1]. Let $C$ be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $X$ and $T, S : C \to C$ be two mean nonexpansive mappings. For $x_1 \in C$, we construct the sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = W(x_n, Sx_n, \gamma_n) \\ y_n = W(Sz_n, Tz_n, \beta_n) \\ x_{n+1} = W(Ty_n, T_n, \alpha_n) \end{cases} \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$.

Several fixed point results and iterative algorithms for approximating the fixed points of nonlinear mappings in Hilbert and Banach spaces. It is easier working with Banach space due to its convex structures. However, metric space do not naturally enjoy this structure. Therefore the need to introduce convex structures to it arises. The concept of convex metric space was first introduced by Takahashi[23] who studied the fixed points for nonexpansive mappings in the setting of convex metric spaces. Since then, several attempts have been made to introduce different convex structures on metric spaces. An example of a metric space with a convex structure is the hyperbolic space. Different convex structures have been introduced on hyperbolic spaces resulting in different definitions of hyperbolic spaces. Although the class of hyperbolic spaces defined by Kohlenbach[14] is slightly more restrictive than the class of hyperbolic spaces introduced in [9], it is however, more general than the class of hyperbolic spaces introduced in[21]. Moreover, it is well-known that Banach spaces and CAT(0) spaces are examples of hyperbolic spaces introduced in[14].

It is worth mentioning that, as far as we know, little to no work has been done on fixed point problems for mean nonexpansive mappings in hyperbolic spaces. Therefore, it is necessary to extend results on fixed point problems for mean non-expansive mappings from uniformly convex Banach spaces and CAT(0) spaces to uniformly convex hyperbolic spaces, since the class of uniformly convex hyperbolic spaces generalizes the class of uniformly convex Banach spaces as well as CAT(0) spaces.

Motivated by all these facts, we study some fixed points results for mean non-expansive mappings in uniformly convex hyperbolic space and establish strong and $\Delta$-convergence theorems for approximating a fixed point of mean nonexpansive mappings in hyperbolic spaces using the iterative scheme introduced by Thakur et al.[24].

## 2. Preliminaries

In this section, we shall discuss some definitions and results to be used in main results. Our study is in hyperbolic space introduced by Kohlenbach[14]:

**Definition 2.1.** [14]A hyperbolic space $(X, d, W)$ is a metric space $(X, d)$ together with a convex mapping $W : X^2 \times [0, 1] \to X$ satisfying:

(i) $d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, x) + \alpha d(u, y)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
(iii) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$,
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$, for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

**Definition 2.2.** [22]Let $X$ be a real Banach space which is equipped with norm $||.||$. Define the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = ||x - y||.$$

Then, we have that $(X, d, W)$ is a hyperbolic space with mapping $W : X^2 \times [0, 1] \to X$ defined by $W(x, y, \alpha) = (1 - \alpha)x + \alpha y$.

**Definition 2.3.** [22]Let $X$ be a hyperbolic space with a mapping $W : X^2 \times [0, 1] \to X$.

(i) A nonempty subset $C$ of $X$ is said to be convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in [0, 1]$.
(ii) $X$ is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$ .
$$d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r$$
provided $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq \epsilon r$.
(iii) A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \epsilon)$ for a given$r > 0$ and $\epsilon \in (0, 2]$, is known as a modulus of uniform convexity of $X$. The mapping $\eta$ is said to be monotone, if it decreases with $r$ (for a fixed $\epsilon$).

**Definition 2.4.** Let $C$ be a nonempty subset of a metric space $X$ and $\{x_n\}$ be any bounded sequence in $C$. For $x \in X$, consider a continuous functional $r(., \{x_n\}) : X \to [0, \infty)$ defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius $r(C, \{x_n\})$ of $\{x_n\}$ with respect to $C$ is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

A point $x \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to $C \subseteq X$ if

$$r(x, \{x_n\}) = \inf\{r(y, \{x_n\}) : y \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to $C$ is denoted by $A(C, \{x_n\})$. If the asymptotic radius and the asymptotic center are taken with respect to $X$, then they are simply denoted by$r(\{x_n\})$ and $A(\{x_n\})$ respectively.

**Definition 2.5.** [15]A sequence $x_n$ in $X$ is said to be $\Delta$-converge to $x \in X$, if $x$ is the unique asymptotic center of $\{x_{nk}\}$ for every subsequence $\{x_{nk}\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$.

**Remark 2.1.** [16]We note that $\Delta$-convergence coincides with the usually weak convergence known in Banach spaces with the usual Opial property.

**Lemma 2.0.1.** [17]*Let $X$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Then every bounded sequence $\{x_n\}$ in $X$ has a unique asymptotic center with respect to any nonempty closed convex subset $C$ of $X$.*

**Lemma 2.0.2.** [6]*Let $X$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$ and let $\{x_n\}$ be a bounded sequence in $X$ with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{nk}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_{nk}\}) = \{x_1\}$ and $\{d(x_n, x_1)\}$ converges, then $x = x_1$.*

**Lemma 2.0.3.** [13]*Let $(X, d, W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $x^* \in X$ and $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\limsup_{n\to\infty} d(x_n, x^*) \leq c$,$\limsup_{n\to\infty} d(y_n, x^*) \leq c$ and $\lim_{n\to\infty} d(W(x_n, y_n, t_n), x^*) = c$, for some $c > 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

**Definition 2.6.** Let $C$ be a nonempty subset of a hyperbolic space $X$ and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is called a Fejer monotone sequence with respect to $C$ if for all $x \in C$ and $n \in N$,

$$d(x_{n+1}, x) \leq d(x_n, x).$$

**Proposition 2.1.** [11] *Let $\{x_n\}$ be a sequence in $X$ and $C$ be a nonempty subset of $X$. Suppose that $T : C \to C$ is any nonlinear mapping and the sequence $\{x_n\}$ is Fejer monotone with respect to $C$, then we have the following:*

(i) *$\{x_n\}$ is bounded.*
(ii) *The sequence $\{d(x_n, x^*)\}$ is decreasing and converges for all $x^* \in F(T)$.*
(iii) *$\lim_{n\to\infty} d(x_n, F(T))$ exists.*

**Theorem 2.1.1.** [8]*Let $C$ be a nonempty closed and convex subset of a hyperbolic space $X$. Let $T : C \to C$ be a mean nonexpansive mapping with $b < 1$ and $F(T) \neq \phi$, then $F(T)$ is closed and convex.*

**Theorem 2.1.2.** [8]*Let $C$ be a nonempty closed and convex subset of complete uniformly convex hyperbolic space $X$ with monotone modulus of convexity $\eta$. Let $T : C \to C$ be mean nonexpansive mapping with $b < 1$ and $\{x_n\}$ be a bounded sequence in $C$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x^*$. Then $x^* \in F(T)$.*

## 3. Main Results

First, we extend the Thakur et al.[24] iteration process into the hyperbolic metric spaces. For $x_1 \in C$, we construct the sequence $\{x_n\}$ as follows:

$$\begin{cases} x_{n+1} = W(Tx_n, Ty_n, \alpha_n) \\ y_n = W(z_n, Tz_n, \beta_n) \\ z_n = W(x_n, Tx_n, \gamma_n), n \in N. \end{cases} \tag{3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$.

### 3.1. Strong and $\Delta$-convergence Theorems for Mean Nonexpansive Mappings.

**Lemma 3.1.1.** *Let $C$ be a nonempty closed and convex subset of a hyperbolic space $X$. Let $T : C \to C$ be a mean nonexpansive mappings with $F(T) \neq \phi$ and the sequence $\{x_n\}$ is defined by (3) then $\lim_{n\to\infty} d(x_n, x^*)$ exists for each $x^* \in F(T)$.*

*Proof.* Let $x^* \in F(T)$, then from (3) we have

$$\begin{aligned} d(z_n, x^*) &= d(W(x_n, Tx_n, \gamma_n), x^*) \\ &\leq (1-\gamma_n)d(x_n, x^*) + \gamma_n d(Tx_n, x^*) \\ &\leq (1-\gamma_n)d(x_n, x^*) + \gamma_n[ad(x_n, x^*) + bd(x_n, x^*)] \\ &= (1-\gamma_n + a\gamma_n + b\gamma_n)d(x_n, x^*) \\ &\leq d(x_n, x^*) \end{aligned} \tag{4}$$

$$\begin{aligned} d(y_n, x^*) &= d(W(z_n, Tz_n, \beta_n), x^*) \\ &\leq (1-\beta_n)d(z_n, x^*) + \beta_n d(Tz_n, x^*) \\ &\leq (1-\beta_n)d(z_n, x^*) + \beta_n[ad(z_n, x^*) + bd(z_n, x^*)] \\ &= (1-\beta_n + a\beta_n + b\beta_n)d(z_n, x^*) \\ &\leq d(z_n, x^*) \end{aligned}$$

From equation (4) , we have

$$d(y_n, x^*) \leq d(x_n, x^*) \tag{5}$$

$$\begin{aligned} d(x_{n+1}, x^*) &= d(W(Tx_n, Ty_n, \alpha_n), x^*) \\ &\leq (1-\alpha_n)d(Tx_n, x^*) + \alpha_n d(Ty_n, x^*) \\ &\leq (1-\alpha_n)[ad(x_n, x^*) + bd(x_n, x^*)] + \alpha_n[ad(y_n, x^*) + bd(y_n, x^*)] \\ &= (1-\alpha_n)(a+b)d(x_n, x^*) + \alpha_n(a+b)d(y_n, x^*) \end{aligned}$$

From equation (5), we have

$$\begin{aligned} d(x_{n+1}, x^*) &\leq (a+b)d(x_n, x^*) \\ &\leq d(x_n, x^*) \end{aligned} \tag{6}$$

It follows that $d(x_n, x^*)$ is non-increasing and bounded. Hence $\lim_{n\to\infty} d(x_n, x^*)$ exist □

**Lemma 3.1.2.** *Let $C$ be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $T : C \to C$ be a mean nonexpansive mappings with $F(T) \neq \phi$ and the sequence$\{x_n\}$ is defined by (3). Then $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

*Proof.* From Lemma 3.1.1, we have that $\lim_{n\to\infty} d(x_n, x^*)$ exists for each $x^* \in F(T)$. Suppose that

$$\lim_{n\to\infty} d(x_n, x^*) = c.$$

**Case I**: If $c = 0$, then we are done.
**Case II**: If $c > 0$. From equations (4) and (5) in Lemma 3.1.1, we have

$$\limsup_{n\to\infty} d(z_n, x^*) \leq c \tag{7}$$

$$\limsup_{n\to\infty} d(y_n, x^*) \leq c \tag{8}$$

Since $T$ is mean nonexpansive mapping, it follows that

$$\begin{aligned} d(Tx_n, x^*) &\leq ad(x_n, x^*) + bd(x_n, x^*) \\ &= (a + b)d(x_n, x^*) \\ &\leq d(x_n, x^*) \end{aligned} \tag{9}$$

and

$$d(Ty_n, x^*) \leq d(y_n, x^*). \tag{10}$$

Taking lim sup of both sides, we have

$$\limsup_{n\to\infty} d(Tx_n, x^*) \leq c \tag{11}$$

$$\limsup_{n\to\infty} d(Ty_n, x^*) \leq c. \tag{12}$$

Since

$$\begin{aligned} c &= \lim_{n\to\infty} d(x_{n+1}, x^*) \\ &= \lim_{n\to\infty} d(W(Tx_n, Ty_n, \alpha_n), x^*) \\ &\leq \lim_{n\to\infty} (1 - \alpha_n)d(Tx_n, x^*) + \alpha_n d(Ty_n, x^*). \end{aligned} \tag{13}$$

By Lemma 2.0.3 and equation (13), we obtain

$$\lim_{n\to\infty} d(Tx_n, Ty_n) = 0. \tag{14}$$

Now

$$\begin{aligned} d(x_{n+1}, x^*) &= d(W(Tx_n, Ty_n, \alpha_n), x^*) \\ &\leq (1 - \alpha_n)d(Tx_n, x^*) + \alpha_n d(Ty_n, x^*) \\ &= d(Tx_n, x^*) + \alpha_n d(Tx_n, Ty_n) \end{aligned} \tag{15}$$

From equation (14) and (15), we have

$$c \leq \liminf_{n\to\infty} d(Tx_n, x^*) \tag{16}$$

From equations (11) and (16), we get

$$\lim_{n\to\infty} d(Tx_n, x^*) = c. \tag{17}$$

On the other hand, we have

$$\begin{aligned} d(Tx_n, x^*) &\leq d(Tx_n, Ty_n) + d(Ty_n, x^*) \\ &\leq d(Tx_n, Ty_n) + d(y_n, x^*) \end{aligned}$$

We obtain

$$c \leq \liminf_{n\to\infty} d(y_n, x^*) \tag{18}$$

From equations (12) and (18), we have

$$c = \lim_{n\to\infty} d(y_n, x^*).$$

Since $T$ is mean nonexapnsive mapping, we have

$$\begin{aligned} d(Tz_n, x^*) &\leq ad(z_n, x^*) + bd(z_n, x^*) \\ &= (a+b)d(z_n, x^*) \\ &\leq d(z_n, x^*) \end{aligned}$$

Taking lim sup of both sides, we have

$$\limsup_{n\to\infty} d(Tz_n, x^*) \leq c \tag{19}$$

From equation (7), (19) and Lemma 2.0.3, we get

$$\lim_{n\to\infty} d(z_n, Tz_n) = 0. \tag{20}$$

Since

$$\begin{aligned} d(y_n, x^*) &= d(W(z_n, Tz_n, \beta_n), x^*) \\ &\leq (1-\beta_n)d(z_n, x^*) + \beta_n(Tz_n, x^*) \\ &= d(z_n, x^*) + \beta_n d(Tz_n, z_n) \end{aligned}$$

yields

$$c \leq \liminf_{n\to\infty} d(z_n, x^*) \tag{21}$$

From equations (7) and (21), we get

$$\begin{aligned} c &= \lim_{n\to\infty} d(z_n, x^*) \\ &= \lim_{n\to\infty} d(W(x_n, Tx_n, \gamma_n), x^*). \end{aligned}$$

From Lemma 2.0.3, we get

$$d(x_n, Tx_n) = 0.$$

□

**Theorem 3.1.1.** *Let $C$ be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $T: C \to C$ be a mean nonexpansive mappings s.t. $b < 1$. Suppose $F(T) \neq \phi$ and the sequence$\{x_n\}$ is defined by (3) then $\{x_n\}$ $\Delta$-convergence to a fixed point of $T$.*

*Proof.* From Lemma 3.1.2 $\{x_n\}$ is a bounded sequence. Thus $\{x_n\}$ has a $\Delta$-convergent subsequences. Now we have to show that every $\Delta$-convergent subsequences of $\{x_n\}$ has a unique $\Delta$-limits in $F(T)$. Let $w$ and $z$ be $\Delta$-limits of subsequences $\{w_n\}$ and $\{z_n\}$ of $\{x_n\}$ respectively. From Lemma 2.0.1, we have $A(C, \{w_n\}) = \{w\}$ and $A(C, \{z_n\}) = \{z\}$. By Lemma 3.1.2, we get $\lim_{n\to\infty} d(w_n, Tw_n) = 0$ and $\lim_{n\to\infty} d(z_n, Tz_n) = 0$. By Theorem 2.1.2, gives that $w, z \in F(T)$. Now we have to show that $w = z$. Suppose $w \neq z$ and so by the uniqueness of an asymptotic center we have

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, w) &= \limsup_{n\to\infty} d(w_n, w) \\
&\leq \limsup_{n\to\infty} d(w_n, z) \\
&= \limsup_{n\to\infty} d(x_n, z) \\
&= \limsup_{n\to\infty} d(z_n, z) \\
&\leq \limsup_{n\to\infty} d((z_n, w) \\
&= \limsup_{n\to\infty} d((x_n, w)
\end{aligned}$$

Which is contradiction. Thus $w = z$ then $\{x_n\}$ $\Delta$- convergence to a fixed point of $T$. □

**Theorem 3.1.2.** *Let $C$ be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $T : C \to C$ be a mean nonexpansive mappings s.t.$b < 1$. Suppose $F(T) \neq \phi$ and the sequence$\{x_n\}$ is defined by (3) then $\{x_n\}$ converges strongly to a fixed point of $T$ iff $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf\{d(x_n, x^*) : x^* \in F(T)\}$.*

*Proof.* Suppose that the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$ and since $0 \leq d(x_n, F(T)) \leq d(x_n, F(T))$. It follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.
**Conversely:** Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Then from Lemma 3.1.1, we obtain $\lim_{n\to\infty} d(x_n, F(T))$ exists.Therefore $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Next we have to show that $\{x_n\}$ is Cauchy sequence in $C$. Let for each $\epsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for any given $\epsilon > 0$, there is $n_0 \in N$ s.t.

$$d(x_n, F(T)) < \frac{\epsilon}{2}, \forall n \geq n_0$$

In particular, $\inf\{d(x_{n_0}, x^*); x^* \in F(T)\} < \frac{\epsilon}{2}$. Then there exist $x_1^* \in F(T)$ s.t. $d(x_{n_0}, x_1^*) < \frac{\epsilon}{2}$. For any $m, n \geq n_0$, we get

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_1^*) + d(x_1^*, x_n) \\
&\leq d(x_{n_0}, x_1^*) + d(x_1^*, x_{n_0}) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in $C$. Since $C$ is a closed subset of a complete hyperbolic space $X$, $C$ is complete. Then $\{x_n\}$ must converge to a point in $C$ i.e. $\lim_{n\to\infty} x_n = x^*$. By Theorem 2.1.1, $F(T)$ is closed, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives $\lim_{n\to\infty} d(x^*, F(T)) = 0$ i.e. $x^* \in F(T)$. □

3.2. **Numerical example.** In this subsection, we construct the following example of a mean nonexpansive mapping which is not nonexpansive mapping.

**Example 3.1.** *Let $X = R$ with the usual metric and $C = [0, 1]$. Define a mapping $T : C \to C$ by*

$$Tx = \begin{cases} \frac{x}{9}, & x \in [0, \frac{1}{2}) \\ \frac{x}{10}, & x \in [\frac{1}{2}, 1] \end{cases}$$

*Clearly $x = 0$ is the fixed point of $T$. Then the following;*

(1) *Because $T$ is not continuous at the point $x = \frac{1}{2}$, $T$ is not a nonexpansive mapping.*
(2) *Now we prove that $T$ is mean nonexpansive mapping. For this purpose, let $a = \frac{1}{7}$, $b = \frac{2}{7}$ and consider the following cases;*
**Case I***: If $x, y \in [0, \frac{1}{2})$, By definition of $T$,*

$$\begin{aligned} |Tx - Ty| &= |\frac{x}{9} - \frac{y}{9}| \\ &= \frac{1}{8}|\frac{8x}{9} - \frac{8y}{9}| \\ &= \frac{1}{8}|x - \frac{y}{9} + \frac{y}{9} - \frac{x}{9} - (y - x + x - \frac{y}{9})| \\ &\leq \frac{1}{8}|x - y| + \frac{1}{4}|x - Ty| + \frac{1}{8}|Tx - Ty| \end{aligned}$$

*This Implies that $|Tx - Ty| \leq \frac{1}{7}|x - y| + \frac{2}{7}|x - Ty|$.*
**Case II** *: If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, By definition of $T$,*

$$\begin{aligned} |Tx - Ty| &= |\frac{x}{9} - \frac{y}{10}| \\ &= |\frac{x}{9} - \frac{Tx}{9} + \frac{Tx}{9} - \frac{Ty}{9} + \frac{Ty}{9} - \frac{y}{10}| \\ &\leq \frac{1}{9}|x - Tx| + \frac{1}{9}|Tx - Ty| + \frac{1}{9}|y - Ty| \\ &\leq \frac{1}{9}|x - y| + \frac{2}{9}|x - Ty| + \frac{2}{9}|Tx - Ty| \end{aligned}$$

*This Implies that $|Tx - Ty| \leq \frac{1}{7}|x - y| + \frac{2}{9}|x - Ty|$.*
**Case III***: If $x \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2})$, The proof is the same as in Case II.*
**Case IV***: If $x, y \in [\frac{1}{2}, 1]$, The proof is the same as in Case I.*
*Hence, $T$ is mean nonexpansive by taking $a = \frac{1}{7}$,$b = \frac{2}{7}$.*

*In what follows, we numerically compare the Thakur iteration process with other iteration processes like Abbas, Agarwal, Noor, Ishikawa and Mann. We take $\alpha_n = 0.85$, $\beta_n = 0.65$, $\gamma_n = 0.45$ and initial value $x_0 = 0.4$.The comparison table 1 shows that the Thakur*[24] *iteration converges to $x^* = 0$ faster than Abbas* [1]*, Agarwal*[2]*, Noor*[19]*, Ishikawa*[12] *and Mann* [18]*. The convergence behavior of these iteration processes are represented in Figure 1*

| *Iteration* | *Thakur* | *Abbas* | *Agarwal* | *Noor* | *Ishikawa* | *Mann* |
|---|---|---|---|---|---|---|
| *0* | *0.40000000* | *0.40000000* | *0.40000000* | *0.40000000* | *0.40000000* | *0.40000000* |
| *1* | *0.01623704* | *0.02321481* | *0.02261728* | *0.06310617* | *0.07595062* | *0.09777778* |
| *2* | *0.00065910* | *0.00134732* | *0.00127885* | *0.00995597* | *0.01442124* | *0.02390123* |
| *3* | *0.00002675* | *0.00007819* | *0.00007231* | *0.00157071* | *0.00273826* | *0.00584252* |
| *4* | *0.00000109* | *0.00000454* | *0.00000409* | *0.00024780* | *0.00051993* | *0.00142817* |
| *5* | *0.00000004* | *0.00000026* | *0.00000023* | *0.00003909* | *0.00009872* | *0.00034911* |
| *6* | *0.00000000* | *0.00000002* | *0.00000001* | *0.00000617* | *0.00001875* | *0.00008534* |
| *7* | *0.00000000* | *0.00000000* | *0.00000000* | *0.00000097* | *0.00000356* | *0.00002086* |
| *8* | *0.00000000* | *0.00000000* | *0.00000000* | *0.00000015* | *0.00000068* | *0.00000510* |
| *9* | *0.00000000* | *0.00000000* | *0.00000000* | *0.00000002* | *0.00000013* | *0.00000125* |
| *10* | *0.00000000* | *0.00000000* | *0.00000000* | *0.00000000* | *0.00000002* | *0.00000030* |

Table 1. Comparison of the rate of convergence with different iteration processes.

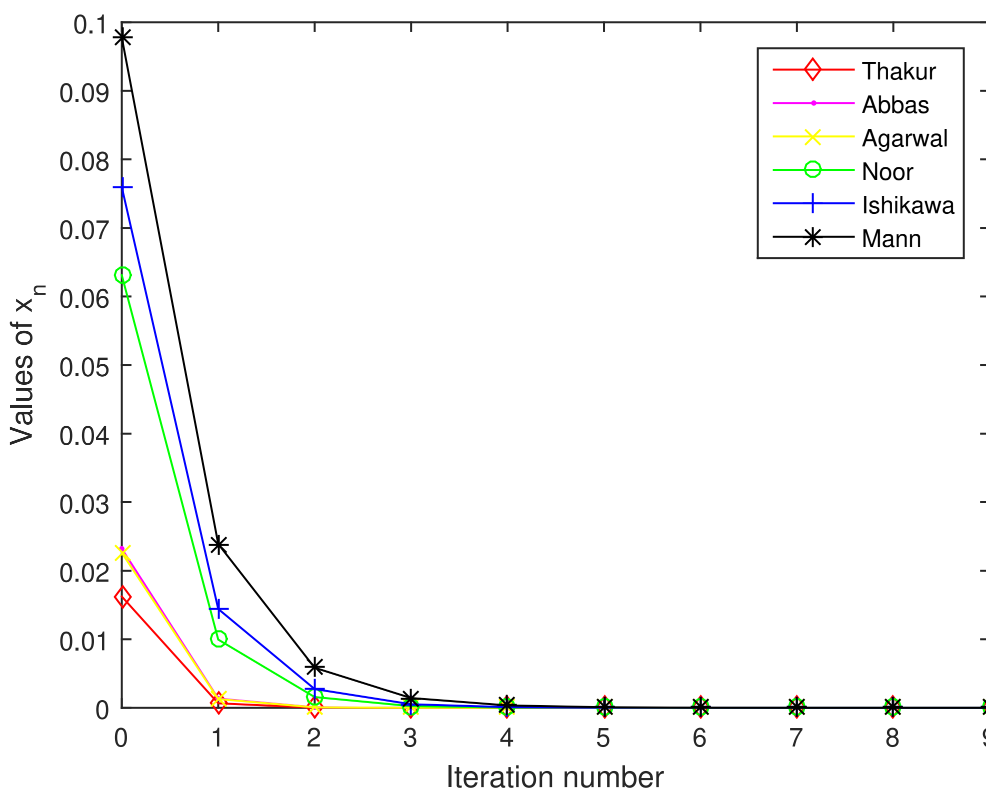


Figure 1. Graphical representation of convergence of iterative method

## References

[1] Abbas M., Nazir T., A new faster iteration process applied to constrained minimization and feasibility problems, Mat. Vesnik, Vol.66, No.2, 223–234, 2014.

[2] Agarwal R.P., O'Regan D., Sahu D.R., Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex Anal., Vol.8, No.1, 61-79, 2007.
[3] Ahmad J., Ullah K., Arshad M., de la Sen M., Approximation of fixed points for mean nonexpansive mappings in Banach spaces, J. Funct. Spaces, Vol.2021, Article ID 1934274.
[4] Akbar A., Rastgoo M., On approximation of fixed points of mean nonexpansive mappings in CAT(0) spaces, Facta Universitatis, Vol.33, No.3, 481-496, 2018.
[5] Chen L., Gao L., Chen D., Fixed point theorems of mean nonexpansive set-valued mappings in Banach spaces, J. Fixed Point Theory Appl., Vol.19, 2129–2143, 2017.
[6] Chang S.S., Wang G., Wang L., Tang Y.K., Ma Z.L., $\Delta$-convergence theorems for multivalued nonexpansive mappings in hyperbolic spaces, Appl. Math. Comp., Vol.249, 535–540, 2014.
[7] Chuadchawna P., Farajzadeh A., Kaewcharoen A., Fixed point approximations of generalized nonexpansive mappings via generalized M-iteration process in Hyperbolic spaces, Int. J. Math. Math. Sci., Vol.2020, Art. ID 6435043.
[8] Ezeora J.N., Izuchukwu C., Mebawondu A.A.,Mewomo O.T., Approximating common fixed points of mean nonexpansive mappings in hyperbolic spaces, Int. J. Nonlinear Anal. Appl. Vol.13, No.2, 459–471, 2022.
[9] Goebel K., Kirk W.A., Iteration processes for nonexpansive mappings, In Topological Methods in Nonlinear Functional Analysis, S. P. Singh, S. Thomeier, and B.Watson, Eds., Vol. 21 of Contemporary Mathematics, 115-123, American Mathematical Society, Providence, RI, USA, 1983.
[10] Gu Z., Li Y., Approximating fixed points of mean nonexpansive mapping in Banach spaces, Int. J. Pure Appl. Math., Vol.40, No.2, 201-208, 2007.
[11] Imdad M., Dashputre S., Fixed point approximation of Picard normal S-iteration process for generalized nonexpansive mappings in hyperbolic spaces, Math. Sci. Vol.10, 131—138,2016.
[12] Ishikawa S., Fixed points by a new iteration method. Proc. Am. Math. Soc., Vol.44, 147-150, 1974.
[13] Khan A.R., Fukhar-ud-din H., Khan M.A.A, An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, Fixed Point Theory Appl. Vol.2012, Art.ID 54.
[14] Kohlenbach U., Some logical metathorems with applications in functional analysis, Trans. Amer. Math. Soc., Vol.357, No.1, 89–128, 2005.
[15] Kirk W.A., Panyanak B., A concept of convergence in geodesic spaces, Nonlinear Anal. Theory Meth.Appl. Ser. A 68, No.12, 3689–3696, 2008.
[16] Kuczumow T., An almost convergence and its applications, Ann. Univ. Mariae Curie-Sklodowska, Sect. A 32, 79–88, 1978.
[17] Leustean L., Nonexpansive iteration in uniformly convex W-hyperbolic spaces, Nonlinear Anal. Optim., Vol.51, No.3, 193–209, 2010.
[18] Mann W.R., Mean value methods in iteration. Proc. Am. Math. Soc., Vol.4, 506-510, 1953.
[19] Noor M.A., New approximation schemes for general variational inequalities. J. Math. Anal. Appl., Vol.251, No.1, 217-229,2000.
[20] Ouahab A., Mbarki A., Masude J., Rahmoune M., A fixed point theorems for mean nonexpansive mappings semigroups in uniformly convex Banach spaces, Int. J. Math. Anal., Vol.6, 101-109, 2012.
[21] Reich S.,Shafrir I., Nonexpansive iterations in hyperbolic spaces, Nonlinear Anal., Vol.15, 537–558, 1990.
[22] Suanoom C., Klin-eam C., Remark on fundamentally non-expansive mappings in hyperbolic spaces, J. Nonlinear Sci. Appl., Vol.9, 1952–1956, 2016.
[23] Takahashi W.A., A convexity in metric space and nonexpansive mappings, I. Kodai Math. Sem. Rep., Vol.22, 142–149, 1970.
[24] Thakur D., Thakur B.S., Postolache M., New iteration scheme for numerical reckoning fixed points of nonexpansive mappings, J. Inequal Appl. 2014.
[25] Wu C.X., Zhang L.J., Fixed points for mean nonexpansive mappings, Acta Math. Appl. Sin., Vol.23,No.3, 489–494, 2007.
[26] Yang Y.S., Cui Y.A., Viscosity approximation methods for mean non-expansive mappings in Banach spaces, Appl. Math. Sci., Vol. 2, No.13, 627–638, 2008.
[27] Zhang S.S., About fixed point theorem for mean nonexpansive mapping in Banach spaces, J. Sichuan Univ., Vol.2, 67–68, 1975.

[28] Zhao H., The existence theorem of fixed point for mean non-expansive mapping in Banach space, Acta. Math. Sinica, Vol. 22, No.4, 459-469,1979.
[29] Zhou J., Cui Y., Fixed point theorems for mean nonexpansive mappings in CAT(0) spaces, Numer. Funct. Anal. Optim. Vol.36, No.9, 1224–1238, 2015.
[30] Zuo Z., Fixed point theorems for mean nonexpansive mappings in Banach spaces, Abstr. Appl. Anal. Vol. 2014, Art. ID 746291.

Omprakash Sahu
Department of Mathematics, Babu Pandhri Rao Kridatt Govt. College Silouti, Dhamtari (C.G.), India
*Email address*: om2261995@yahoo.com

Amitabh Banerjee
Principal, Govt. J. Y. Chhattisgarh College Raipur, India
*Email address*: amitabh_61@yahoo.com