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ON THE AMBARTSUMIAN FUNCTIONAL EQUATION

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Abstract. In this work, we define the pantograph functional equation with parameter and study the existence of solutions in two classes $x \in C[0,T]$ and $x \in L_1[0,T]$, we use the technique of the Banach fixed point theorem and Schauder fixed point Theorem. In both cases we study the continuous dependence of the unique solution on the pantograph functional equation. The Hyers–Ulam stability will be studied. Additionally, we give an example to illustrate our outcomes.

keywords: Pantograph equation; Banach fixed point Theorem; Schauder fixed point Theorem; existence of solutions; continuous dependence; Hyers–Ulam stability.

1. Introduction

A pantograph (or "pan" or "panto") is an apparatus mounted on the roof of an electric train, tram or electric bus to collect power through contact with an overhead line. The pantograph equation is a fundamental mathematical model in the field of delay differential equations. It is well known that the pantograph differential equation is given by

$$\frac{dx}{dt} = f\Bigg(t,\ x(t),\ x(\gamma t)\Bigg),$$

a special case of the pantograph equation is well known as the Ambartsumian delay equation which has a particular application in Astrophysics.

$$\frac{dx}{dt} = ax(t) + \ \lambda x(\gamma\ t).$$

For papers studying such kind of problems (see [4, 11, 12]) and Pantograph differential equations have been studied in many papers and monographs [10].

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Key words and phrases. Pantograph equation; Banach fixed point Theorem; Schauder fixed point Theorem; existence of solutions; continuous dependence; Hyers–Ulam stability.

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In this study, we define the pantograph functional equation with parameter,

$$x(t) = f\Bigg(t,\ x(t),\ \lambda\ x(\gamma t)\ \Bigg),\ t \in [0, T] \tag{1.1}$$

and its special case, the Ambartsumian delay equation

$$x(t) = ax(t) +\ \lambda x(\gamma\ t).$$

Where $\lambda > 0$, $\gamma \in (0,\ 1)$ and a is constant. Our aim here is study the existence of solutions $x \in C[0, T]$ and $x \in L_1[0, T]$ of equation (1.1). Moreover, the continuous dependence of the unique solution on the functions f, γ and on the parameter $\lambda > 0$ will be proved. The Hyers – Ulam stability of (1.1) will be given.

The paper is organized as follows: Section 2 contains the solvability of unique solution $x \in C[0, T]$ by Banach fixed point and discuss some stability facts of the of (1.1). Moreover, the Hyers – Ulam stability of (1.1) will be studied. In Section 3, the solvability for the existence of the solutions $x \in L_1[0, T]$ by Schauder fixed point Theorem and the continuous dependence of the unique solution $x \in L_1[0, T]$ on the parameter $\lambda\ \geq 0$ and on the function f. Some general discussion and examples in Section 4.

2. Solution in $C[0, T]$

Let $C = C(I)$, be the class of continuous functions on $I = [0, T]$, $T < \infty$, with the standard norm

$$\|x\| = \sup_{t\in I} |x(t)|.$$

Consider the pantograph functional equation (1.1) under the following assumptions:

(i) $f : I \times R \times R \to R$ is continuous in $t \in I$ and satisfies Lipschitz condition,

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq a\ |x_1 - y_1| +\ \lambda\ |x_2 - y_2|\ \forall\ t \in I,\ x_i, y_i \in R, i = 1,\ 2.$$

(ii) $(a +\ \lambda) < 1$.

Now, we have the following existences theorem.

Theorem 2.1. *Assume that (i) and (ii) be satisfied, then the pantograph functional equation (1.1) has a unique solution $x \in C(I)$.*

Proof. Define the operator F by

$$Fx(t) = f\Bigg(t,\ x(t),\ \lambda\ x(\gamma t)\ \Bigg).$$

Now, let $x \in C(I)$ and $t_1, t_2 \in I$, $t_1 < t_2$, $|t_2 - t_1| < \delta$ and denote
$\theta_f(\delta) = \sup_{x,y\in C(I)} \{|f(t_2, x(t), y(t)) - f(t_1, x(t), y(t))| : t_1, t_2 \in I,\ t_1 < t_2,\ |t_2 - t_1| < \delta\}$,

then we have

$$\begin{aligned}
|Fx(t_2)-Fx(t_1)| &= \bigg|f\bigg(t_2,\ x(t_2),\ \lambda\, x(\gamma t_2)\ \bigg)-f\bigg(t_1,\ x(t_1),\ \lambda\, x(\gamma t_1)\ \bigg)\bigg| \\
&\le \bigg|f\bigg(t_2,\ x(t_2),\ \lambda\, x(\gamma t_2)\ \bigg)-f\bigg(t_1,\ x(t_2),\ \lambda\, x(\gamma t_2)\ \bigg) \\
&+ f\bigg(t_1,\ x(t_2),\ \lambda\, x(\gamma t_2)\ \bigg)-f\bigg(t_1,\ x(t_1),\ \lambda\, x(\gamma t_2)\ \bigg) \\
&+ f\bigg(t_1,\ x(t_1),\ \lambda\, x(\gamma t_2)\ \bigg)-f\bigg(t_1,\ x(t_1),\ \lambda\, x(\gamma t_1)\ \bigg)\bigg| \\
&\le \theta_f(\delta)+a\ |x(t_2)-x(t_1)|+\ \lambda\ |x(\gamma t_2)-x(\gamma t_1)|.
\end{aligned}$$

This means that $F: C(I)\to C(I)$.
Now let $x_1, x_2\in C(I)$, then

$$\begin{aligned}
|Fx_2(t)-Fx_1(t)| &= \bigg|f\bigg(t,\ x_2(t),\ \lambda\, x_2(\gamma t)\ \bigg)-f\bigg(t,\ x_1(t),\ \lambda\, x_1(\gamma t)\ \bigg)\bigg| \\
&\le \bigg|f\bigg(t,\ x_2(t),\ \lambda\, x_2(\gamma t)\ \bigg)-f\bigg(t,\ x_1(t),\ \lambda\, x_2(\gamma t)\ \bigg) \\
&+ f\bigg(t,\ x_1(t),\ \lambda\, x_2(\gamma t)\ \bigg)-f\bigg(t,\ x_1(t),\ \lambda\, x_1(\gamma t)\ \bigg)\bigg| \\
&\le a|x_2(t)-x_1(t)|+\ \lambda|x_2(\gamma t)-x_1(\gamma t)|,
\end{aligned}$$

then

$$\begin{aligned}
\|Fx_2-Fx_1\| &\le a\|x_2-x_1\|+\ \lambda\|x_2-x_1\| \\
&\le (a\ +\ \lambda)\ \|x_2-x_1\|.
\end{aligned}$$

Since $(a+\ \lambda)<1$, then F is a contraction and by Banach fixed point Theorem [8] there exists a unique solution $x\in C(I)$ of the equation (1.1).

2.1. **Continuous dependence.**

Theorem 2.2. *Let the assumptions of Theorem 2.1 be satisfied for $f,\ f^*,\ \lambda,\ \lambda^*,\ \gamma$ and γ^*. Then the unique solution $x\in C(I)$ depends continuously on $f,\ \lambda$ and γ in the sense that*

$$\forall\epsilon>0,\ \exists\ \delta(\epsilon)\ such\ that$$

$$max\{|\lambda-\lambda^*|,\ |f(t,x,y)-f^*(t,x,y)|,\ |\gamma-\gamma^*|\}<\delta, then\ \|x-x^*\|<\epsilon.$$

where x^* is the solution of

$$x^*(t)=f^*\bigg(t,\ x^*(t),\ \lambda^*\ x^*(\gamma^* t)\ \bigg).$$

Proof.

$$\begin{aligned}
|x(t)-x^*(t)| &= \left|f\Big(t,\ x(t),\ \lambda\ x(\gamma t)\ \Big)-f^*\Big(t,\ x^*(t),\ \lambda^*\ x^*(\gamma^* t)\ \Big)\right| \\
&\leq \left|f\Big(t,\ x(t),\ \lambda\ x(\gamma t)\ \Big)-f^*\Big(t,\ x(t),\ \lambda\ x(\gamma t)\ \Big)\right. \\
&+ \left. f^*\Big(t,\ x(t),\ \lambda\ x(\gamma t)\ \Big)-f^*\Big(t,\ x^*(t),\ \lambda^*\ x^*(\gamma^* t)\ \Big)\right| \\
&\leq \delta+a\ \Big|x(t)-x^*(t)\Big|+\Big|\ \lambda\ x(\gamma t)-\ \lambda^*\ x^*(\gamma^* t)\Big| \\
&\leq \delta+a\Big|x(t)-x^*(t)\Big|+\Big|\ \lambda\ x(\gamma t)-\ \lambda^*\ x(\gamma t) \\
&+ \quad \lambda^*\ x(\gamma t)-\ \lambda^*\ x^*(\gamma t)+\ \lambda^*\ x^*(\gamma t)-\ \lambda^*\ x^*(\gamma^* t)\Big| \\
&\leq \delta+a|x(t)-x^*(t)|+|\lambda-\ \lambda^*||x(\gamma t)|+\ \lambda^*|x(\gamma t)-x^*(\gamma t)| \\
&+ \quad \lambda^*|x^*(\gamma t)-x^*(\gamma^* t)| \\
&\leq \delta+a|x(t)-x^*(t)|+|\lambda-\ \lambda^*||x(\gamma t)|+\ \lambda^*|x(\gamma t)-x^*(\gamma t)| \\
&+ \quad \lambda^*\ \epsilon^*,
\end{aligned}$$

then

$$\|x-x^*\| \quad \leq \quad \delta+a\ \|x-x^*\|+\ \delta\ |x(\gamma t)|+\ \lambda^*\ \|x-x^*\|+\ \lambda^*\ \epsilon^*.$$

Hence

$$\|x-x^*\| \quad \leq \quad \frac{\delta+\ \delta\ |x(\gamma t)|+\ \lambda^*\ \epsilon^*}{1-(a\ +\ \lambda^*)}=\epsilon.$$

2.2. **Hyers-Ulam stability.**

Definition 2.3. [6, 9] Let the solution $x \in C(I)$ of (1.1) be exists. then equation (1.1) is Hyers - Ulam stable if $\forall \epsilon > 0,\ \exists\ \delta(\epsilon)$ *such that* for any $\delta-approximate$ solution x_s satisfies,

$$\left|x_s(t)-f\Big(t,\ x_s(t),\ \lambda\ x_s(\gamma t)\ \Big)\right|<\delta, \tag{2.1}$$

implies $\|x-x_s\|<\epsilon$.

Theorem 2.4. *Let the assumptions of Theorem 2.1 be satisfied, then (1.1) is Hyers - Ulam stable.*

Proof. From (2.1), we have

$$\left|x_s(t)-f\Big(t,\ x_s(t),\ \lambda\ x_s(\gamma t)\ \Big)\right|<\delta,$$

Now,

$$\begin{aligned}
|x(t)-x_s(t)| &= \Bigg|f\Bigg(t,\ x(t),\ \lambda\, x(\gamma t)\Bigg)-x_s(t)\Bigg| \\
&\leq \Bigg|f\Bigg(t,\ x(t),\ \lambda\, x(\gamma t)\Bigg)\ -\ f\Bigg(t,\ x_s(t),\ \lambda\, x_s(\gamma t)\Bigg) \\
&+\ x_s(t)-f\Bigg(t,\ x_s(t),\ \lambda\, x_s(\gamma t)\Bigg)\Bigg| \\
&\leq a\ |x_(t)-x_s(t)|+\ \lambda|x(\gamma t)-x_s(\gamma t)|+\Bigg|x_s(t)-f\Bigg(t,\ x_s(t),\ \lambda\, x_s(\gamma t)\Bigg)\Bigg|,
\end{aligned}$$

then

$$\|x-x_s\| \quad\leq\quad \|x-x_s\|+\ \lambda\ \|x-x_s\|+\delta,$$

Hence

$$\|x-x_s\| \quad\leq\quad \frac{\delta}{1-(a+\ \lambda)}=\epsilon.$$

3. Solution in $L_1(I)$

Let $L_1=L_1(I)$, be the class of Lebesgue integrable functions on $I=[0,T]$, $T<\infty$, with the standard norm

$$\|x\|_1=\int_0^T |x(t)|dt.$$

Take into account the following assumptions:

(iii) $f: I\times R\to R$ is measurable in $t\in I$ for any $x\in R$ and continuous in $x\in R$ for all $t\in I$. Moreover, there exist a bounded measurable function $m: I\to R$ and a positive constant b_2 such that

$|f(t,x,y)|\leq |m(t)|+a\ |x|+\lambda\ |y|\ for\ each\ t\in I\ and\ forall\ x,\ y\in R\ ,\ i=1,\ 2.$

(iv) $(a+\frac{\lambda}{\gamma})<1.$

Theorem 3.1. *Let the assumptions $(iii)-(iv)$ be satisfied, then the pantograph functional equation (1.1) has at least one solution $x\in L_1(I)$.*

Proof. Let Q_r be the closed ball

$$Q_r=\{x\in L_1(I): \|x\|_1\leq r\}.$$

Associate the operator

$$Fx(t)=f\Bigg(t,\ x(t),\ \lambda\, x(\gamma t)\Bigg).$$

Now, let $x\in Q_r$, then

$$\begin{aligned}
|Fx(t)| &= \Bigg|f\Bigg(t,\ x(t),\ \lambda\, x(\gamma t)\Bigg)\Bigg| \\
&\leq |m(t)|+b_2\Bigg(|x(t)|+|\lambda x(\gamma t)|\Bigg),
\end{aligned}$$

then

$$\int_0^T |Fx(t)|dt \quad\leq\quad \int_0^T |m(t)|dt+a\int_0^T |x(t)|dt+\frac{\lambda}{\gamma}\int_0^T |x(\theta)|d\theta$$

and

$$\begin{aligned}\|Fx\|_1 &\leq \|m\|_1 + a\ \|x\|_1 + \frac{\lambda}{\gamma}\|x\|_1\\ &\leq \|m\|_1 + a\ r\ + \frac{\lambda}{\gamma}\ r\\ &\leq \|m\|_1 + a\ r + \frac{\lambda\ r}{\gamma}\ =\ r.\end{aligned}$$

Then the class of functions $\{Fx\}$ is uniformly bounded on Q_r.
Now, let $x \in Q_r$, then

$$\begin{aligned}\|(Fx)_h - (Fx)\|_1 &= \int_0^T |(Fx(s))_h - (Fx(s))|ds\\ &= \int_0^T \frac{1}{h}\Big|\int_t^{t+h} |(Fx(\theta))d\theta - (Fx(s))\Big|ds\\ &\leq \int_0^T \frac{1}{h}\int_t^{t+h} |Fx(\theta) - Fx(s)|\ d\theta ds\\ &\leq \int_0^T \frac{1}{h}\int_t^{t+h} \Big|f\Big(\theta,\ x(\theta),\ \lambda\ x(\gamma\theta)\Big) - f\Big(s,\ x(s),\ \lambda\ x(\gamma s)\Big)\Big|\ d\theta ds.\end{aligned}$$

Since $F \in L_1(I)$, then

$$\frac{1}{h}\int_t^{t+h} \Big|f\Big(\theta,\ x(\theta),\ \lambda\ x(\gamma\theta)\Big) - f\Big(s,\ x(s),\ \lambda\ x(\gamma s)\Big)\Big|\ d\theta ds \to 0,\ ash \to 0.$$

This means that $Fx(t)_h \to (Fx)$ uniformly in $L_1(I)$. Thus the class of functions $\{Fx\}$ is relatively compact [15]. Hence F is compact operator.
Now, let $\{x_n\} \subset Q_r$, and $x_n \to x$, then

$$Fx_n(t) = f\Big(t,\ x_n(t),\ \lambda\ x_n(\gamma t)\Big)$$

and

$$\lim_{n\to\infty} Fx_n(t) = \lim_{n\to\infty} f\Big(t,\ x_n(t),\ \lambda\ x_n(\gamma t)\Big).$$

Applying Lebesgue dominated convergence Theorem [15], then from our assumptions we get

$$\begin{aligned}\lim_{n\to\infty} Fx_n(t) &= f\Big(t,\ \lim_{n\to\infty}\ x_n(t),\ \lambda\ \lim_{n\to\infty}\ x_n(\gamma t)\Big)\\ &= f\Big(t,\ x(t),\ \lambda\ x(\gamma t)\Big)\ =\ Fx(t).\end{aligned}$$

This means that $Fx_n(t) \to Fx(t)$. Hence the operator F is continuous. Now, by Schauder fixed point Theorem [15] there exists at least one solution $x \in L_1(I)$ of (1.1).

3.1. **Uniqueness of the solution.** Now, replace the assumption (iii) by $(iii)^*$ as follows:

$(iii)^*$ $f : I \times R \to R$ is measurable in $t \in I\ \forall x \in R$ and satisfies Lipschitz condition,

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq a\ |x_1 - y_1|\ +\ \lambda\ |x_2 - y_2|\ \forall\ t \in I,\ x_i, y_i \in R, i = 0,\ 1.$$

So, we have the following Lemma.

Lemma 3.2. *The assumption $(iii)^*$ implies the assumption (iii).*

Proof. From the assumption $(iii)^*$ let $y_1 = y_2 = 0$, then we have

$$|f(t, x_1, x_2)| - |f(t, 0, 0)| \leq |f(t, x_1, x_2) - f(t, 0, 0)| \leq a\ |x_1|\ +\ \lambda\ |x_2|,$$

$$|f(t, x_1, x_2)| \leq |f(t, 0, 0)| + a\ |x_1|\ +\ \lambda\ |x_2|$$

and

$$|f(t, x_1, x_2)| \leq |m(t)| + a\ |x_1|\ +\ \lambda\ |x_2|,\ \textit{where}\ |m(t)| = \sup_{t \in I} |f(t, 0, 0)|.$$

Theorem 3.3. *Let the assumptions $(iii)^*$ and (iv) be satisfied, then the solution of a pantograph functional equation (1.1) is unique.*

Proof. Let $x,\ y$ be two solutions in Q_r of (1.1), then

$$\begin{aligned} |x(t) - y(t)| &= \Bigg| f\Bigg(t,\ x(t),\ \lambda\ x(\gamma t)\ \Bigg) - f\Bigg(t,\ y(t),\ \lambda\ y(\gamma t)\ \Bigg)\Bigg| \\ &\leq \Bigg| f\Bigg(t,\ x(t),\ \lambda\ x(\gamma t)\ \Bigg) - f\Bigg(t,\ y(t),\ \lambda\ x(\gamma t)\ \Bigg) \\ &+ f\Bigg(t,\ y(t),\ \lambda\ x(\gamma t)\ \Bigg) - f\Bigg(t,\ y(t),\ \lambda\ y(\gamma t)\ \Bigg)\Bigg| \\ &\leq a\ |x(t) - y(t)| +\ \lambda |x(\gamma t) - y(\gamma t)|, \end{aligned}$$

then

$$\begin{aligned} \int_0^T |x(t) - y(t)| dt &\leq a \int_0^T |x(t) - y(t)| dt + \lambda \int_0^T |x(\gamma t) - y(\gamma t)| dt \\ \|x - y\|_1 &\leq a\ \|x - y\|_1 + \frac{\lambda}{\gamma} \int_0^T |x(\theta) - y(\theta)| dt \\ &\leq a\ \|x - y\|_1 + \frac{\lambda}{\gamma} \|x - y\|_1. \end{aligned}$$

Hence

$$\|x - y\|_1 \Bigg(1 - \Bigg(a\ + \frac{\lambda}{\gamma}\Bigg)\Bigg) \quad \leq \quad 0.$$

then $x =\ y$ and the solution of (1.1) is unique.

3.2. **Continuous dependence.**

Theorem 3.4. *Let the assumptions of Theorem 3.3 be satisfied for $f,\ f^*,\ \lambda$ and λ^*. Then the unique solution $x \in L_1(I)$ depends continuously on f and λ in the sense that*

$$\forall \epsilon > 0,\ \exists\ \delta(\epsilon)\ \textit{such that}$$

$$\textit{max}\ \{\ |\lambda - \lambda^*|,\ |f(t, x, y) - f^*(t, x, y)|\ \}\ <\ \delta,\ \textit{then}\ \|x - x^*\|_1\ <\ \epsilon.$$

where x^* be a solution of

$$x^*(t) = f^*\Bigg(t,\ x^*(t),\ \lambda^*\ x^*(\gamma t)\ \Bigg).$$

Proof.

$$\begin{aligned}
|x(t)-x^*(t)| &= \left| f\Big(t,\ x(t),\ \lambda\, x(\gamma t)\ \Big) - f^*\Big(t,\ x^*(t),\ \lambda^*\ x^*(\gamma t)\ \Big)\right| \\
&\leq \left| f\Big(t,\ x(t),\ \lambda\, x(\gamma t)\ \Big) - f^*\Big(t,\ x(t),\ \lambda\, x(\gamma t)\ \Big)\right. \\
&+ \left. f^*\Big(t,\ x(t),\ \lambda\, x(\gamma t)\ \Big) - f^*\Big(t,\ x^*(t),\ \lambda^*\ x^*(\gamma t)\ \Big)\right| \\
&\leq \left| f\Big(t,\ x(t),\ \lambda\, x(\gamma t)\ \Big) - f^*\Big(t,\ x(t),\ \lambda\, x(\gamma t)\ \Big)\right| \\
&+ a\left|x(t)-x^*(t)\right| + \left|\ \lambda\, x(\gamma t) -\ \lambda^*\ x^*(\gamma t)\right| \\
&\leq \delta + a\left|x(t)-x^*(t)\right| + \left|\ \lambda\, x(\gamma t) -\ \lambda^*\ x(\gamma t)\right. \\
&+ \left.\ \lambda^*\ x(\gamma t) -\ \lambda^*\ x^*(\gamma t)\right| \\
&\leq \delta + a\ |x(t)-x^*(t)| + |\lambda -\ \lambda^*|\ |x(\gamma t)| \\
&+ \ \lambda^* |x(\gamma t) - x^*(\gamma t)|
\end{aligned}$$

then

$$\begin{aligned}
\int_0^T |x(t)-x^*(t)|dt &\leq \delta + a\int_0^T |x(t)-x^*(t)|dt + \ \frac{\delta}{\gamma}\ \int_0^T |x(\theta)|d\theta \\
&+ \ \frac{\lambda^*}{\gamma}\int_0^T |x(\theta)-x^*(\theta)|d\theta \\
\|x-x^*\|_1 &\leq \delta + a\ \|x-x^*\|_1 + \ \frac{\delta}{\gamma}\ r\ + \frac{\lambda^*}{\gamma}\ \|x-x^*\|_1.
\end{aligned}$$

Hence

$$\|x-x^*\|_1 \quad \leq \quad \frac{\delta + \frac{\delta}{\gamma}\ r}{1-(a\ +\ \frac{\lambda^*}{\gamma})} = \epsilon.$$

3.3. **Hyers-Ulam stability.**

Definition 3.5. [6, 9] Let the solution $x \in L_1(I)$ of (1.1) be exists, then equation (1.1) is Hyers - Ulam stable if $\forall \epsilon > 0,\ \exists\ \delta(\epsilon)$ *such that* for any $\delta - approximate$ solution x_s satisfies,

$$\left\| x_s(t) - f\Big(t,\ x_s(t),\ \lambda\, x_s(\gamma t)\ \Big)\right\|_1 < \delta, \tag{3.1}$$

implies $\|x - x_s\|_1 < \epsilon$.

Theorem 3.6. *Let the assumptions of Theorem 3.3 be satisfied, then (1.1) is Hyers - Ulam stable.*

Proof. From (3.1), we have

$$\left\| x_s(t) - f\Big(t,\ x_s(t),\ \lambda\, x_s(\gamma t)\ \Big)\right\|_1 < \delta,$$

Now,

$$\begin{aligned}|x(t)-x_s(t)| &= \left|f\Big(t,\ x(t),\ \lambda\, x(\gamma t)\ \Big)-x_s(t)\right|\\ &\leq \left|f\Big(t,\ x(t),\ \lambda\, x(\gamma t)\ \Big)\ -\ f\Big(t,\ x_s(t),\ \lambda\, x_s(\gamma t)\ \Big)\right.\\ &+ \left. x_s(t)-f\Big(t,\ x_s(t),\ \lambda\, x_s(\gamma t)\ \Big)\right|\end{aligned}$$

then

$$\begin{aligned}\int_0^T |x(t)-x_s(t)|dt &\leq a\int_0^T |x_2(t)-x_1(t)|dt+\ \lambda\int_0^T |x_2(\gamma t)-x_1(\gamma t)|dt\\ &+ \int_0^T \left|x_s(t)-f\Big(t,\ x_s(t),\ \lambda\, x_s(\gamma t)\ \Big)\right|dt\\ \|x-x_s\|_1 &\leq a\ \|x-x_s\|_1+\frac{\lambda}{\gamma}\ \|x-x_s\|_1+\delta.\end{aligned}$$

Hence

$$\|x-x_s\|_1 \quad \leq \quad \frac{\delta}{1-(a+\ \frac{\lambda}{\gamma})}=\epsilon.$$

4. General discussion and examples

1- Let $\lambda\ and\ \gamma=\frac{1}{q}$, then we have

$$x(t)=f\Big(t,\ x(t),\ \frac{1}{q}\ x(\frac{t}{q})\ \Big),$$

and

$$x(t)=ax(t)+\frac{1}{q}\ x(\frac{t}{q}).$$

2- Let $\lambda\ =\ \gamma$ where $\gamma\in(0,\ 1)$, then we have

$$x(t)=f\Big(t,\ x(t),\ \gamma\ x(\gamma\ t)\ \Big),$$

and

$$x(t)=ax(t)+\ \gamma\ x(\gamma\ t).$$

Or

$$x(t)=f\Big(t,\ x(t),\ \lambda\ x(\lambda\ t)\ \Big),$$

and

$$x(t)=ax(t)+\ \lambda\ x(\lambda\ t).$$

Example 1.

Taking into account the equation

$$x(t)=\frac{t^3}{5}+\frac{1}{2}\Big(|x(t)|+|\frac{x(0.5t)}{4}|\Big),\ t\in[0,1]. \tag{4.1}$$

Here

$$f\Big(t,\ x(t),\ \lambda\ x(\gamma t)\ \Big) = t^3 + \frac{1}{2}\Big(|x(t)| + |\frac{x(0.5t)}{4}|\Big),\ t \in [0,1]$$

It is clear that our assumptions of Theorem (3.3) are satisfied,
then $\|m\| = \frac{1}{5}$, $a\ = \frac{1}{4}$, $\lambda = \frac{1}{4}$ and $\gamma = \frac{1}{2}$.
and r satisfies

$$r \ = \ \frac{\|m\|}{1-(a+\frac{\lambda}{\gamma})}$$

$$r = \frac{4}{5}.$$

and

$$(a + \frac{\lambda}{\gamma}) = \frac{3}{4} < 1.$$

Therefore, by applying to Theorem 3.3, the pantograph functional equation (4.1) has a unique solution.

Example 2.
Taking into account the equation

$$x(t) = sint + \frac{1}{3}\Big(|x(t)| + |\frac{x(0.5t)}{5}|\Big),\ t \in [0,T]. \tag{4.2}$$

Here

$$f\Big(t,\ x(t),\ \lambda\ x(\gamma t)\ \Big) = sint + \frac{1}{3}\Big(|x(t)| + |\frac{x(0.5t)}{5}|\Big),\ t \in [0,T]$$

It is clear that our assumptions of Theorem (3.3) are satisfied,
then $\|m\| = 1$, $a = \frac{1}{3}$, $\lambda = \frac{1}{5}$ and $\gamma = \frac{1}{2}$.
and r satisfies

$$r \ = \ \frac{\|m\|}{1-(a+\frac{\lambda}{\gamma})}$$

$$r = \frac{15}{4}.$$

and

$$(a + \frac{\lambda}{\gamma}) = \frac{11}{15} < 1.$$

By applying to Theorem 3.3, the pantograph functional equation (4.2) has a unique solution.

5. Conclusions

In this investigation, we have conducted a thorough examination of the pantograph functional equation. Firstly, we define the pantograph functional equation (1.1) and its special case, the Ambartsumian delay equation (1.2) then, we discussed two cases for study investigated the solvability of (1.1): In the first case, we studied the existence of unique solution x on the class $C[0,T]$, we employed the Banach fixed point theorem [8]. Then, we studied the existence of solutions

x on the class $L_1[0,T]$, for(1.1) by applying Schauder fixed point Theorem [15]. Moreover, we have discussed the continuous dependence of the unique solution on parameter λ and on the functions f, γ. Furthermore, we thoroughly investigated the Hyers–Ulam stability of (1.1). Finally, we provided some illustrative examples to demonstrate the practical application and validity of our obtained results.

References

[1] Abbasbandy, S. and Bervillier, C. Analytic continuation of Taylor series and the boundary value problems of some nonlinear ordinary differential equations, Applied Mathematics and Computation, vol. 218, no. 5, pp. 2178–2199, 2011.

[2] Benchohra, M. and Lazreg, J. E. Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. *Romanian journal of Mathematics and Computer Science* **2014**, *4*, 60–72.

[3] Benhammouda, B. Vazquez-Leal, H. and Hernandez-Martinez, L. Procedure for exact solutions of nonlinear pantograph delay differential equations, British Journal of Mathematics and Computer Science, vol. 4, no. 19, pp. 2738–2751, 2014.

[4] Carr, J. and Dyson, J. The functional differential equation $y'(x) = ay(\lambda x) + by(x)$, Proc. Roy. Soc. Edinburgh Sect. A 74 (1974-75), 165-174.

[5] El-Sayed, A. M. A. Hamdallah, E. M. A. and Ba-Ali, Malak. M. S. On the De Blasi Measure of Noncompactness and Solvability of a Delay Quadratic Functional Integro-Differential Equation. *Mathematics* 2022, 10, 1362. https://doi.org/10.3390/ math10091362

[6] El-Sayed, A. M. A. Ba-Ali, Malak. M. S. and Hamdallah, E. M. A. An Investigation of a Nonlinear Delay Functional Equation with a Quadratic Functional Integral Constraint. Mathematics 2023, 1, 14475. https://doi.org/10.3390/math11214475

[7] El-Sayed, A. M. A. Hamdallah, E. M. A. and Ba-Ali, Malak M. S. Qualitative Study for a Delay Quadratic Functional Integro-Differential Equation of Arbitrary (Fractional) Orders. *Symmetry* 2022, 14, 784. https:// doi.org/10.3390/sym14040784

[8] Goebel, K. and Kirk, W. A. Topics in Metric Fixed point theory, *Cambridge University Press, Cambridge* (1990).

[9] Hyers, D. H. *On the stability of the linear functional equation*, Proceedings of the National Academy of Sciences **27.4** (1941), 222–224.

[10] Iserles, A. On the generalized pantograph functional-differential equation, European J. Appl. Math. 4 (1992), 1 38.

[11] Kato, T. Asymptotic behaviour of solutions of the functional-differential equations $y'(x) = ay(\lambda x) + by(x)$, in Delay and functional differential equations and their applications (K. Schmitt, ed.), Academic Press, New York.

[12] Kato, T. and McLeod, J. B. The functional-differential equation $y'(x) = ay(\lambda x)+by(x)$, Bull. Amer. Math. Soc. 77 (1971), 891-937.

[13] Liu, M. Z. and Properties, D. Li of analytic solution and numerical solution of multi-pantograph equation, Applied Mathematics and Computation, vol. 155, no. 3, pp. 853–871, 2004.

[14] Muroya, Y. Ishiwata, E. and Brunner, H. On the attainable order of collocation methods for pantograph integro-differential equations, Journal of Computational and Applied Mathematics, vol. 152, no. 1-2, pp. 347–366, 2003.

[15] Podlubny, I. Fractional differential equations, San Diego: Academic Press, 1999.

[16] Raja, M. A. Z. Numerical treatment for boundary value problems of Pantograph functional differential equation using computational intelligence algorithms, Applied Soft Computing, vol. 24, pp. 806–821, 2014.

[17] Sezer, M. and Akyüz-Daşcıoğlu, A. A Taylor method for numerical solution of generalized pantograph equations with linear functional argument, Journal of Computational and Applied Mathematics, vol. 200, no. 1, pp. 217–225, 2007.

[18] Sedaghat, S. Ordokhani, Y. and Dehghan, M. Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials, Communications in Nonlinear Science and Numerical Simulation, vol. 17, no. 12, pp. 4815–4830, 2012.

[19] Trif, D. Direct operatorial tau method for pantograph-type equations, Applied Mathematics and Computation, vol. 219, no. 4, pp. 2194–2203, 2012.

[20] Yalçınbaş, S. Aynigül, M. and Sezer, M. A collocation method using Hermite polynomials for approximate solution of pantograph equations, Journal of the Franklin Institute, vol. 348, no. 6, pp. 1128–1139, 2011.

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