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A STUDY OF GENERALIZED PARTIAL MOCK THETA FUNCTIONS

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Abstract. S. Ramanujan in his last letter to G.H. Hardy has introduced seventeen mock theta functions of differenct orders (3,5 and 7) without explicitly mentioning the reason for his levelling of order, later Watson added to this set three more third order mock theta function. Watson also defined bilateral form of some mock theta function of order five and expressed them in terms of lerch transendetal functions . Patial mock theta functions of sixth order were studied by Y.S. Choi and partial mock theta functions of third order were studied by G.E. Andrews. In this paper we have given generalization of partial sixth and third order mock theta functions and it is shown that these generalized partial mock theta functions are F_q functions. q-integral representation of these generalized partial mock theta functions are also given. we have also expressed some bilateral mock theta functions in terms of lerch transcendental functions $f(x, \xi; q, p)$.

1. Introduction:

S.Ramanujan in his last letter to G.H. Hardy[14,pp 354-355] indroduced seventeen functions whom he called mock theta functions, as they were not theta functions. He stated two conditions for a function to be a mock theta function:

(a) For every root of unity ζ, there is θ- function $\theta_\zeta(q)$ such that difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially.

(b) There is no single θ-function which works for all ζ i.e., for every θ-function $\theta(q)$ there is some root of unity ζ for which difference $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ radially.

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of the seventeen mock theta functions,four were of third order,ten of fifth order in two groups with five functions in each group and three of seventh order. Ramanujan did not specify what he meant by the order of a mock theta function.Later Watson [18] added three more third order mock theta functions,making the four third order mock theta functions to seven.
G.E. Andrews [4] while visiting Trinity College Cambridge University discovered some notebooks of Ramanujan , and called it the " Lost " Notebook. In the Notebook Andrews found more mock theta functions and some identities and Andrews and Hickeson [5] called them sixth order.
The partial sixth order mock theta functions of Ramanujan are:

$$\phi_{0,N}(q) = \sum_{n=0}^{N} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q)_{2n}},$$

$$\psi_{0,N}(q) = \sum_{n=0}^{N} \frac{(-1)^n q^{(n+1)^2} (q;q^2)_n}{(-q)_{2n+1}},$$

$$\rho_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{\frac{n(n+1)}{2}} (-q)_n}{(q;q^2)_{n+1}},$$

$$\sigma_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{\frac{(n+1)(n+2)}{2}} (-q)_n}{(q;q^2)_{n+1}},$$

$$\lambda_{0,N}(q) = \sum_{n=0}^{N} \frac{(-1)^n q^n (q;q^2)_n}{(-q)_n},$$

$$\mu_{0,N}(q) = \sum_{n=0}^{N} \frac{(-1)^n (q;q^2)_n}{(-q)_n},$$

$$\gamma_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2} (q)_n}{(q^3;q^3)_n},$$

The partial third order mock theta functions of Ramanujan are:

$$f_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{(-q)_n^2},$$

$$\phi_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{(-q^2;q^2)_n},$$

$$\psi_{0,N}(q) = \sum_{n=1}^{N} \frac{q^{n^2}}{(q;q^2)_n},$$

$$\chi_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{(1-q+q^2).......(1-q^n+q^{2n})},$$

$$\omega_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2},$$

$$\upsilon_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}},$$

$$\rho_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{2n(n+1)}}{(1+q+q^2).........(1+q^{2n+1}+q^{4n+2})},$$

We give a generalization of the partial sixth order and partial third order mock theta functions. The generalized partial sixth order mock theta functions are:

$$\phi_{0,N}(t,\alpha,z;q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q};q^2)_n}{(\frac{-z^2}{q};q)_{2n}},$$

$$\psi_{0,N}(t,\alpha,z;q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-1)+n\alpha} z^{2n+1} (\frac{z^2}{q};q^2)_n}{(\frac{-z^2}{q};q^2)_{2n+1}},$$

$$\rho_{0,N}(t,\alpha,z;q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n q^{\frac{n(n-3)}{2}+n\alpha} z^n (-z;q)_n}{(\frac{z^2}{q};q^2)_{n+1}},$$

$$\sigma_{0,N}(t,\alpha,z;q) = \frac{1}{2(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n q^{\frac{n(n-1)}{2}+n\alpha} z^{n+1} (\frac{-z}{q};q)_n}{(\frac{z^2}{q};q^2)_{n+1}},$$

$$\lambda_{0,N}(t,\alpha,z;q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n (-1)^n q^{n\alpha} (\frac{q^3}{z^2};q^2)_n}{(\frac{-q^2}{z};q^2)_n},$$

$$\mu_{0,N}(t,\alpha,z;q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n (-1)^n q^{n(\alpha-1)} (\frac{q^3}{z^2};q^2)_n}{(\frac{-q^2}{z};q)_n},$$

and

$$\gamma_{0,N}(t,\alpha,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{n(n-3)+n\alpha}z^{2n}}{(\upsilon^2 z;q)_n(\upsilon^4 z;q)_n},$$

For $t=0$,$\alpha=1$,we have the generalized partial functions of Choi[9].
The generalized partial third order mock theta function are:

$$f_{0,N}(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{n^2-4n+n\beta}\alpha^n z^{2n}}{(-z;q)_n(\frac{-\alpha z}{q};q)_n}$$

$$\phi_{0,N}(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{n^2-3n+n\beta}z^{2n}}{(\frac{-\alpha z^2}{q};q^2)_n}$$

$$\psi_{0,N}(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{n^2-n+n\beta}z^{2n+1}}{(\frac{\alpha z^2}{q^2};q^2)_{n+1}}$$

$$\upsilon_{0,N}(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{n^2-2n+n\beta}z^{2n}}{(\frac{-\alpha^2 z^2}{q^3};q^2)_{n+1}}$$

$$\omega_{0,N}(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{2n^2-5n-4+n\beta}\alpha^{2n}z^{4(n+1)}}{(\frac{z^2}{q};q^2)_{n+1}(\frac{\alpha^2 z^2}{q^3};q^2)_{n+1}}$$

$$\chi_{0,N}(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{n^2-3n+n\beta}z^{2n}}{(\upsilon z;q)_n(-\upsilon^2 z;q)_n}$$

and

$$\rho_{0,N}(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{2n^2-3n+n\beta}z^{4n}}{(\frac{\upsilon^2 z^2}{q};q^2)_{n+1}(\frac{\upsilon^{-2} z^2}{q};q^2)_{n+1}}$$

where

$$\upsilon=e^{\frac{\pi i}{3}}$$

For $\beta=1$ and $z=q$ the first five functions namely $f_{0,N}$,$\phi_{0,N}$,$\psi_{0,N}$,$\upsilon_{0,N}$,$\omega_{0,N}$ are generalized third order partial mock theta functions of Andrews[6].For $t=0$,$\beta=1$,$\alpha=q$ the generalized functions $f_{0,N}$,$\phi_{0,N}$,$\psi_{0,N}$ and $\chi_{0,N}$ reduce to the third order mock theta functions of Ramanujan and $\omega_{0,N}$,$\nu_{0,N}$ and $\rho_{0,N}$ reduces to the third order mock theta functions of Watson[18].
In this study we will show that these generalized functions are F_q-functions.

2. Notation:

We use the following q-notation. Suppose q and z are complex numbers and n is an integer. If $n \geq 0$ we define

$$(z)_n = (z;q)_n = \prod_{i=0}^{n-1}(1-q^i z) \ if \ n \ \leq 0 \text{ and } (z)_{-n} = (z;q)_{-n} = \frac{(-z)^{-n} q^{\frac{n(n+1)}{2}}}{\left(\frac{q}{z};q\right)_n}$$

and more generally $(z_1, z_2, \cdots, z_r; q)_n = (z_1)_n (z_2)_n \cdots (z_r)_n$.

For $\left|q^k\right| < 1$ let us define $(z;q^k)_n = (1-z)(1-zq^k)\cdots(1-zq^{k(n-1)})$, $n \geq 1$ $(z;q^k)_0 = 1$ and $(z;q^k)_\infty = \lim_{n\to\infty}(z;q^k)_n = \prod_{i\geq 0}(1-q^{ki}z)$ and even more generally,

$$(z_1, z_2 \cdots z_r; q^k)_\infty = (z_1;q^k)_\infty \cdots (z_r;q^k)_\infty$$

A basic hypergeometric series ${}_{r+1}\Phi_r$ on base q^k is defined as

$${}_{r+1}\Phi_r \left[\begin{matrix} a_1, a_2 & \cdots & a_{r+1} \\ b_1, b_2 & \cdots & b_r \end{matrix} ; \ q^k; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{r+1}; q^k)_n z^n}{(q^k;q^k)_n (b_1, b_2, \cdots b_r; q^k)_n}, (|z| < 1) \tag{1}$$

and a bilateral basic hypergeometric series ${}_r\Psi_r$ is defined as

$${}_r\Psi_r \left[\begin{matrix} a_1, & \cdots & a_r \\ b_1, & \cdots & b_r \end{matrix} ; q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, \cdots, a_r; q)_n \ z^n}{(b_1 \cdots b_r; q)_n}, (\left|\frac{b_1 \cdots b_r}{a_1 \cdots a_r}\right| < |z| < 1) \tag{2}$$

The Lerch transcendental function $f(x, \xi; q, p)$ is defined by:

$$f(x, \xi; q, p) = \sum_{-\infty}^{\infty} \frac{(pq)^{n^2} (x\xi)^{-2n}}{(-p\xi^{-2}; p^2)_n} \tag{3}$$

and by

$$f(x, \xi; q, p) = \sum_{-\infty}^{\infty} (-\xi^2 p; p^2)_n q^{n^2} x^{2n} \tag{4}$$

3. The Generalized Functions are F_q - Functions:

We show that these partial generalized functions are F_q - Function.

Theorem 1

The generalized partial sixth order mock theta functions $\phi_{0,N}(t,\alpha,z;q)$,$\psi_{0,N}(t,\alpha,z;q)$,$\rho_{0,N}(t,\alpha,z;q)$, $\sigma_{0,N}(t,\alpha,z;q)$ and the generalized partial third order mock theta functions $f_{0,N}(t,\alpha,\beta,z;q)$,$\phi_{0,N}(t,\alpha,\beta,z;q)$,$\psi_{0,N}(t,\alpha,\beta,z;q)$,$\nu_{0,N}(t,\alpha,\beta,z;q)$,$\chi_{0,N}(t,\alpha,z;q)$,$\rho_{0,N}(t,\beta,z;q)$, $\omega_{0,N}(t,\alpha,\beta,z;q)$ are F_q -Functions.

Proof:

We shall give the proof for $\phi_{0,N}(t,\alpha,z;q)$ only .The proofs for the other generalized partial mock theta functions are similar,hence omitted.

Applying the difference operator $D_{q,t}$ to $\phi_{0,N}(t,\alpha,z;q)$,we have :

t $D_{q,t}$ $\phi_{0,N}(t,\alpha,z;q) = \phi_{0,N}(t,\alpha,z;q)$ - $\phi_{0,N}(tq,\alpha,z;q)$

$$= \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_n}{(\frac{-z^2}{q}; q)_{2n}} - \frac{1}{(tq)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (tq)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_n}{(\frac{-z^2}{q}; q)_{2n}}$$

$$= \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_n}{(\frac{-z^2}{q}; q)_{2n}} - \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_n (1 - tq^n)}{(\frac{-z^2}{q}; q)_{2n}}$$

$$= \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n(\alpha+1)} z^{2n} (\frac{z^2}{q}; q^2)_n}{(\frac{-z^2}{q}; q)_{2n}}$$

$$= t\phi_{0,N}(t, \alpha + 1, z; q).$$

so

$$D_{q,t}\phi_{0,N}(t, \alpha, z; q) = \phi_{0,N}(t, \alpha + 1, z; q).$$

Hence $\phi_{0,N}(t, \alpha, z; q)$ is a F_q -function.
As stated earlier the proofs for other partial generalized functions are similar,so omitted.

4. Relation between the Generalized Partial Sixth Order Mock Theta Functions and Generalized Partial Third Order Mock Theta Functions:

Theorem 2

$(i)\phi_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_n}{(\frac{-z^2}{q}; q)_{2n}} + \frac{z}{q}\psi_{0,N}(t, \alpha, z; q).$
$(ii)\sigma_{0,N}(t, \alpha, z; q) = \frac{z}{2}(1 + \frac{z}{q})D_{q,t}\rho_{0,N}(t, \alpha, z; q).$
$(iii)D_{q,t}\phi_{0,N}(t, \alpha^2, \beta, z; q) = (1 + \frac{\alpha^2 z^2}{q^3})\nu_{0,N}(t, \alpha, \beta, z; q).$
$(iv)\psi_{0,N}(t, \frac{-\alpha^2}{q}, \beta, z; q) = zD_{q,t}\nu_{0,N}(t, \alpha, \beta, z; q).$

Proof of (i)

$\phi_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_n (1 + z^2 q^{2n-1})}{(\frac{-z^2}{q}; q)_{2n+1}}$

$$= \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_n}{(\frac{-z^2}{q}; q)_{2n+1}} + \frac{z}{q} \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_n}{(\frac{-z^2}{q}; q)_{2n+1}}$$

$$= \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_n}{(\frac{-z^2}{q}; q)_{2n+1}} + \frac{z}{q}\psi_{0,N}(t, \alpha, z; q)$$

Which proves Theorem 2 (i).

Proof of (ii)

$$\rho_{0,N}(t, \alpha, z; q) = \frac{z(1 + \frac{z}{q})}{2(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n q^{\frac{n(n-1)}{2} + n\alpha} z^n (-z; q)_n}{(\frac{z^2}{q}; q^2)_{n+1}}$$
$$= \frac{z}{2}(1 + \frac{z}{q})D_{q,t}\rho_{0,N}(t, \alpha, z; q).$$

Which proves Theorem 2 (ii).

Proof of (iii)
Writing α^2 for α in $\phi_{0,N}(t,\alpha,\beta,z;q)$, we have

$$D_{q,t}\phi_{0,N}(t,\alpha^2,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{n^2-2n+n\beta}z^{2n}}{(\frac{-\alpha^2z^2}{q};q^2)_n}=(1+\frac{\alpha^2z^2}{q^3})\nu_{0,N}(t,\alpha,\beta,z;q)$$

Which proves Theorem 2 (iii).

Proof of (iv)
Writing $\frac{-\alpha}{q}$ for α and then α^2 for α in $\psi_{0,N}(t,\alpha,\beta,z;q)$, we have

$$\psi_{0,N}(t,\alpha^2,\beta,z;q)=\frac{z}{(t)_\infty}\sum_{n=0}^{N}\frac{(t)_n q^{n^2-2n+n\beta}z^{2n}}{(\frac{-\alpha^2z^2}{q^3};q^2)_{n+1}}=zD_{q,t}\nu_{0,N}(t,\alpha,\beta,z;q)$$

which proves Theorem 2(iv).

5. q-Integral Representation for the Generalized Partial Sixth Order Mock Theta Function and Generalized Partial Third Order Mock Theta Functions:

The q-integral was defined by Thomas and Jackson[11,p.23] as

$$\int_0^1 f(t)d_qt=(1-q)\sum_{n=0}^{\infty}f(q^n)q^n$$

We now give the q-integral representation for the generalized sixth order mock theta functions and also for generalized third order mock theta functions.

Theorem 3(a)

$$(i)\phi_{0,N}(q^t,\alpha,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1\omega^{t-1}(\omega q;q)_\infty\phi_{0,N}(0,a\omega,z;q)d_q\omega.$$

$$(ii)\psi_{0,N}(q^t,\alpha,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1\omega^{t-1}(\omega q;q)_\infty\psi_{0,N}(0,a\omega,z;q)d_q\omega.$$

$$(iii)\rho_{0,N}(q^t,\alpha,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1\omega^{t-1}(\omega q;q)_\infty\rho_{0,N}(0,a\omega,z;q)d_q\omega.$$

$$(iv)\gamma_{0,N}(q^t,\alpha,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1\omega^{t-1}(\omega q;q)_\infty\gamma_{0,N}(0,a\omega,z;q)d_q\omega.$$

$$(v)\sigma_{0,N}(q^t,\alpha,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1\omega^{t-1}(\omega q;q)_\infty\sigma_{0,N}(0,a\omega,z;q)d_q\omega.$$

Proof

We shall give the detailed proof for $\phi_{0,N}(q^t,\alpha,z;q)$. The proof for the other functions are similar,so omitted.
Limiting case of q-beta integral [11,p.23(1.11.7)] is

$$\frac{1}{(q^x;q)_\infty} = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{x-1}(tq;q)_\infty d_q t.$$

Now

$$\phi_{0,N}(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{N}\frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n}(\frac{z^2}{q};q^2)_n}{(\frac{-z^2}{q};q)_{2n}},$$

Replacing t by q^t and q^α by a,we have

$$\phi_{0,N}(q^t,\alpha,z;q) = \frac{1}{(q^t)_\infty}\sum_{n=0}^{N}\frac{(-1)^n (q^t)_n q^{n(n-3)+n\alpha} z^{2n}(\frac{z^2}{q};q^2)_n}{(\frac{-z^2}{q};q)_{2n}},$$

$$= \sum_{n=0}^{N}\frac{(-1)^n q^{n(n-3)+n\alpha} z^{2n}(\frac{z^2}{q};q^2)_n}{(\frac{-z^2}{q};q)_{2n}}\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{n+t-1}(\omega q;q)_\infty d_q\omega,$$

$$= \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{t-1}(\omega q;q)_\infty \sum_{n=0}^{N}\frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n}(\frac{z^2}{q};q^2)_n}{(\frac{-z^2}{q};q)_{2n}}(a\omega)^n d_q\omega \quad (5)$$

But

$$\phi_{0,N}(0,\alpha,z;q) = \sum_{n=0}^{N}\frac{(-1)^n q^{n(n-3)+n\alpha} z^{2n}(\frac{z^2}{q};q^2)_n}{(\frac{-z^2}{q};q)_{2n}},$$

and since $q^\alpha = a$,

$$\phi_{0,N}(0,a,z;q) = \sum_{n=0}^{N}\frac{(-1)^n (a)^n q^{n(n-3)} z^{2n}(\frac{z^2}{q};q^2)_n}{(\frac{-z^2}{q};q)_{2n}},$$

Hence

$$\phi_{0,N}(0,a\omega,z;q) = \sum_{n=0}^{N}\frac{(-1)^n (a\omega)^n q^{n(n-3)} z^{2n}(\frac{z^2}{q};q^2)_n}{(\frac{-z^2}{q};q)_{2n}}, \quad (6)$$

By using (5.1),(5.2) can be written as

$$\phi_{0,N}(q^t,\alpha,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{t-1}(\omega q;q)_\infty \phi_{0,N}(0,a\omega,z;q)d_q\omega.$$

which proves (i). The proofs for all other functions are similar.

Theorem 3(b):

The q-integral representation for the generalized partial third order mock theta functions:

$$(i) f_{0,N}(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{t-1}(\omega q;q)_\infty f_{0,N}(0,a,a\omega,z;q)d_q\omega.$$

$$(ii)\phi_{0,N}(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{t-1}(\omega q;q)_\infty\phi_{0,N}(0,a,a\omega,z;q)d_q\omega.$$

$$(iii)\psi_{0,N}(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{t-1}(\omega q;q)_\infty\psi_{0,N}(0,a,a\omega,z;q)d_q\omega.$$

$$(iv)\nu_{0,N}(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{t-1}(\omega q;q)_\infty\nu_{0,N}(0,a,a\omega,z;q)d_q\omega.$$

$$(v)\chi_{0,N}(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{t-1}(\omega q;q)_\infty\chi_{0,N}(0,a,a\omega,z;q)d_q\omega.$$

$$(vi)\rho_{0,N}(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{t-1}(\omega q;q)_\infty\rho_{0,N}(0,a,a\omega,z;q)d_q\omega.$$

$$(vii)\omega_{0,N}(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 \omega^{t-1}(\omega q;q)_\infty\omega_{0,N}(0,a,a\omega,z;q)d_q\omega.$$

Proof:
The proofs are similar to given above for $\phi_{0,N}(q^t,\alpha,z;q)$,so the Theorem 3(b) follows.

6. Repersentation in terms of Lerch Transcendental Function

The bilateral mock theta functions corresponding to third order mock theta functions were studied by S.D. Prasad [13] .we now express some of these functions in terms of Lerch trascendental functions by means of the following lemma of M Ahmad [3] .

Lemma 6.1 For $\epsilon=\pm1$,

$$\sum_{n=-\infty}^{\infty}(-1)^{rn}\frac{q^{\alpha n^2}q^{\beta n}}{(\epsilon q^\gamma;q^\delta)_n}=f(i^r(-\epsilon)^{-1/2}q^{\frac{2\gamma-2\beta-\delta}{4}},(-\epsilon)^{1/2}q^{\frac{\delta-2\gamma}{4}};q^{\frac{2\alpha-\delta}{2}},q^{\frac{\delta}{2}}).\qquad(7)$$

and

$$\sum_{n=-\infty}^{\infty}(-1)^{rn}(-q;q^\gamma)_n q^{\alpha n^2}q^{\beta n}=f(i^r q^{\frac{\beta}{2}},q^{\frac{2-\gamma}{4}};q^\alpha,q^{\frac{\gamma}{2}}).\qquad(8)$$

Proof. The proof follows from direct substitution and use of basic hypergeometric transformations. □

we now express the following bilateral 3rd order mock theta functions as Lerch trascendental functions
A mock theta function is expressed in terms of a series from 0 to ∞, whereas the correponding bilateral mock theta is the same series from $-\infty$ to ∞.
$\phi_{0,c}(q)=\sum_{-\infty}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}=f(-q^{\frac{1}{2}},q^{\frac{-1}{2}};1,q)$ by taking $\alpha=1$,$\beta=0$, $\epsilon=-1$, $\gamma=2,\delta=2,r=2$ in equation (3.1)
$\psi_{0,c}(q)=\sum_{-\infty}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}=f(i,i;1,q)$ by taking $\alpha=1$,$\beta=0$, $\epsilon=1$, $\gamma=1,\delta=2,r=2$ in equation (3.1)

$\upsilon 0, c(q) = \frac{1}{1+q}\sum_{-\infty}^{\infty} \frac{q^{n^2+n}}{(-q^3;q^2)_n} = \frac{1}{1+q} f(-q^{\frac{1}{2}}, q^{-1}; 1, q)$ by taking $\alpha = 1$,$\beta = 1$, $\epsilon = -1$, $\gamma = 3, \delta = 2, r = 2$ in equation (3.1)

Conclusions: Mock theta functions are mysterious functions.These investigations will be helpful in understanding more about these partial mock theta functions.Being shown that they belong to the class of F_q-functions and properties are established for the partial mock theta functions and relations between these partial mock theta functions may also be derived.

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