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SOME REMARKS ON THE HIGHER REGULARITY OF MINIMIZERS OF ANISOTROPIC FUNCTIONALS

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Abstract. We consider the anisotropic integral functional of the calculus of variations

$$\int_\Omega \left[\sum_{i=1}^n c_i |D_i u|^{p_i}\right] dx,$$

where $c_i \geq 0$ and $2 \leq p_i \leq p_{i+1}$ for every $i = 1, \ldots n-1$.
We exhibit a minimizer of such functional, for an opportune choice of the exponents p_i, which turns out to be bounded everywhere and Lipschitz continuous (or even of class C^1) in an opportune subset of Ω.

1. Introduction

Since the second half of the eighties, many people interested to the study of the regularity properties of minimizers of functionals of the calculus of variations of the type

$$\mathcal{I}(u, \Omega) = \int_\Omega \sum_{i=1}^n c_i f_i(D_i u) dx, \tag{1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$, is an open bounded set, $u : \Omega \to \mathbb{R}$, $c_i \geq 0$ for $i = 1, \ldots, n$ are constants and $f_i : \mathbb{R} \to [0, +\infty)$ are functions satisfying, for every $t \in \mathbb{R}$ the following non standard growth condition:

$$\lambda |t|^{p_i} \leq f_i(t) \leq \Lambda |t|^{p_i}$$

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for every $i = 1, \dots, n$ and some positive constants $0 < \lambda < \Lambda$. Moreover, some further assumption, such as a *strong ellipticity* condition of the kind

$$\sum_{i=1}^{n} f_i''(\xi_i)\eta_i\eta_i \geq \nu \sum_{i=1}^{n} |\xi_i|^{p_i-2}|\eta_i|^2,$$

is assumed for every $\xi, \eta \in \mathbb{R}^n$ and some $\nu > 0$.
Authors often refer to such a functional as the *othotropic functional*.

A minimizer of functional (1) is a function u which belongs to the *anisotropic Sobolev space*

$$W^{1,(p_i)}(\Omega) = \left\{ v \in W^{1,2}(\Omega) : \ D_i v \in L^{p_i}(\Omega), \ \forall \ i = 1, \dots, n \right\}, \tag{2}$$

and such that

$$\mathcal{I}(u, \Omega) \leq \mathcal{I}(v, \Omega),$$

for every $v \in W^{1,(p_i)}(\Omega)$ such that $v = u$ on $\partial\Omega$.

As it is well known, there exists minimizers of functional (1), that are not bounded, if the exponents p_i are too far apart (see [18], [11], [13]).
To be more precise about this point, it is opportune to introduce the *harmonic mean* of the exponents p_i, that is

$$\bar{p} = n \left[\sum_{i=1}^{n} \frac{1}{p_i} \right]^{-1}$$

and its *Sobolev conjugate* $\bar{p}^*$ defined as

$$\bar{p}^* = \frac{n\bar{p}}{n - \bar{p}} \qquad \text{if} \quad \bar{p} < n,$$

while $\bar{p}^*$ is any number strictly greater than $\bar{p}$ otherwise.

In [18] and [11] for instance it is proved that, if $n \geq 6$, $p_i = 2$ for every $i = 1, \dots, n-1$ and $p_n = 4$, then there exists a minimizer of functional (1) which is not bounded at some point of the unit ball in $\Omega = B_1(0) \subset \mathbb{R}^n$.
This result can be generalized obtaining that, if $p_i > \bar{p}^*$ for some i, then the minimizer u may not be bounded.
On the other hand, it has also been proved (see [3], [10], [25]) that the minimizers of functional (1) are bounded, provided

$$\max_{1 \leq i \leq n} p_i \leq \bar{p}^*. \tag{3}$$

A question arise at this point:

"*is the bound (3) sufficient to get higher regularity (namely, Lipschitz regularity) for the minimizers of functional (1)?*"

In [1], the same problem is taken into account in the vectorial case, that is for $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ $(N > 1)$, and a very important result is proved under assumption (3) :

"*there exists an open subset Ω_0 of Ω, such that $|\Omega \setminus \Omega_0| = 0$ and $u \in C^{1,\alpha}_{loc}(\Omega_0, \mathbb{R}^N)$ for some $\alpha \in (0, 1)$*".

Therefore, if condition (3) holds in the vectorial case, we have Hölder regularity of minimizers, up to a set of measure zero. In literature we often refer to this property as *partial regularity* of minimizers.

In the scalar case, higher regularity results have been obtained by paying the price of a stronger bound for the exponents p_i (see for instance [19], [22] and the references quoted there).
Moreover, in [16] and [17] it is proven that any bounded minimizer of functional (1) is Lipschitz continuous, but this proof is known to have some flaws, while in [4] the same result is obtained for any choice of the exponents $2 \leq p_1, \leq \cdots \leq p_n$.

The goal of this paper is to exhibit an example of a minimizer u of functional (1).
For the construction of the minimizer we will choose the exponents $p_1, \ldots, p_n$ in such a way that they satisfy bound (3) as well as some further lower bound. Moreover we will make some assumption about the dependences of u from the variables $x_1, \ldots, x_n$, that will force us to slightly reduce the set where the minimizer itself is defined.

2. Statements and assumptions

Let $k \in \mathbb{N}$ such that $1 \leq k < n$ and consider the anisotropic functional (1) in the case of $p_1 = p_2 = \cdots = p_k = p \geq 2$ and $p_{k+1} = p_{k+2} = \cdots = p_n = q > p$. Moreover we assume that $c_1 = c_2 = \cdots = c_k = 1$ while $c_{k+1} = c_{k+2} = \cdots = c_n = A > 0$:

$$\int_{B_1(0)} \left[\sum_{i=1}^{k} |D_i u|^p + A \sum_{i=k+1}^{n} |D_i u|^q \right] dx, \tag{4}$$

We assume also that a minimizer of (4) assumes a prescribed value u_0 at the boundary of the unit ball $B_1(0)$.
Our purpose is to determine an explicit a minimizer of functional (4) for suitable values of A, p, q, k, n.

Let $W^{p,q}$ be the space defined by (2), when $p_i = p$ for every $i = 1, \ldots, k$, $p_i = q$ for every $i = k+1, \ldots, n$ and $\Omega = B_1(0)$.
As we can easily see, in this case

$$\bar{p} = \frac{npq}{qk + p(n-k)}$$

Furthermore, if $p \leq k$ we have that $\bar{p} < n$ while, if $p > k$, the same condition holds according to

$$q < \frac{n-k}{p-k} p. \tag{5}$$

Hence, if q satisfies condition (5), we can set

$$\bar{p}^* = \frac{npq}{k(q-p) + p(n-q)}.$$

Then, from now on, we will assume that $p > k$ and (5) holds. Under such assumptions, it easily follows that $q \leq \bar{p}^*$. For our convenience, we will also assume that $q > n - k$.

By the regularity results stated above, it follows that condition (5) for q, ensures that the minimizers of functional (4) are bounded.
It is also remarkable that a sufficient upper bound for q to get Lipschitz regularity is

$$q < \frac{n+2}{n}p \tag{6}$$

(see [19] and [22]). Although more recent results slightly improved bound (6) for Lipschitz regularity, and that in [4] Lipschitz regularity is obtained just assuming that (3) holds, for our convenience in the following we will always assume that

$$k < p \leq \max\left\{\frac{n+2}{n}p, \quad n-k\right\} < q < \frac{n-k}{p-k}p. \tag{7}$$

Remark 1. *The chain of inequalities (7) stands according to*

$$k < p < \frac{n-k}{p-k}p \quad \Longrightarrow \quad k < p < n$$

and that k satisfies inequalities $k < p \leq n-k$ if and only if $k < n/2$.
It is also opportune to point out that not every bound in (7) is necessary; for instance we could do without assuming that $p > k$, $p < n-k$ nor (5), but we will use these conditions to ease our further calculations.

3. Main result

We recall that a function $u \in W^{p,q}$ is a minimizer of functional (4) if and only if u satisfy the Euler equation:

$$\int_{B_1(0)} \left[p\sum_{i=1}^{k} |D_i u|^{p-2} D_i u D_i \varphi + Aq \sum_{i=k+1}^{n} |D_i u|^{q-2} D_i u D_i \varphi\right] dx = 0, \tag{8}$$

for every $\varphi \in W_0^{p,q}$, that is the subset of $W^{p,q}$, consisting of those functions who vanish at $\partial B_1(0)$.
If we integrate by parts in (8), it follows that a classical solution of such equation should be a function satisfying the second order partial differential equation

$$\sum_{i=1}^{k} |D_i u|^{p-2} D_i(D_i u) + A\frac{q(q-1)}{p(p-1)} \sum_{i=k+1}^{n} |D_i u|^{q-2} D_i(D_i u) = 0. \tag{9}$$

Our first purpose is to perform a suitable change of variables in equation (9), in order to turn it into an ordinary differential equation. To this aim we set

$$r = \left(\sum_{i=1}^{k} |x_i|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}, \quad R = \left(\sum_{i=k+1}^{n} |x_i|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \quad \text{and} \quad \rho = \frac{r^p}{R^q} \tag{10}$$

and we will look for a solution u of (9) such that $u = u(\rho)$. First of all we prove the following result

Lemma 3.1. *Let r, R and ρ be defined by (10). Then a function $u = u(\rho)$ satisfies equation (9) if and only if it is a solution to the following second order ordinary differential equation*

$$\begin{aligned} p(p-1)\rho u'' &\left[1 + B\rho^{q-p+1}|u'|^{q-p}\right] \\ &+u'\left[p^2 - 2p + k + B\frac{p(p-1)(q^2-n+k)}{q(q-1)}\rho^{q-p+1}|u'|^{q-p}\right] = 0 \end{aligned} \tag{11}$$

for a suitable constant $B = B(A, p, q) > 0$.

Proof: By assuming that $u(x_1, \dots, x_n) = u(\rho)$, for every $i = 1, \dots, n$ we have (for our convenience, in the following we will use the notation ρ_{x_i} for $D_i\rho$ ...)

$$D_i u = u'(\rho)\rho_{x_i} \quad \text{and} \quad D_i(D_i u) = u''(\rho)\rho_{x_i}^2 + u'(\rho)\rho_{x_i x_i} \tag{12}$$

and, by putting derivatives (12) into equation (9) we obtain that:

$$\sum_{i=1}^{k} |u_{x_i}|^{p-2}u_{x_i x_i} = |u'(\rho)|^{p-2}\left[u''(\rho)\sum_{i=1}^{k}|\rho_{x_i}|^p + u'(\rho)\sum_{i=1}^{k}|\rho_{x_i}|^{p-2}\rho_{x_i x_i}\right]$$

and

$$\sum_{i=k+1}^{n} |u_{x_i}|^{q-2}u_{x_i x_i} = |u'(\rho)|^{q-2}\left[u''(\rho)\sum_{i=k+1}^{n}|\rho_{x_i}|^q + u'(\rho)\sum_{i=k+1}^{n}|\rho_{x_i}|^{q-2}\rho_{x_i x_i}\right].$$

Moreover, as we can easily see, if $i = 1, \dots, k$:

$$\rho_{x_i} = p\frac{r_{x_i}}{r}\rho \quad \text{and} \quad \rho_{x_i x_i} = p(p-1)\frac{r_{x_i}^2}{r^2}\rho + p\frac{r_{x_i x_i}}{r}\rho \tag{13}$$

while, if $i = k+1, \dots, n$:

$$\rho_{x_i} = -q\frac{R_{x_i}}{R}\rho \quad \text{and} \quad \rho_{x_i x_i} = q(q+1)\frac{R_{x_i}^2}{R^2}\rho - q\frac{R_{x_i x_i}}{R}\rho. \tag{14}$$

By definition (10) it follows that

$$r_{x_i} = \frac{x_i|x_i|^{\frac{2-p}{p-1}}}{r^{\frac{1}{p-1}}}, \qquad r_{x_i x_i} = \frac{1}{p-1}\frac{|x_i|^{\frac{2-p}{p-1}}}{r^{\frac{2}{p-1}}}\left[1 - \left(\frac{|x_i|}{r}\right)^{\frac{p}{p-1}}\right] \tag{15}$$

and, analogously

$$R_{x_i} = \frac{x_i|x_i|^{\frac{2-q}{q-1}}}{R^{\frac{1}{q-1}}}, \qquad R_{x_i x_i} = \frac{1}{q-1}\frac{|x_i|^{\frac{2-q}{q-1}}}{R^{\frac{2}{q-1}}}\left[1 - \left(\frac{|x_i|}{R}\right)^{\frac{q}{q-1}}\right] \tag{16}$$

So, by (13), (14), (15) and (16) we have

$$\sum_{i=1}^{k}|\rho_{x_i}|^p = \left(\frac{p}{r}\rho\right)^p\sum_{i=1}^{k}|r_{x_i}|^p = \left(\frac{p}{r}\rho\right)^p\frac{1}{r^{\frac{p}{p-1}}}\sum_{i=1}^{k}|x_i|^{p+\frac{p(2-p)}{p-1}} = \left(\frac{p}{r}\rho\right)^p$$

and similarly

$$\sum_{i=k+1}^{n}|\rho_{x_i}|^q = \left(\frac{q}{R}\rho\right)^q.$$

Moreover

$$\sum_{i=1}^{k}|\rho_{x_i}|^{p-2}\rho_{x_i x_i} = \left(\frac{p}{r^{\frac{p}{p-1}}}\rho\right)^{p-1}\left(\frac{p^2-2p+k}{p-1}\right)$$

and

$$\sum_{i=k+1}^{n} |\rho_{x_i}|^{q-2}\rho_{x_i x_i} = \left(\frac{q}{R^{\frac{q}{q-1}}}\rho\right)^{q-1}\left(\frac{q^2-n+k}{q-1}\right).$$

Finally, equation (9) becomes

$$\begin{aligned}
&|u'|^{p-2}\left[u''\left(\frac{p}{r}\rho\right)^{p} + u'\left(\frac{p}{r^{\frac{p}{p-1}}}\rho\right)^{p-1}\left(\frac{p^2-2p+k}{p-1}\right)\right] \\
&+\frac{Aq(q-1)}{p(p-1)}|u'|^{q-2}\left[u''\left(\frac{q}{R}\rho\right)^{q} + u'\left(\frac{q}{R^{\frac{q}{q-1}}}\rho\right)^{q-1}\left(\frac{q^2-n+k}{q-1}\right)\right] = 0 \\
&\Rightarrow \quad (p\rho)^{p-1}|u'|^{p-2}\left[p(p-1)\rho u'' + (p^2-2p+k)u'\right] \\
&\quad +A\frac{(q\rho)^q}{p}|u'|^{q-2}\left[q(q-1)\rho u'' + (q^2-n+k)u'\right] = 0 \\
&\Rightarrow \quad p(p-1)\rho u''\left[1 + B\rho^{q-p+1}|u'|^{q-p}\right] \\
&\qquad +u'\left[p^2-2p+k+B\frac{(q^2-n+k)p(p-1)}{q(q-1)}\rho^{q-p+1}|u'|^{q-p}\right] = 0
\end{aligned}$$

where we set

$$B = \frac{A(q-1)q^{q+1}}{(p-1)p^{p+1}} > 0.$$

In order to simplify equation (11) we prove the following

Lemma 3.2. *Let $\psi = \psi(\rho)$ be the positive function defined by*

$$\psi(\rho) = B\rho^{q-p+1}|u'(\rho)|^{q-p}. \tag{17}$$

Then equation (11) turns to

$$\frac{1+\psi}{\psi(D+C\psi)}\psi' = \frac{1}{p(p-1)\rho}, \tag{18}$$

for some positive constants $C = C(n,k,p,q)$ and $D = D(k,p,q)$.

Proof: By definition (17) of ψ it follows that

$$\psi'(\rho) = (q-p+1)\frac{\psi(\rho)}{\rho} + \frac{(q-p)\psi(\rho)}{|u'|^2}u'u''.$$

Hence

$$\frac{u'u''}{|u'|^2} = \frac{1}{q-p}\left(\frac{\psi'(\rho)}{\psi(\rho)} - \frac{q-p+1}{\rho}\right).$$

Thus, if we multiply equation (11) by $u'/|u'|^2$, we get

$$\begin{aligned}
&\frac{p(p-1)}{q-p}\rho\left[\frac{\psi'}{\psi}-\frac{q-p+1}{\rho}\right](1+\psi)+\\
&\qquad+\left[p^2-2p+k+\frac{p(p-1)(q^2-n+k)}{q(q-1)}\psi\right]=0\\
\Rightarrow\quad&\frac{p(p-1)}{q-p}\rho(1+\psi)\psi'+\left[p^2-2p+k-\frac{p(p-1)(q-p+1)}{q-p}\right]\psi\\
&\qquad+\left[\frac{p(p-1)(q^2-n+k)}{q(q-1)}-\frac{p(p-1)(q-p+1)}{q-p}\right]\psi^2=0\\
\Rightarrow\quad&p(p-1)\rho(1+\psi)\psi'+\left[k(q-p)-p(q-1)\right]\psi\\
&\qquad-\frac{p(p-1)}{q(q-1)}[q(p-1)+(n-k)(q-p)]\psi^2=0.
\end{aligned}$$

Finally, by setting

$$C=\frac{p(p-1)}{q(q-1)}\left[(n-k)(q-p)+q(p-1)\right]>0$$

and

$$D=q(p-k)+p(k-1)>0,$$

the latter equation turns to

$$p(p-1)\rho(1+\psi)\psi'-(D+C\psi)\,\psi=0,$$

which easily leads to (18).

Remark 2. *It is not difficult to show that, under assumptions (7), we have*

$$C<p(p-1)<D.$$

We now integrate the first order ordinary differential equation (18) obtaining that

$$\psi(\rho)\left(D+C\psi(\rho)\right)^{\frac{D}{C}-1}=E\rho^{\frac{D}{p(p-1)}}\tag{19}$$

where E is a positive constant, coming out from the integration.

By investigating equation (19) we can easily see that, if we let $r\to 0$ while $R>0$ (i.e. if $\rho\to 0$), then

$$\psi(\rho)\to 0\qquad\text{as}\qquad\rho^{\frac{D}{p(p-1)}},\tag{20}$$

that is $\psi(\rho)$ is an infinitesimal of degree $D/p(p-1)$ as $\rho\to 0$.
On the other hand, if we let $R\to 0$ and $r>0$ (i.e. if $\rho\to+\infty$), then

$$\psi(\rho)\to+\infty\qquad\text{as}\qquad\rho^{\frac{C}{p(p-1)}},\tag{21}$$

hence $\psi(\rho)$ is an infinite of order $C/p(p-1)$ as $\rho\to+\infty$.

Let us denote by B_s^k the $k-$dimensional ball of radius s and fix $\varepsilon\in(0,1)$. We set

$$S_\varepsilon^k=B_\varepsilon^k\times\mathbb{R}^{n-k}=\{x\in\mathbb{R}^n:\ r<\varepsilon\}$$

and

$$S_\varepsilon^{n-k}=\mathbb{R}^k\times B_\varepsilon^{n-k}=\{x\in\mathbb{R}^n:\ R<\varepsilon\}.$$

Because of (20) and (21), from now on we will consider u as it is defined in the set

$$\Omega = B_1 \setminus (S_\varepsilon^k \cup S_\varepsilon^{n-k}).$$

By (20) and (21) it follows that, as far as $x \in \Omega$, ψ turns out to be bounded, i.e. we can set

$$M = \sup_{\Omega} \ \psi(\rho) < \infty. \tag{22}$$

Now, since $D_i u = u'(\rho) D_i \rho$, by (17) we obtain

$$|D_i u| = \frac{p}{B^{\frac{1}{q-p}}} \ \frac{R^{\frac{q}{q-p}}}{r^{\frac{1}{p-1}+\frac{q}{q-p}}} \ |x_i|^{\frac{1}{p-1}} \ \psi^{\frac{1}{q-p}} \qquad \text{for every} \quad i = 1, \dots, k \tag{23}$$

and

$$|D_i u| = \frac{q}{B^{\frac{1}{q-p}}} \ \frac{R^{\frac{p}{q-p}-\frac{1}{q-1}}}{r^{\frac{p}{q-p}}} \ |x_i|^{\frac{1}{q-1}} \ \psi^{\frac{1}{q-p}} \qquad \text{for every} \quad i = k+1, \dots, n. \tag{24}$$

At this point, we are in condition to prove the following result:

Theorem 3.1. *Let $\psi : [0, +\infty) \to [0, +\infty)$ be the solution of equation (18) and let $u : B_1(0) \to \mathbb{R}$ be the function related to ψ by (17). Then the following properties hold:*

i) *$u \in W^{p,q} \cap C^1(\Omega)$.*
ii) *u is a classical solution to equation (9) in Ω and then is a class $C^1(\Omega)$ minimizer of functional (1) .*

Remark 3. *More generally, we may assert that u is a bounded and C^1 minimizer of functional (1) in the set $\mathbb{R}^n \setminus (S_\varepsilon^k \cup S_\varepsilon^{n-k})$.*

Proof of i): (23) and (24) easily lead to

$$\sum_{i=1}^{k} |D_i u|^p = \left(\frac{p}{B^{\frac{1}{q-p}}} \right)^p \frac{R^{\frac{pq}{q-p}}}{r^{\frac{pq}{q-p}}} \ \psi^{\frac{p}{q-p}}, \tag{25}$$

$$\sum_{i=k+1}^{n} |D_i u|^q = \left(\frac{q}{B^{\frac{1}{q-p}}} \right)^q \frac{R^{\frac{pq}{q-p}}}{r^{\frac{pq}{q-p}}} \ \psi^{\frac{q}{q-p}}. \tag{26}$$

Then the assertion follows from (22) and the fact that r and R are far from zero.

Proof of ii): Since in the set $\Omega = B_1 \setminus (S_\varepsilon^k \cup S_\varepsilon^{n-k})$ ψ solves equation (18), which is obtained by transforming Euler equation (9) for functional (4), assertion *ii)* follows.

4. The example

In this section we collect the results of Lemma 3.1 and 3.2 and of Theorem 3.1, to define a minimizer of functional (1) in a particular case.

Let us assume, in the above discussion, to set $D = 2C$.
It is easy to see that this happens, for instance, if we set $n = 13$, $k = 5$, $p = 6$ and

$q = 16$. Indeed in this case, we have that $D = 40$, $C = 20$ and inequalities (7) turn out to the following:

$$5 < 6 \le \max\left\{\frac{90}{13},\ 8\right\} = 8 < 16 < 48.$$

We recall that, under such assumptions, the minimizers of functional

$$\int_{B_1(0)} \left[\sum_{i=1}^{5} |D_i u|^6 + A \sum_{i=6}^{13} |D_i u|^{16}\right] dx, \tag{27}$$

are bounded in $B_1(0)$.

Moreover, let us fix the boundary condition u_0 in such a way that $E = C$. Then by (19) we deduce that

$$\psi(\rho)\,(2 + \psi(\rho)) = \rho^{\frac{D}{p(p-1)}}$$

and then, by solving this equation with respect to ψ:

$$\psi(\rho) = \sqrt{\rho^{\frac{D}{p(p-1)}} + 1} - 1. \tag{28}$$

Furthermore in this case we have

$$\frac{D}{p(p-1)} = \frac{4}{3} \qquad \text{and} \qquad B = \frac{2^{61}}{3^6} A.$$

Thus, by fixing the value of the coefficient $A > 0$ in such a way that $B = 1$, by (28) and (17), it follows that the minimizer u of functional (27) satisfy:

$$|u'(\rho)| = \left[\frac{\sqrt{\rho^{4/3}+1} - 1}{\rho^{11}}\right]^{\frac{1}{10}} = \frac{1}{\rho^{29/30}\left[\sqrt{\rho^{4/3}+1} + 1\right]^{1/10}}.$$

Finally, since $\rho = r^6/R^{16}$, by (23) and (24) it follows that

$$|D_i u| = 6\ \frac{R^{8/15}}{r}\ \frac{|x_i|^{1/5}}{\left[\sqrt{r^8 + R^{64/3}} + R^{32/3}\right]^{1/10}} \qquad \text{for} \quad i = 1, \dots, 5, \tag{29}$$

$$|D_i u| = 16\ \frac{r^{1/5}}{R^{8/15}}\ \frac{|x_i|^{1/15}}{\left[\sqrt{r^8 + R^{64/3}} + R^{32/3}\right]^{1/10}} \qquad \text{for} \quad i = 6, \dots, 13. \tag{30}$$

Equations (29) and (30) and the thesis of theorem 3.1 confirm that u is a minimizer of functional (27) that is bounded everywhere and is of class C^1 in the set $\Omega = B_1 \setminus (S^5_\varepsilon \cup S^8_\varepsilon)$.

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