ON THE AMBARTSUMIAN FUNCTIONAL EQUATION

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ABSTRACT. In this work, we define the pantograph functional equation with parameter and study the existence of solutions in two classes \(x \in C[0, T]\) and \(x \in L_1[0, T]\); we use the technique of the Banach fixed point theorem and Schauder fixed point theorem. In both cases we study the continuous dependence of the unique solution on the pantograph functional equation. The Hyers–Ulam stability will be studied. Additionally, we give an example to illustrate our outcomes.

keywords: Pantograph equation; Banach fixed point Theorem; Schauder fixed point Theorem; existence of solutions; continuous dependence; Hyers–Ulam stability.

1. INTRODUCTION

A pantograph (or "pan" or "panto") is an apparatus mounted on the roof of an electric train, tram or electric bus to collect power through contact with an overhead line. The pantograph equation is a fundamental mathematical model in the field of delay differential equations. It is well known that the pantograph differential equation is given by

\[
\frac{dx}{dt} = f(t, x(t), x(\gamma t)),
\]

a special case of the pantograph equation is well known as the Ambartsumian delay equation which has a particular application in Astrophysics.

\[
\frac{dx}{dt} = ax(t) + \lambda x(\gamma t).
\]

For papers studying such kind of problems (see [4, 11, 12]) and Pantograph differential equations have been studied in many papers and monographs [10].
In this study, we define the pantograph functional equation with parameter,

\[ x(t) = f \left(t, x(t), \lambda x(\gamma t) \right), \quad t \in [0, T] \]  

(1.1)

and its special case, the Ambartsumian delay equation

\[ x(t) = ax(t) + \lambda x(\gamma t). \]

Where \( \lambda > 0, \gamma \in (0, 1) \) and \( a \) is constant. Our aim here is study the existence of solutions \( x \in C[0, T] \) and \( x \in L^1[0, T] \) of equation (1.1). Moreover, the continuous dependence of the unique solution on the functions \( f, \gamma \) and on the parameter \( \lambda > 0 \) will be proved. The Hyers – Ulam stability of (1.1) will be given.

The paper is organized as follows: Section 2 contains the solvability of unique solution \( x \in C[0, T] \) by Banach fixed point and discuss some stability facts of the of (1.1). Moreover, the Hyers – Ulam stability of (1.1) will be studied. In Section 3, the solvability for the existence of the solutions \( x \in L^1[0, T] \) by Schauder fixed point Theorem and the continuous dependence of the unique solution \( x \in L^1[0, T] \) on the parameter \( \lambda \geq 0 \) and on the function \( f \). Some general discussion and examples in Section 4.

### 2. Solution in \( C[0, T] \)

Let \( C = C(I) \), be the class of continuous functions on \( I = [0, T], T < \infty \), with the standard norm

\[ \|x\| = \sup_{t \in I} |x(t)|. \]

Consider the pantograph functional equation (1.1) under the following assumptions:

(i) \( f : I \times R \times R \rightarrow R \) is continuous in \( t \in I \) and satisfies Lipschitz condition,

\[ |f(t, x_1, x_2) - f(t, y_1, y_2)| \leq a |x_1 - y_1| + \lambda |x_2 - y_2| \quad \forall t \in I, x_i, y_i \in R, i = 1, 2. \]

(ii) \( (a + \lambda) < 1. \)

Now, we have the following existences theorem.

**Theorem 2.1.** Assume that (i) and (ii) be satisfied, then the pantograph functional equation (1.1) has a unique solution \( x \in C(I) \).

**Proof.** Define the operator \( F \) by

\[ Fx(t) = f \left(t, x(t), \lambda x(\gamma t) \right). \]

Now, let \( x \in C(I) \) and \( t_1, t_2 \in I, t_1 < t_2, |t_2 - t_1| < \delta \) and denote

\[ \theta_f(\delta) = \sup_{x,y \in C(I)} \left\{ |f(t_2, x(t), y(t)) - f(t_1, x(t), y(t))| : t_1, t_2 \in I, t_1 < t_2, |t_2 - t_1| < \delta \right\}. \]
then we have
\[
|F(x(t_2) - F(x(t_1))| = |f(t_2, x(t_2), \lambda x(\gamma t_2)) - f(t_1, x(t_1), \lambda x(\gamma t_1))| \\
\leq |f(t_2, x(t_2), \lambda x(\gamma t_2)) - f(t_1, x(t_1), \lambda x(\gamma t_2))| \\
+ |f(t_1, x(t_1), \lambda x(\gamma t_1)) - f(t_1, x(t_1), \lambda x(\gamma t_1))| \\
\leq \theta f(\delta) + a |x(t_2) - x(t_1)| + \lambda |x(\gamma t_2) - x(\gamma t_1)|.
\]

This means that $F : C(I) \to C(I)$.

Now let $x_1, x_2 \in C(I)$, then
\[
|F(x_2(t) - F(x_1(t))| = |f(t, x_2(t), \lambda x_2(\gamma t)) - f(t, x_1(t), \lambda x_1(\gamma t))| \\
\leq |f(t, x_2(t), \lambda x_2(\gamma t)) - f(t, x_1(t), \lambda x_2(\gamma t))| \\
+ |f(t, x_1(t), \lambda x_2(\gamma t)) - f(t, x_1(t), \lambda x_1(\gamma t))| \\
\leq a|x_2(t) - x_1(t)| + \lambda|x_2(\gamma t) - x_1(\gamma t)|,
\]

then
\[
\|F(x_2) - F(x_1)\| \leq a\|x_2 - x_1\| + \lambda\|x_2 - x_1\| \\
\leq (a + \lambda)\|x_2 - x_1\|.
\]

Since $(a + \lambda) < 1$, then $F$ is a contraction and by Banach fixed point Theorem [8]
there exists a unique solution $x \in C(I)$ of the equation (1.1).

2.1. Continuous dependence.

**Theorem 2.2.** Let the assumptions of Theorem 2.1 be satisfied for $f, f^*, \lambda, \lambda^*, \gamma$
and $\gamma^*$. Then the unique solution $x \in C(I)$ depends continuously on $f, \lambda$ and $\gamma$
in the sense that

\[
\forall \epsilon > 0, \exists \delta(\epsilon) \text{ such that}
\]

\[
\max\{|\lambda - \lambda^*|, |f(t, x, y) - f^*(t, x, y)|, |\gamma - \gamma^*|\} < \delta, \text{ then } \|x - x^*\| < \epsilon.
\]

where $x^*$ is the solution of
\[
x^*(t) = f^*(t, x^*(t), \lambda^* x^*(\gamma^* t)).
\]
Proof.

\[
|x(t) - x^*(t)| = \left| f\left( t, x(t), \lambda x(\gamma t) \right) - f^*\left( t, x^*(t), \lambda^* x^*(\gamma^* t) \right) \right|
\]

\[
\leq \left| f\left( t, x(t), \lambda x(\gamma t) \right) - f^*\left( t, x(t), \lambda x(\gamma t) \right) \right|
\]

\[
+ \left| f^*\left( t, x(t), \lambda x(\gamma t) \right) - f^*\left( t, x^*(t), \lambda^* x^*(\gamma^* t) \right) \right|
\]

\[
\leq \delta + a \left| x(t) - x^*(t) \right| + \left| \lambda x(\gamma t) - \lambda^* x^*(\gamma^* t) \right|
\]

\[
\leq \delta + a \left| x(t) - x^*(t) \right| + \lambda x(\gamma t) - \lambda^* x(\gamma t)
\]

\[
+ \lambda^* x(\gamma t) - \lambda^* x^*(\gamma t) + \lambda^* x^*(\gamma^* t) - \lambda^* x^*(\gamma^* t)
\]

\[
\leq \delta + a \left| x(t) - x^*(t) \right| + \left| \lambda - \lambda^* \right| x(\gamma t) + \lambda^* x(\gamma t) - x^*(\gamma t)
\]

\[
+ \lambda^* x^*(\gamma t) - x^*(\gamma^* t)
\]

\[
\leq \delta + a \left| x(t) - x^*(t) \right| + \left| \lambda - \lambda^* \right| x(\gamma t) + \lambda^* x(\gamma t) - x^*(\gamma t)
\]

\[
+ \lambda^* \epsilon^*
\]

then

\[
\|x - x^*\| \leq \delta + a \|x - x^*\| + \delta \left| x(\gamma t) \right| + \lambda^* \|x - x^*\| + \lambda^* \epsilon^*
\]

Hence

\[
\|x - x^*\| \leq \frac{\delta + \delta \left| x(\gamma t) \right| + \lambda^* \epsilon^*}{1 - (a + \lambda^*)} = \epsilon.
\]

2.2. Hyers-Ulam stability.

Definition 2.3. [6, 9] Let the solution \( x \in C(I) \) of (1.1) be exists. then equation (1.1) is Hyers - Ulam stable if \( \forall \epsilon > 0, \exists \delta(\epsilon) \) such that for any \( \delta \) – approximate solution \( x_s \) satisfies,

\[
\left| x_s(t) - f\left( t, x_s(t), \lambda x_s(\gamma t) \right) \right| < \delta,
\]

implies \( \|x - x_s\| < \epsilon \).

Theorem 2.4. Let the assumptions of Theorem 2.1 be satisfied, then (1.1) is Hyers - Ulam stable.

Proof. From (2.1), we have

\[
\left| x_s(t) - f\left( t, x_s(t), \lambda x_s(\gamma t) \right) \right| < \delta,
\]
Now,
\[ |x(t) - x_s(t)| = \left| f\left( t, x(t), \lambda x(\gamma t) \right) - x_s(t) \right| \]
\[ \leq \left| f\left( t, x(t), \lambda x(\gamma t) \right) - f\left( t, x_s(t), \lambda x_s(\gamma t) \right) \right| \]
\[ + |x_s(t) - f\left( t, x_s(t), \lambda x_s(\gamma t) \right)| \]
\[ \leq a |x(t) - x_s(t)| + \lambda |x(\gamma t) - x_s(\gamma t)| + \left| x_s(t) - f\left( t, x_s(t), \lambda x_s(\gamma t) \right) \right|, \]
then
\[ \|x - x_s\| \leq \|x - x_s\| + \lambda \|x - x_s\| + \delta, \]
Hence
\[ \|x - x_s\| \leq \frac{\delta}{1 - (a + \lambda)} = \epsilon. \]

3. Solution in \( L_1(I) \)

Let \( L_1 = L_1(I) \), be the class of Lebesgue integrable functions on \( I = [0, T] \), \( T < \infty \), with the standard norm
\[ \|x\|_1 = \int_0^T |x(t)|dt. \]

Take into account the following assumptions:

(iii) \( f : I \times R \to R \) is measurable in \( t \in I \) for any \( x \in R \) and continuous in \( x \in R \) for all \( t \in I \). Moreover, there exist a bounded measurable function \( m : I \to R \) and a positive constant \( b_2 \) such that
\[ |f(t, x, y)| \leq |m(t)| + a |x| + \lambda |y| \text{ for each } t \in I \text{ and for all } x, y \in R, \ i = 1, 2. \]

(iv) \( (a + \frac{\lambda}{\gamma}) < 1. \)

**Theorem 3.1.** Let the assumptions (iii) - (iv) be satisfied, then the pantograph functional equation (1.1) has at least one solution \( x \in L_1(I) \).

**Proof.** Let \( Q_r \) be the closed ball
\[ Q_r = \{ x \in L_1(I) : \|x\|_1 \leq r \}. \]

Associate the operator
\[ Fx(t) = f\left( t, x(t), \lambda x(\gamma t) \right). \]

Now, let \( x \in Q_r \), then
\[ |Fx(t)| = \left| f\left( t, x(t), \lambda x(\gamma t) \right) \right| \]
\[ \leq |m(t)| + b_2 \left( |x(t)| + |\lambda x(\gamma t)| \right), \]
then
\[ \int_0^T |Fx(t)|dt \leq \int_0^T |m(t)|dt + a \int_0^T |x(t)|dt + \frac{\lambda}{\gamma} \int_0^T |x(\theta)|d\theta \]
and
\[
\|Fx\|_1 \leq \|m\|_1 + a \|x\|_1 + \frac{\lambda}{\gamma} \|x\|_1 \\
\leq \|m\|_1 + a r + \frac{\lambda}{\gamma} r \\
\leq \|m\|_1 + a r + \frac{\lambda r}{\gamma} = r.
\]

Then the class of functions \( \{Fx\} \) is uniformly bounded on \( Q_r \).

Now, let \( x \in Q_r \), then
\[
\|(Fx)_h - (Fx)\|_1 = \int_0^T \left| (Fx(s))_h - (Fx(s)) \right| ds \\
= \int_0^T \frac{1}{h} \int_t^{t+h} \left| (Fx(\theta))d\theta - (Fx(s)) \right| ds \\
\leq \int_0^T \frac{1}{h} \int_t^{t+h} \left| Fx(\theta) - Fx(s) \right| d\theta ds \\
\leq \int_0^T \frac{1}{h} \int_t^{t+h} \left| f(\theta, x(\theta), \lambda x(\gamma\theta)) - f(s, x(s), \lambda x(\gamma s)) \right| d\theta ds.
\]

Since \( F \in L_1(I) \), then
\[
\frac{1}{h} \int_t^{t+h} \left| f(\theta, x(\theta), \lambda x(\gamma\theta)) - f(s, x(s), \lambda x(\gamma s)) \right| d\theta ds \to 0, \text{ as } h \to 0.
\]

This means that \( Fx(t)_h \to Fx(t) \) uniformly in \( L_1(I) \). Thus the class of functions \( \{Fx\} \) is relatively compact [15]. Hence \( F \) is compact operator.

Now, let \( \{x_n\} \subset Q_r \), and \( x_n \to x \), then
\[
Fx_n(t) = f(t, x_n(t), \lambda x_n(\gamma t))
\]
and
\[
\lim_{n \to \infty} Fx_n(t) = \lim_{n \to \infty} f(t, x_n(t), \lambda x_n(\gamma t)).
\]

Applying Lebesgue dominated convergence Theorem [15], then from our assumptions we get
\[
\lim_{n \to \infty} Fx_n(t) = f(t, \lim_{n \to \infty} x_n(t), \lambda \lim_{n \to \infty} x_n(\gamma t))
\]
\[
= f(t, x(t), \lambda x(\gamma t)) = Fx(t).
\]

This means that \( Fx_n(t) \to Fx(t) \). Hence the operator \( F \) is continuous. Now, by Schauder fixed point Theorem [15] there exists at least one solution \( x \in L_1(I) \) of (1.1).

### 3.1. Uniqueness of the solution

Now, replace the assumption \((iii)\) by \((iii)^*\) as follows:

\((iii)^*\) \( f : I \times R \to R \) is measurable in \( t \in I \) \( \forall x \in R \) and satisfies Lipschitz condition,
\[
|f(t, x_1, y_2) - f(t, y_1, y_2)| \leq a |x_1 - y_1| + \lambda |x_2 - y_2| \forall t \in I, \ x_i, y_i \in R, i = 0, 1.
\]
Lemma 3.2. The assumption (iii)* implies the assumption (iii).

Proof. From the assumption (iii)* let \( y_1 = y_2 = 0 \), then we have

\[
|f(t, x_1, x_2)| - |f(t, 0, 0)| \leq |f(t, x_1, x_2) - f(t, 0, 0)| \leq a |x_1| + \lambda |x_2|,
\]

and

\[
|f(t, x_1, x_2)| \leq |f(t, 0, 0)| + a |x_1| + \lambda |x_2|
\]

then

\[
x = y
\]

implies the assumption (iii)*. Then the unique solution \( x \) of (1.1) is unique.

Theorem 3.3. Let the assumptions (iii)* and (iv) be satisfied, then the solution of a pantograph functional equation (1.1) is unique.

Proof. Let \( x, y \) be two solutions in \( Q_r \) of (1.1), then

\[
|x(t) - y(t)| = \left| f\left( t, x(t), \lambda x(\gamma t) \right) - f\left( t, y(t), \lambda y(\gamma t) \right) \right|
\]

\[
\leq |f\left( t, x(t), \lambda x(\gamma t) \right) - f\left( t, y(t), \lambda x(\gamma t) \right)| + |f\left( t, y(t), \lambda x(\gamma t) \right) - f\left( t, y(t), \lambda y(\gamma t) \right)|
\]

\[
\leq a |x(t) - y(t)| + \lambda|x(\gamma t) - y(\gamma t)|,
\]

then

\[
\int_0^T |x(t) - y(t)| dt \leq a \int_0^T |x(t) - y(t)| dt + \lambda \int_0^T |x(\gamma t) - y(\gamma t)| dt
\]

\[
\|x - y\|_1 \leq a \|x - y\|_1 + \frac{\lambda}{\gamma} \int_0^T |x(\theta) - y(\theta)| dt
\]

\[
\leq a \|x - y\|_1 + \frac{\lambda}{\gamma} \|x - y\|_1.
\]

Hence

\[
\|x - y\|_1 \left( 1 - \left( a + \frac{\lambda}{\gamma} \right) \right) \leq 0.
\]

then \( x = y \) and the solution of (1.1) is unique.

3.2. Continuous dependence.

Theorem 3.4. Let the assumptions of Theorem 3.3 be satisfied for \( f, f^*, \lambda \) and \( \lambda^* \). Then the unique solution \( x \in L_1(I) \) depends continuously on \( f \) and \( \lambda \) in the sense that

\[
\forall \epsilon > 0, \exists \delta(\epsilon) \text{ such that}
\]

\[
\max \{ |\lambda - \lambda^*|, |f(t, x, y) - f^*(t, x, y)| \} < \delta, \text{ then } \|x - x^*\|_1 < \epsilon.
\]

where \( x^* \) be a solution of

\[
x^*(t) = f^*\left( t, x^*(t), \lambda^* x^*(\gamma t) \right).
\]
Proof.

\[ |x(t) - x^*(t)| = \left| f \left( t, x(t), \lambda x(\gamma t) \right) - f^* \left( t, x^*(t), \lambda^* x^*(\gamma t) \right) \right| \]
\[ \leq \left| f \left( t, x(t), \lambda x(\gamma t) \right) - f^* \left( t, x(t), \lambda x(\gamma t) \right) \right| \]
\[ + f^* \left( t, x(t), \lambda x(\gamma t) \right) - f^* \left( t, x^*(t), \lambda^* x^*(\gamma t) \right) \]
\[ \leq \left| f \left( t, x(t), \lambda x(\gamma t) \right) - f^* \left( t, x(t), \lambda x(\gamma t) \right) \right| \]
\[ + a \left| x(t) - x^*(t) \right| + \left| \lambda x(\gamma t) - \lambda^* x^*(\gamma t) \right| \]
\[ \leq \delta + a \left| x(t) - x^*(t) \right| + \left| \lambda x(\gamma t) - \lambda^* x(\gamma t) \right| \]
\[ + \lambda^* \left| x(\gamma t) - x^*(\gamma t) \right| \]

then

\[ \int_0^T |x(t) - x^*(t)| dt \leq \delta + a \int_0^T |x(t) - x^*(t)| dt + \frac{\delta}{\gamma} \int_0^T |x(\theta)| d\theta \]
\[ + \frac{\lambda^*}{\gamma} \int_0^T |x(\theta) - x^*(\theta)| d\theta \]

\[ \|x - x^*\|_1 \leq \delta + a \|x - x^*\|_1 + \frac{\delta}{\gamma} r + \frac{\lambda^*}{\gamma} \|x - x^*\|_1. \]

Hence

\[ \|x - x^*\|_1 \leq \frac{\delta + \frac{a}{\gamma} r}{1 - (a + \frac{\lambda^*}{\gamma})} \leq \epsilon. \]

3.3. Hyers-Ulam stability.

Definition 3.5. [6, 9] Let the solution \( x \in L_1(I) \) of (1.1) be exists, then equation (1.1) is Hyers - Ulam stable if \( \forall \epsilon > 0, \exists \delta(\epsilon) \) such that for any \( \delta \)-approximate solution \( x_s \) satisfies,

\[ \left\| x_s(t) - f \left( t, x_s(t), \lambda x_s(\gamma t) \right) \right\|_1 < \delta, \]  
(3.1)

implies \( \|x - x_s\|_1 < \epsilon. \)

Theorem 3.6. Let the assumptions of Theorem 3.3 be satisfied, then (1.1) is Hyers - Ulam stable.

Proof. From (3.1), we have

\[ \left\| x_s(t) - f \left( t, x_s(t), \lambda x_s(\gamma t) \right) \right\|_1 < \delta, \]
Now,
\[
|x(t) - x_s(t)| = \left| f\left(t, x(t), \lambda x(\gamma t)\right) - x_s(t)\right|
\leq \left| f\left(t, x(t), \lambda x(\gamma t)\right) - f\left(t, x_s(t), \lambda x_s(\gamma t)\right)\right|
+ \left| x_s(t) - f\left(t, x_s(t), \lambda x_s(\gamma t)\right)\right|
\]
then
\[
\int_0^T |x(t) - x_s(t)| dt \leq a \int_0^T |x_2(t) - x_1(t)| dt + \lambda \int_0^T |x_2(\gamma t) - x_1(\gamma t)| dt
+ \int_0^T \left| x_s(t) - f\left(t, x_s(t), \lambda x_s(\gamma t)\right)\right| dt
\]
\[
\|x - x_s\|_1 \leq a \|x - x_s\|_1 + \frac{\lambda}{\gamma} \|x - x_s\|_1 + \delta.
\]
Hence
\[
\|x - x_s\|_1 \leq \frac{\delta}{1 - (a + \frac{\lambda}{\gamma})} = \epsilon.
\]

4. General discussion and examples

1- Let \( \lambda \) and \( \gamma = \frac{1}{q} \), then we have
\[
x(t) = f\left(t, x(t), \frac{1}{q} x\left(\frac{t}{q}\right)\right),
\]
and
\[
x(t) = ax(t) + \frac{1}{q} x\left(\frac{t}{q}\right).
\]
2- Let \( \lambda = \gamma \) where \( \gamma \in (0, 1) \), then we have
\[
x(t) = f\left(t, x(t), \gamma x(\gamma t)\right),
\]
and
\[
x(t) = ax(t) + \gamma x(\gamma t).
\]
Or
\[
x(t) = f\left(t, x(t), \lambda x(\lambda t)\right),
\]
and
\[
x(t) = ax(t) + \lambda x(\lambda t).
\]

Example 1.
Taking into account the equation
\[
x(t) = \frac{t^3}{5} + \frac{1}{2} \left| x(t) + \frac{x(0.5t)}{4} \right|, \quad t \in [0, 1].
\]
Here
\[ f \left( t, x(t), \lambda x(\gamma t) \right) = t^3 + \frac{1}{2} \left( |x(t)| + \left| \frac{x(0.5t)}{4} \right| \right), \quad t \in [0, 1] \]

It is clear that our assumptions of Theorem (3.3) are satisfied, then \( \|m\| = \frac{1}{5}, a = \frac{1}{3}, \lambda = \frac{1}{5} \) and \( \gamma = \frac{1}{2} \).

and \( r \) satisfies
\[ r = \frac{\|m\|}{1 - (a + \frac{\lambda}{\gamma})} \]
\[ r = \frac{4}{5} \]

and
\[ (a + \frac{\lambda}{\gamma}) = \frac{3}{4} < 1. \]

Therefore, by applying to Theorem 3.3, the pantograph functional equation (4.1) has a unique solution.

**Example 2.**

Taking into account the equation
\[ x(t) = \sin t + \frac{1}{3} \left( |x(t)| + \left| \frac{x(0.5t)}{5} \right| \right), \quad t \in [0, T]. \] (4.2)

Here
\[ f \left( t, x(t), \lambda x(\gamma t) \right) = \sin t + \frac{1}{3} \left( |x(t)| + \left| \frac{x(0.5t)}{5} \right| \right), \quad t \in [0, T] \]

It is clear that our assumptions of Theorem (3.3) are satisfied, then \( \|m\| = 1, a = \frac{1}{3}, \lambda = \frac{1}{5} \) and \( \gamma = \frac{1}{2} \).

and \( r \) satisfies
\[ r = \frac{\|m\|}{1 - (a + \frac{\lambda}{\gamma})} \]
\[ r = \frac{15}{4} \]

and
\[ (a + \frac{\lambda}{\gamma}) = \frac{11}{15} < 1. \]

By applying to Theorem 3.3, the pantograph functional equation (4.2) has a unique solution.

5. **Conclusions**

In this investigation, we have conducted a thorough examination of the pantograph functional equation. Firstly, we define the pantograph functional equation (1.1) and its special case, the Ambartsumian delay equation (1.2) then, we discussed two cases for study investigated the solvability of (1.1): In the first case, we studied the existence of unique solution \( x \) on the class \( C[0, T] \), we employed the Banach fixed point theorem [8]. Then, we studied the existence of solutions...
Moreover, we have discussed the continuous dependence of the unique solution on parameter $\lambda$ and on the functions $f, \gamma$. Furthermore, we thoroughly investigated the Hyers–Ulam stability of (1.1). Finally, we provided some illustrative examples to demonstrate the practical application and validity of our obtained results.

References


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