UNIQUENESS OF GENERAL DIFFERENCE DIFFERENTIAL POLYNOMIALS AND MEROMORPHIC(ENTIRE) FUNCTIONS

HARINA P. WAGHAMORE, MANJUNATH B. E.

Abstract. This study explores the uniqueness of entire and meromorphic functions with equal weights $l \geq 0$ by investigating the general difference-differential polynomial $\Psi(z, f)$. We have extended the findings attributed to [3] and derived a new result. Additionally, we examine the implications when a polynomial of degree $n$ shares a common value with the general difference-differential polynomial. We have also posed an open problem for future research work.

1. Background Information, Definitions and results

A meromorphic function is a non-constant function that exhibits poles as singularities throughout the complex plane. The Nevanlinna theory of meromorphic functions provides standard notations for the discussion, as referenced by [5], [9], and [10]. If $f(z)$ and $g(z)$ share $a(z)$ CM(IM), we refer to $a(z)$ as a small function concerning $f(z)$ if $T(r, a(z)) = S(r, f)$, where $S(r, f)$ is any small quantity satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to \infty$, possibly outside a set of finite linear measure.

We use $N_{k} \left( r, \frac{1}{T-a} \right)$ to represent the count of zeros of $f(z) = a$ with a multiplicity of up to $k$. We use $\overline{N}_{k} \left( r, \frac{1}{T-a} \right)$ to represent the corresponding count where the multiplicity is not considered. Similarly, $N_{k} \left( r, \frac{1}{f-a} \right)$ represents the count of zeros of $f(z) = a$ with a multiplicity greater than or equal to $k$, and $\overline{N}_{k} \left( r, \frac{1}{f-a} \right)$ represents the corresponding count where the multiplicity is not considered.

Let’s say we have a function $f$ and a non-negative integer (or infinity) $k$. We can define $E_{k}(a; f)$ as the set of all points $a$ where $f$ equals $a$. If $a$ appears as an $a$-point of
f with multiplicity m, we count it m times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). When \( E_k(a; f) = E_k(a; g) \), we say that \( f \) and \( g \) share the value \( a \) with weight \( k \).

If \( f \) and \( g \) share \((a, k)\), they also share \((a, p)\) for any \( 0 \leq p \leq k \). Furthermore, \( f \) and \( g \) share a value \( a \) either in terms of identity (IM) or counting multiplicities (CM) only if they share \((a, 0)\) or \((a, \infty)\) respectively.

We denote \( N_L \left( r, \frac{1}{r-r^{(k)}} \right) \) as the counting function of zeros of \( f - 1 \where \( p > q \), with \( N_L \left( r, \frac{1}{r-r^{(k)}} \right) \) representing the reduced counting function. Similarly, \( N_k^{(a)} \left( r, \frac{1}{r-r^{(k)}} \right) \) denotes the counting function of zeros of \( f - 1 \) where \( p = q = 1 \). Suppose \( z_0 \) is a zero of \( f - 1 \) with multiplicity \( p \) and a zero of \( g - 1 \) with multiplicity \( q \). We use \( N_L \left( r, \frac{1}{r-r^{(k)}} \right) \) to count zeros of \( f - 1 \) where \( p \geq q \), and \( N_k^{(a)} \left( r, \frac{1}{r-r^{(k)}} \right) \) follows similarly. Additionally, \( N_k^{(a)} \left( r, \frac{1}{r-r^{(k)}} \right) \) counts those \( 1 \) points of \( f \) where \( p = q \geq 2 \), with \( N_k^{(a)} \left( r, \frac{1}{r-r^{(k)}} \right) \) defined in a parallel manner.

**Definition 1.1.** [12] The difference polynomial and its shifts in \( f(z) \) is defined as

\[
\Psi_{0}(z, f) = \sum_{\lambda \in I} \alpha(z) f(z)^{\lambda,0} f(z+c_1)^{\lambda,1} ... f(z+c_k)^{\lambda,k},
\]

where degree is denoted as \( d(\Psi_0) = \max_{\lambda \in I} \{d(\lambda)\} \) and \( \lambda = \{\lambda_0,0,...,\lambda_k,k\} \), \( I \) is a finite set of the index and meromorphic co-efficients \( \alpha(z) \) are satisfying \( T(r, \alpha(z)) = S(r, f) \), \( \lambda \in I \). \( f(z)^{\lambda,0} f(z+c_1)^{\lambda,1} ... f(z+c_k)^{\lambda,k} \) is monomial in \( f(z) \) and \( f(z+c_1),...,f(z+c_k) \), where \( c_1, ..., c_k \) are distinct non-zero complex constants and \( d(\lambda) = i_{\lambda,0} + ... + i_{\lambda,k} \).

**Definition 1.2.** The definition of the general differential-difference polynomial of \( f(z) \) and its shifts, as provided in [1], is as follows.

\[
\Psi(z, f) = \sum_{\lambda \in I} \alpha(z) f(z)^{\lambda,0} f^{(1)}(z)^{\lambda,1} ... f^{(m)}(z)^{\lambda,m} \times f(z+c_1)^{\lambda,1} f^{(1)}(z+c_1)^{\lambda,1} ... f^{(m)}(z+c_1)^{\lambda,m} ... f(z+c_k)^{\lambda,k} f^{(1)}(z+c_k)^{\lambda,k} ... f^{(m)}(z+c_k)^{\lambda,k} f(z)^{\lambda,0} f^{(1)}(z)^{\lambda,1} ... f^{(m)}(z)^{\lambda,m} f(z+c_1)^{\lambda,1} ... f^{(m)}(z+c_1)^{\lambda,m} f(z+c_k)^{\lambda,k} ... f^{(m)}(z+c_k)^{\lambda,k} = \sum_{\lambda \in I} \alpha(z) \prod_{i=0}^{k} \prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}}
\]

where \( I \) is a finite set of multi-indices \( \lambda = \{\lambda_0,0,...,\lambda_0,m,\lambda_1,0,...,\lambda_1,m,...,\lambda_k,0,...,\lambda_k,m\} \), \( \lambda_0(z) = 0 \) and \( c_1, c_2, ..., c_k \) are distinct complex constants. The growth of \( \alpha(z), \lambda \in I \) is \( S(r, f) \).

\( d(\lambda) = \sum_{i=0}^{k} \sum_{j=0}^{m} \lambda_{i,j} \) denotes the degree of the monomial \( \prod_{i=0}^{k} \prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}} \) of \( \Psi(z, f) \).

Then \( d(\Psi) = \max\{d(\lambda)\}, d^{*}(\Psi) = \min\{d(\lambda)\} \) denote the degree and the lower degree of \( \Psi(z, f) \) respectively.

The differential-difference polynomial \( \Psi(z, f) \) is called a homogeneous if \( d(\Psi) = d^{*}(\Psi) \) otherwise, it is a non-homogeneous.

A study on uniqueness under different conditions was conducted for \( f(z) \) and \( f^{(k)}(z) \) sharing a small function [2, 4, 6, 10, see]. In 2008, Zhang and Lu [11] concluded.

**Theorem A.** [11] Suppose \( k(\geq 1) \) and \( n(\geq 1) \) are integers, and \( f \) is a non-constant meromorphic function. Moreover, consider a small meromorphic function \( a(z) \) concerning \( f \), where \( a(z) \) is distinct from \( 0 \) and \( \infty \). If \( f^{n} \) and \( f^{(k)} \) share the value \( a(z) \) IM and

\[
4\Theta(0, f) + (2k + 6)\Theta(\infty, f) + 2\delta_{k+2}(0, f) > 12 + 2k - n,
\]
or \( f^n \) and \( f^{(k)} \) share the value \( a(z) \) CM and
\[
2\Theta(0, f) + (k + 3)\Theta(\infty, f) + \delta_{k+2}(0, f) > 6 + k - n,
\]
then \( f \equiv f^{(k)} \).

In 2013, Bhoosnurmath and Kabbur extended the above result to a general differential polynomial and obtained the following results.

**Theorem B.** [1] Consider a non-constant meromorphic function \( f \) and a small meromorphic function \( a(z) \) such that \( a(z) \) is not identically equal to 0 or \( \infty \). Let \( \Psi[f] \) represent a non-constant differential polynomial in \( f \). If \( f \) and \( \Psi[f] \) share the value a CM and
\[
(2Q + 6)\Theta(\infty, f) + (2 + 3d(\Psi))\delta(0, f) > 2Q + 2d(\Psi) + 3Q(\Psi) + 7,
\]
then \( f \equiv \Psi[f] \).

**Theorem C.** [1] Given a non-constant meromorphic function \( f \) and a small meromorphic function \( a(z) \) such that \( a(z) \) is not identically equal to 0 or \( \infty \), along with \( \Psi[f] \) denoting a non-constant differential polynomial in \( f \), if \( f \) and \( \Psi[f] \) share the value a CM and
\[
3\Theta(\infty, f) + (d(\Psi) + 1)\delta(0, f) > 4,
\]
then \( f \equiv \Psi[f] \).

**Theorem D.** [1] Suppose \( f \) is a non-constant entire function and \( a(z) \) is a small meromorphic function such that \( a(z) \) is not identically equal to 0 or \( \infty \). Let \( \Psi[f] \) denote a non-constant differential polynomial in \( f \). If \( f \) and \( \Psi[f] \) share the value a CM and
\[
(3d(\Psi) + 2)\delta(0, f) > 2Q(\Psi) + 2,
\]
then \( f \equiv \Psi[f] \).

**Theorem E.** [1] Consider \( f(z) \) as a non-constant entire function and \( a(z) \) as a small meromorphic function such that \( a(z) \) is not identically equal to 0 or \( \infty \). Let \( \Psi[f] \) represent a non-constant differential polynomial in \( f \). If \( f \) and \( \Psi[f] \) share the value a CM and
\[
(d(\Psi) + 1)\delta(0, f) > 1,
\]
then \( f \equiv \Psi[f] \).

In 2020, [3] studied \( \Psi(z, f) \) instead of a differential polynomial in \( f \) and proved some results:

**Theorem F.** [3] Given a non-constant meromorphic function \( f(z) \) and a small meromorphic function \( a(z) \), where \( a(z) \) is not identically equal to 0 or \( \infty \), let \( \Psi(z, f) \) denote a non-constant differential-difference polynomial as defined in (2). If \( f(z) \) and \( \Psi(z, f) \) share the value a CM and
\[
\Theta(\infty, f)(2Q^* + 6) + \delta(0, f)(3d^*(\Psi) + 2) > 2Q^* + 2d(\Psi) + 8,
\]
then \( f(z) \equiv \Psi(z, f) \).

**Theorem G.** [3] Assume \( f(z) \) is a non-constant meromorphic function and \( a(z) \) is a small meromorphic function such that \( a(z) \neq 0, \infty \). Let \( \Psi(z, f) \) be a non-constant differential-difference polynomial as defined in (2). If \( f(z) \) and \( \Psi(z, f) \) share the value a CM and
\[
3\Theta(\infty, f) + \delta(0, f)(d^*(\Psi) + 1) > 4,
\]
then \( f(z) \equiv \Psi(z, f) \).
Theorem H. [3] Consider $f(z)$ as a non-constant entire function and $a(z)$ as a small meromorphic function such that $a(z)$ is not identically equal to 0 or $\infty$. Let $\Psi(z, f)$ denote a non-constant differential-difference polynomial as defined in (2). If $f(z)$ and $\Psi(z, f)$ share the value $a IM and$

$$\Theta(\infty, f)(Q^* + 3) + 2\Theta(0, f) + \delta(0, f)d(\Psi) \geq Q^* + 2d(\Psi) - 2d^*(\Psi) + 5,$$

then $f(z) \equiv \Psi(z, f)$. 

Theorem I. [3] Given $f(z)$ as a non-constant entire function and $a(z)$ as a small meromorphic function, where $a(z)$ is not identically equal to 0 or $\infty$, let $\Psi(z, f)$ represent a non-constant differential-difference polynomial as defined in Definition 1. If $f(z)$ and $\Psi(z, f)$ share the value $a CM$ and

$$(d^*(\Psi) + 1)\delta(0, f) > 1,$$

then $f(z) \equiv \Psi(z, f)$.

Question 1. What happens if the non-constant meromorphic function $f(z)$ and the differential-difference polynomial $\Psi(z, f)$ share a value $a$ with finite weight?

Question 2. When examining a meromorphic function $f$ within a polynomial $p(f)$ and a differential-difference polynomial $\Psi(z, f)$, what conclusions can be drawn regarding the uniqueness of $p(f)$ and $\Psi(z, f)$ when they share a value a CM(IM)?

In this paper, we try to answer these two questions. Indeed, the following theorems are the main results of the paper.

Theorem 1.1. Let $f(z)$ be a non-constant meromorphic function and $l$ be a non-negative integer. Suppose $a(\neq 0, \infty)$ is a meromorphic function satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$ such that $f(z)$ and $\Psi(z, f)$ share $(a, l)$. If $l \geq 2$ and

$$\Theta(\infty, f)(Q^* + 3) + 2\Theta(0, f) + \delta(0, f)d(\Psi) \geq Q^* + 2d(\Psi) - 2d^*(\Psi) + 5,$$

or $l = 1$

$$\Theta(\infty, f)\left(\frac{Q^* + 7}{2}\right) + \Theta(0, f)\frac{5}{2} + \delta(0, f)d(\Psi) \geq 2d(\Psi) + Q^* - d^*(\Psi) + 6,$$

or $l = 0$

$$\Theta(\infty, f)(2Q^* + 6) + 4\Theta(0, f) + \delta(0, f)2d(\Psi) \geq 4d(\Psi) + 2Q^* - 2d^*(\Psi) + 10,$$

then $f(z) \equiv \Psi(z, f)$.

Example 1.1. Let $\Psi(z, f) = -f(z)f^{(1)}$, where $f(z) = e^z$. Then $\Psi(z, f)$ and $f$ share $(0, \infty)$ all the conditions (7) - (9) of Theorem 1.1 are satisfied but $\Psi(z, f) \neq f(z)$.

This example shows that the condition $a \neq 0$ is necessary for Theorem 1.1.

Theorem 1.2. Suppose $f(z)$ is a non-constant meromorphic function and $a(z)$ is a small function where $a(z) \neq 0, \infty$. Let $p(z)$ be a non-zero polynomial of degree $n \geq 1$, and $\Psi(z, f)$ be a non-constant differential-difference polynomial. If $p(f)$ and $\Psi(z, f)$ share the value a IM and

$$\Theta(\infty, f)(2Q^* + 6) + \delta(0, f)(3d^*(\Psi) + 2n) > 2Q^* + 2d(\Psi) + 2n + 6,$$

then $p(f) \equiv \Psi(z, f)$. 


Theorem 1.3. Given a non-constant meromorphic function \( f(z) \) and a small function \( a(z) \) with \( a(z) \neq 0, \infty \), let \( p(z) \) denote a non-zero polynomial of degree \( n \geq 1 \). Additionally, consider \( \Psi(z, f) \) as a non-constant differential-difference polynomial. If \( p(f) \) and \( \Psi(z, f) \) share the value a CM and

\[
3\Theta(\infty, f) + (d^*(\Psi) + n)\delta(0, f) > 3 + n,
\]

then \( p(f) \equiv \Psi(z, f) \).

Theorem 1.4. Considering \( f(z) \) as a non-constant entire function and \( a(z) \) as a small function with \( a(z) \neq 0, \infty \), let \( p(z) \) represent a non-zero polynomial of degree \( n \geq 1 \). Furthermore, let \( \Psi(z, f) \) be a non-constant differential-difference polynomial. If \( p(f) \) and \( \Psi(z, f) \) share the value a CM and

\[
\delta(0, f)(d^*(\Psi) + n) > n,
\]

then \( p(f) \equiv \Psi(z, f) \).

Theorem 1.5. Given \( f(z) \), a non-constant entire function, and \( a(z) \), a small function with \( a(z) \neq 0, \infty \), along with \( p(z) \), a non-zero polynomial of degree \( n \geq 1 \), and \( \Psi(z, f) \), a non-constant differential-difference polynomial, suppose \( p(f) \) and \( \Psi(z, f) \) share the value a IM and

\[
(3d^*(\Psi) + 2n)\delta(0, f) > 2d(\Psi) + 2n,
\]

then \( p(f) \equiv \Psi(z, f) \).

Example 1.2. Let \( p \) be a polynomial of degree one and \( f = e^z \), \( \Psi(z, f) = f^{(2)}(z)^2 f(z + 2\pi i)^2 \). Here, by definition of (1.1) and by \( \Psi(z, f) \) we observe that \( d(\Psi) = \lambda_{0,1} + \lambda_{1,0} = \frac{1}{2} + \frac{1}{2} = 1 \), i.e., \( d(\Psi) = 1 \), \( d^*(\Psi) = \lambda_{0,1} + \lambda_{1,0} = \frac{1}{2} + \frac{1}{2} = 1 \), i.e., \( d^*(\Psi) = 1 \) and \( Q^* = 3\lambda_{0,1} + \lambda_{1,0} = 2 \), i.e., \( Q^* = 2 \). Also \( \overline{N}(r, f) = S(r, f) \) and \( \overline{N}(r, 0; f) = \overline{N}(r, 0; e^z) \sim T(r, f) \). Then \( \Theta(\infty, f) = 1 \) and \( \delta(0, f) = 0 \). The deficiency conditions in (10), (11), (12), and (13) are not satisfied, but \( p(f) \equiv \Psi(z, f) \).

Hence, this example demonstrates that the conditions we have obtained are sufficient but not necessary for ensuring \( p(f) \equiv \Psi(z, f) \), in Theorems 1.1, 1.2, 1.3 and 1.4.

Remark 1. Let’s examine the cases where \( i = 0 \) or \( i = 1 \). Assuming \( c_1 = 0 \), according to the definition of \( \Psi(z, f) \), we obtain

\[
\Psi(z, f) = \sum_{\lambda \in I} a_\lambda(z)f(z)^{\lambda_{0,0} + \lambda_{1,0} + \lambda_{0,1} + \lambda_{1,1} + \cdots + \lambda_{0,m} + \lambda_{1,m}}\psi_{0,0}^{n_{0,0}}\psi_{0,1}^{n_{0,1}}\cdots\psi_{r,m}^{n_{r,m}} = \Psi[f],
\]

where \( n_{0,0} = \lambda_{0,0} + \lambda_{1,0} \), \( n_{0,1} = \lambda_{0,1} + \lambda_{1,1} \), \( n_{r,m} = \lambda_{0,m} + \lambda_{1,m}, i = 0, 1 \). Then taking \( d(\Psi) = d(\Psi) \), and \( d^*(\Psi) = d(\Psi) \), we get

(1) In theorem 1.2, we get

\[
\Theta(\infty, f)(2Q + 6) + \delta(0, f)(3d(\Psi) + 2) > 2Q + 2d(\Psi) + 8,
\]

this signifies an advancement upon the outcome presented in Theorem B.

(2) In Theorem 1.3, we get

\[
3\Theta(\infty, f) + (d(\Psi) + 1)\delta(0, f) > 4,
\]

which aligns with Theorem C.
(3) In Theorem 1.4, we get
\[(d(\Psi) + 1)\delta(0, f) > 1,\]
which aligns with Theorem E.

(4) In Theorem 1.5, we get
\[(3d(\Psi) + 2)\delta(0, f) > 2d(\Psi) + 2,\]
which aligns with Theorem D.

2. Lemmas

Lemma 2.1. [8] Suppose \( f(z) \) is a non-constant meromorphic function.
\[
N\left(r, \frac{1}{f^{(m)}} \right) = N\left(r, \frac{1}{f} \right) + T(r, f^{(k)}) - T(r, f) + S(r, f),
\]  
(14)
\[
N\left(r, \frac{1}{f^{(m)}} \right) \leq kN(r, f) + N\left(r, \frac{1}{f} \right) + S(r, f).
\]  
(15)

Lemma 2.2. [9] Consider the expression \( \varphi = \left( \frac{F''}{F} - 2 \frac{F'}{F} \right) - \left( \frac{G''}{G} - 2 \frac{G'}{G} \right) \), where \( F \) and \( G \) are two non-constant meromorphic functions. If \( F \) and \( G \) share 1 IM and \( \varphi \neq 0 \), then
\[
N_{\varphi}^{(1)}\left(r, \frac{1}{F - 1} \right) \leq N(r, \varphi) + S(r, F) + S(r, G).
\]  
(16)

Lemma 2.3. [7] Suppose \( f(z) \) is a transcendental meromorphic function of zero order, and let \( q \) and \( \eta \) be two non-zero complex constants. Then
\[
T(r, f(qz + \eta)) = T(r, f(z)) + S(r, f),
\]
\[
N(r, \infty; f(qz + \eta)) \leq N(r, \infty; f(z)) + S(r, f),
\]
\[
N(r, 0; f(qz + \eta)) \leq N(r, 0; f(z)) + S(r, f),
\]
\[
N(r, \infty; f(qz + \eta)) \leq N(r, \infty; f(z)) + S(r, f),
\]
\[
N(r, 0; f(qz + \eta)) \leq N(r, 0; f(z)) + S(r, f).
\]

Lemma 2.4. [3] Suppose \( f(z) \) is a meromorphic function and \( \Psi(z, f) \) is a differential-difference polynomial in \( f \). Then
\[
m\left(r, \frac{\Psi(z, f)}{f^{d(\Psi)}} \right) \leq (d(\Psi) - d^*(\Psi))m(r, f) + S(r, f).
\]  
(17)

Lemma 2.5. [3] Consider \( f(z) \) as a meromorphic function and \( \Psi(z, f) \) as a differential-difference polynomial in \( f \). Then
\[
m\left(r, \frac{\Psi(z, f)}{f^{d(\Psi)}} \right) \leq (d(\Psi) - d^*(\Psi))m\left(r, \frac{1}{f} \right) + S(r, f).
\]  
(18)

Lemma 2.6. [3] Consider \( f(z) \) as a meromorphic function and \( \Psi(z, f) \) as a differential-difference polynomial in \( f \). Then
\[
N(r, \Psi(z, f)) \leq d(\Psi)N(r, f) + Q^*N(r, f) + S(r, f).
\]  
(19)

Lemma 2.7. [3] Consider \( f(z) \) as a meromorphic function and a differential-difference polynomial \( \Psi(z, f) \) in \( f \). Then
\[
N\left(r, \frac{\Psi(z, f)}{f^{d(\Psi)}} \right) \leq Q^*\left(N(r, f) + N\left(r, \frac{1}{f} \right)\right) + (d(\Psi) - d^*(\Psi))N\left(r, \frac{1}{f} \right) + S(r, f)
\]  
(20)
Lemma 2.8. [3] Consider $f(z)$ as a meromorphic function and a differential-difference polynomial $\Psi(z, f)$ in $f$. Then

$$N\left(r, \frac{\Psi(z, f)}{f^d(\Psi)}\right) \leq (d(\Psi) - d^*(\Psi))N(r, f) + Q^* \left(N(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right) + S(r, f).$$ \hspace{1cm} (21)

Lemma 2.9. [3] Consider $f(z)$ as a meromorphic function and a differential-difference polynomial $\Psi(z, f)$ in $f$. Then

$$T(r, \Psi(z, f)) \leq d(\Psi)T(r, f) + Q^* \overline{N}(r, f) + S(r, f),$$ \hspace{1cm} (22)

where $Q^* = \max_{0 \leq k, \lambda \leq I} \{\lambda_1 + 2\lambda, ... + m\lambda_{1,m}\}$.

Lemma 2.10. [3] Consider $f(z)$ as a meromorphic function and a differential-difference polynomial $\Psi(z, f)$ in $f$. If $\Psi(z, f) \neq 0$, then we have

$$N\left(r, \frac{1}{\Psi(z, f)}\right) \leq T(r, \Psi(z, f)) - T(r, f^d(\Psi)) + (d(\Psi) - d^*(\Psi))m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^d(\Psi)}\right) + S(r, f),$$ \hspace{1cm} (23)

$$N\left(r, \frac{1}{\Psi(z, f)}\right) \leq Q^* \overline{N}(r, f) + (d(\Psi) - d^*(\Psi))m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^d(\Psi)}\right) + S(r, f).$$ \hspace{1cm} (24)

where $Q^* = \max_{0 \leq k, \lambda \leq I} \{\lambda_1 + 2\lambda, ... + m\lambda_{1,m}\}$.

Lemma 2.11. [3] Consider $f(z)$ as a meromorphic function and a differential-difference polynomial $\Psi(z, f)$ in $f$ of degree $d$ and let $Q^* = \lambda_0, 1 + 2\lambda, ... + m\lambda_{0,m}$. Then

$$T(r, \Psi(z, f)) = O(T(r, f)), S(r, \Psi(z, f)) = S(r, f).$$

Lemma 2.12. [3] Consider $f$ and $g$ a non constant meromorphic functions

i) if $f$ and $g$ share $(0, 1)$, then

$$N_L\left(r, \frac{1}{g - 1}\right) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r),$$ \hspace{1cm} (25)

Here, as $r$ approaches infinity, $S(r) = o(T(r))$, where $T(r) = \max\{T(r, f), T(r, g)\}$.

ii) if $f$ and $g$ share $(1, 1)$, then

$$2N_L\left(r, \frac{1}{g - 1}\right) + 2N_L\left(r, \frac{1}{f - 1}\right) - N_{f \geq 2}\left(r, \frac{1}{g - 1}\right) - N_{g \geq 2}\left(r, \frac{1}{f - 1}\right) \leq N\left(r, \frac{1}{g - 1}\right) - \overline{N}\left(r, \frac{1}{g - 1}\right).$$ \hspace{1cm} (26)

3. Proof of Main Results

Proof of Theorem 1.1. Consider $F = \frac{f}{a}$ and $G = \frac{\Psi(z, f)}{a}$. Then $F - 1 = \frac{f - a}{a}$ and $G - 1 = \frac{\Psi(z, f) - a}{a}$.

Given that $f(z)$ and $\Psi(z, f)$ share $(a, l)$, we can conclude that $F$ and $G$ share $(1, l)$ except at the zeros and poles of $a$. Additionally, observe that

$$\overline{N}(r, F) = \overline{N}(r, f),$$

$$\overline{N}(r, G) = \overline{N}(r, \Psi(z, f)) = \overline{N}(r, f) + S(r, f).$$
Define,
\[ \phi = \left( F'' \frac{F'}{F-1} - \frac{2F'}{F-1} \right) - \left( G'' \frac{G'}{G-1} - \frac{2G'}{G-1} \right), \tag{27} \]

Claim \( \phi = 0 \),
suppose on the contrary that \( \phi \neq 0 \). Therefore from (27), we have
\[ m(r, f) = S(r, f). \]

By the Nevanlinna Second fundamental theorem of, we have
\[ T(r, G) + T(r, F) \leq N\left(r, \frac{1}{F}\right) + 2N(r, f) + N\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{F-1}\right) \]
\[ + \frac{N}{L}\left(r, \frac{1}{G-1}\right) - N_0\left(r, \frac{1}{F}\right) - N_0\left(r, \frac{1}{G}\right) + S(r, f). \tag{28} \]

\( N_0\left(r, \frac{1}{F}\right) \) represents the counting function of zeros of \( F' \) that are distinct from the zeros of \( F(F-1) \). Similarly \( N_0\left(r, \frac{1}{G}\right) \) is defined.

**Case 1.** From (28), when \( l \geq 1 \), we have
\[ N_{E}^l\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{1}{\phi}\right) + S(r, f) \leq N\left(\frac{1}{\phi}, r\right) + S(r, f) \]
\[ \leq N_2\left(r, \frac{1}{F}\right) + \frac{N}{L}(r, F) + N_2\left(r, \frac{1}{G}\right) + N_L\left(r, \frac{1}{F-1}\right) \]
\[ + \frac{N}{L}\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F}\right) + N_0\left(r, \frac{1}{G}\right) + S(r, f), \]

and so,
\[ \frac{N}{L}\left(r, \frac{1}{G-1}\right) + \frac{N}{L}\left(r, \frac{1}{F-1}\right) = N_{E}^1\left(r, \frac{1}{F-1}\right) + \frac{N}{L}\left(r, \frac{1}{F-1}\right) + \frac{N}{L}\left(r, \frac{1}{G-1}\right) + S(r, f) \]
\[ \leq N_2\left(r, \frac{1}{F}\right) + \frac{N}{L}(r, F) + 2N_L\left(r, \frac{1}{F-1}\right) + N_2\left(r, \frac{1}{G}\right) \]
\[ + 2N_L\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F}\right) + N_0\left(r, \frac{1}{G}\right) \]
\[ + \frac{N}{L}\left(r, \frac{1}{F-1}\right) + \frac{N}{L}\left(r, \frac{1}{G-1}\right) + S(r, f). \tag{29} \]

**Subcase 1.1.** When \( l = 1 \), we have,
\[ \frac{N}{L}\left(r, \frac{1}{F-1}\right) \leq \frac{1}{2} N\left(r, \frac{1}{F'} \mid F \neq 0 \right) \leq \frac{1}{2} N\left(r, \frac{1}{F}\right) + \frac{1}{2} N(r, F), \tag{30} \]

Here, \( N\left(r, \frac{1}{F'} \mid F \neq 0 \right) \) represents the zeros of \( F' \) excluding those of \( F \). Combining (26) and (30), we obtain
\[ 2N_L\left(r, \frac{1}{F-1}\right) + N_{E}^2\left(r, \frac{1}{F-1}\right) + 2N_L\left(r, \frac{1}{G-1}\right) + N\left(r, \frac{1}{G-1}\right) \]
\[ \leq N\left(r, \frac{1}{G-1}\right) + \frac{1}{2} \left( N\left(r, \frac{1}{F}\right) + N(r, F) \right) + S(r, f). \tag{31} \]
Thus, from (31) and (30), we have
\[
\mathcal{N}\left(r, \frac{1}{F-1}\right) + \mathcal{N}\left(r, \frac{1}{G-1}\right) \leq \mathcal{N}_{(2)}\left(r, \frac{1}{F}\right) + \mathcal{N}(r, f) + \mathcal{N}_{(2)}\left(r, \frac{1}{G}\right) + \frac{1}{2} \mathcal{N}(r, f) \\
+ T(r, G) + N_0\left(r, \frac{1}{F}\right) + \frac{1}{2} \mathcal{N}\left(r, \frac{1}{f}\right) + N_0\left(r, \frac{1}{G}\right) \\
+ S(r, f).
\]

From (28), (32), and using (24), we have
\[
T(r, F) \leq \frac{7}{2} \mathcal{N}(r, f) + \mathcal{N}\left(r, \frac{1}{G}\right) + \frac{5}{2} \mathcal{N}\left(r, \frac{1}{f}\right) + S(r, f),
\]
\[
T(r, f) \leq \left[ (1 - \Theta(\infty, f)) \left( Q^* + \frac{7}{2} \right) + (1 - \Theta(0, f)) \frac{5}{2} + (1 - \delta(0, f)) d(\Psi) + (d(\Psi) - d^*(\Psi)) \right] \\
T(r, f) + S(r, f),
\]
\[
\Theta(\infty, f) \left( Q^* + \frac{7}{2} \right) + \Theta(0, f) \frac{5}{2} + \delta(0, f) d(\Psi) \leq Q^* + 2d(\Psi) - d^*(\Psi) + 5,
\]
This contradicts the assertion in (8).

Subcase 1.2. For \( l \geq 2 \), under these circumstances, we have
\[
2\mathcal{N}_L\left(r, \frac{1}{F-1}\right) + 2\mathcal{N}_L\left(r, \frac{1}{G-1}\right) + \mathcal{N}_E^2\left(r, \frac{1}{F-1}\right) + \mathcal{N}\left(r, \frac{1}{G-1}\right) \\
\leq N\left(r, \frac{1}{G-1}\right) + S(r, f).
\]

Derived from (29), we acquire,
\[
\mathcal{N}\left(r, \frac{1}{F-1}\right) + \mathcal{N}\left(r, \frac{1}{G-1}\right) \leq \mathcal{N}(r, f) + \mathcal{N}_{(2)}\left(r, \frac{1}{G}\right) + \mathcal{N}\left(r, \frac{1}{G-1}\right) + \mathcal{N}_{(2)}\left(r, \frac{1}{F}\right) \\
+ N_0\left(r, \frac{1}{G}\right) + N_0\left(r, \frac{1}{G}\right) \\
\leq \mathcal{N}(r, f) + \mathcal{N}_{(2)}\left(r, \frac{1}{G}\right) + \mathcal{N}_{(2)}\left(r, \frac{1}{F}\right) + T(r, G) \\
+ N_0\left(r, \frac{1}{G}\right) + N_0\left(r, \frac{1}{F}\right) + S(r, f).
\]

Now from (28), (24) and (33), we obtain
\[
T(r, F) \leq 3\mathcal{N}(r, f) + 2\mathcal{N}\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + S(r, f) \\
\leq (Q^* + 3) \mathcal{N}(r, f) + 2\mathcal{N}\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) d(\Psi) + m\left(r, \frac{1}{f}\right) (d(\Psi) - d^*(\Psi)) \\
+ S(r, f),
\]
\[
T(r, f) \leq \left[ (1 - \Theta(\infty, f)) (Q^* + 3) + 2(1 - \Theta(0, f)) + (1 - \delta(0, f)) d(\Psi) + (d(\Psi) - d^*(\Psi)) \right] \\
T(r, f) + S(r, f),
\]
\[
\Theta(\infty, f) (Q^* + 3) + 2 \Theta(0, f) + \delta(0, f) d(\Psi) \leq 2d(\Psi) + Q^* - d^*(\Psi) + 4,
\]
This contradicts the assertion in (7).

Case 2. In the case where \( l = 0 \), we then have:
\[
\mathcal{N}_E^1\left(r, \frac{1}{F-1}\right) = \mathcal{N}_E^1\left(r, \frac{1}{G-1}\right) + S(r, f), \quad \mathcal{N}_E^2\left(r, \frac{1}{G-1}\right) = \mathcal{N}_E^2\left(r, \frac{1}{F-1}\right) + S(r, f).
\]
And also, from (3.2), we have
\[
N \left( r, \frac{1}{F - 1} \right) + N \left( r, \frac{1}{G - 1} \right) \leq N_E^1 \left( r, \frac{1}{F - 1} \right) + N_E^2 \left( r, \frac{1}{F - 1} \right) + N_L \left( r, \frac{1}{F - 1} \right) + N \left( r, \frac{1}{G - 1} \right) + S(r,f),
\]
\[
+ N \left( r, \frac{1}{G - 1} \right) + N \left( r, \frac{1}{F - 1} \right) \leq N(r,F) + N_E^1 \left( r, \frac{1}{F - 1} \right) + N_E^2 \left( r, \frac{1}{F - 1} \right) + N_L \left( r, \frac{1}{F - 1} \right) + N \left( r, \frac{1}{G - 1} \right) + S(r,f),
\]
From (25), (26), (28), and (8), we get
\[
T(r,G) + T(r,F) \leq 2N(r,f) + N \left( r, \frac{1}{G} \right) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - 1} \right) + N \left( r, \frac{1}{G - 1} \right)
\]
\[
- \left( N_0 \left( r, \frac{1}{F} \right) + N_0 \left( r, \frac{1}{G} \right) \right) + S(r,f),
\]
\[
T(r,F) \leq 4N \left( r, \frac{1}{f} \right) + 6N \left( r, \frac{1}{G} \right) + S(r,f),
\]
\[
T(r,f) \leq [(1 - \Theta(\infty,f)) (2Q^* + 6) + (1 - \Theta(0,f)) 4 + (1 - \delta(0,f)) 2d(\Psi) + 2(d(\Psi)^2 + d^2(\Psi))]
\]
\[
T(r,f) + S(r,f).
\]
We obtain,
\[
\Theta(\infty,f) (2Q^* + 6) + \Theta(0,f) 4 + \delta(0,f) 2d(\Psi) \leq 4d(\Psi) + 2Q^* - 2d^2(\Psi) + 9.
\]
This contradicts the assertion in (9).

This confirms the assertion, demonstrating that \( \varphi \equiv 0 \). Thus, according to (27), we deduce that
\[
\frac{G'}{G} = \frac{2G'}{G - 1} = \frac{F''}{F'} - \frac{2F'}{F - 1},
\]
so on integrating twice, we obtain
\[
\frac{1}{F - 1} = \frac{A}{G - 1} + B.
\]

\( A \neq 0 \) and \( B \) are constant.

In this context, three possible cases can emerge:

**Subcase 1.1.** When \( B \neq 0, -1 \), from (35), we get
\[
\frac{F - 1}{B + 1 - BF} = \frac{G - 1}{A}, \quad N \left( r, \frac{1}{F - \frac{g + 1}{B}} \right) = N \left( r, G \right).
\]

Under these conditions, the Nevanlinna Second fundamental theorem provides:
\[
T(r,f) = T(r,F) + S(r,f),
\]
\[
\leq N \left( r, \frac{1}{F} \right) + N \left( r, F \right) + N \left( r, \frac{1}{F - \frac{g + 1}{B}} \right) + S(r,f),
\]
\[
\leq [(1 - \Theta(0,f)) + 2 (1 - \Theta(\infty,f))] T(r,f) + S(r,f),
\]
\[
\Theta(0,f) + 2\Theta(\infty,f) \leq 2.
\]
This contradicts the assertion in (7), (8) and (9).
**Subcase 1.2.** Assuming $B = 0$, according to (35), we get:

$$G = AF - (A - 1). \tag{36}$$

Our assertion is that $A = 1$. Suppose $A \neq 1$. Then, based on (36), we obtain:

$$\mathcal{N}(r, G) = \mathcal{N} \left( r, \frac{1}{F - \frac{A - 1}{A}} \right).$$

Using the Nevanlinna second fundamental theorem and (24), we obtain

$$T(r, f) = T(r, F) + S(r, f)$$

$$\leq \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N}(r, F) + \mathcal{N} \left( r, \frac{1}{F - \frac{A - 1}{A}} \right) + S(r, f),$$

$$\leq [(1 - \Theta(\infty, f)) (Q^* + 1) - \Theta(0, f) + (1 - \delta(0, f)) d(\Psi) + d(\Psi) - d^*(\Psi) + 1]T(r, f) + S(r, f),$$

$$\Theta(\infty, f)(Q^* + 1) + \Theta(0, f) + \delta(0, f)d(\Psi) \leq 2d(\Psi) + Q^* - d^*(\Psi) + 1.$$  

Thus $A = 1$, and in this case, from (3.11)

$$F = G,$$

and so $f(z) = \Psi(z, f)$.

**Subcase 1.3.** Suppose $B = -1$ from (35),

$$\frac{1}{F - 1} = \frac{A}{G - 1} - 1, \tag{37}$$

$$\implies F = \frac{A}{A - G + 1}. \tag{38}$$

If $A \neq -1$

$$\mathcal{N} \left( r, \frac{1}{F - \frac{A - 1}{A}} \right) = \mathcal{N} \left( r, \frac{1}{G} \right).$$

Applying the same reasoning as in subcase 1.2 leads to a contradiction. Hence, $A = -1$.

From (38), we have:

$$GF \equiv 1,$$

i.e., $f(z). [\Psi(z, f)] \equiv a^2. \tag{39}$

Therefore, under these conditions, we have

$$\mathcal{N}(r, f) + \mathcal{N} \left( r, \frac{1}{f} \right) = S(r, f),$$

Based on (38) and (39), along with the first fundamental theorem,

$$(1 + d(\Psi)) T(r, f) = T \left( r, \frac{1}{f(d(\Psi) + 1)} \right),$$

$$\leq m \left( r, \frac{\Psi(z, f)}{f(d(\Psi))} \right) + \mathcal{N} \left( r, \frac{\Psi(z, f)}{f(d(\Psi))} \right) + S(r, f),$$

$$\leq T(r, f) (d(\Psi) - d^*(\Psi)) + S(r, f),$$

$$(1 + d^*(\Psi)) T(r, f) \leq S(r, f).$$

Which is a contradiction.
Proof of theorem 1.2.

\[ F = \frac{\Psi(z, f)}{a}, \quad G = \frac{p(f)}{a}. \] (40)

Given that \( p(f) \) and \( \Psi(z, f) \) share a IM, it implies that \( F \) and \( G \) also share 1 IM.

Now, utilizing Lemma 2.11 and from (1), we can deduce

\[ T(r, G) \leq T(r, f) + S(r, f), \] (41)

\[ \overline{N}(r, F) = \overline{N}(r, \Psi(z, f)) = \overline{N}(r, f) + S(r, f), \] (42)

\[ \overline{N}(r, G) = \overline{N}(r, f) + S(r, f), \] (43)

\[ \overline{N}_{E}^{1}(r, \frac{1}{F - 1}) = \overline{N}_{E}^{1}(r, \frac{1}{G - 1}) + S(r, f), \] (44)

\[ \overline{N}_{E}^{2}(r, \frac{1}{F - 1}) = \overline{N}_{E}^{2}(r, \frac{1}{G - 1}) + S(r, f), \] (45)

\[ \overline{N}_{E}(r, \frac{1}{F - 1}) \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, F) + S(r, f). \] (46)

Suppose that \( \varphi \neq 0 \). Then we have,

\[ N(r, \varphi) \leq N_{2}(r, \frac{1}{F}) + \overline{N}(r, G) + N_{2}(r, \frac{1}{G}) + N_{L}(r, \frac{1}{F - 1}) \]

\[ + N_{L}(r, \frac{1}{G - 1}) + N_{0}(r, \frac{1}{F}) + N_{0}(r, \frac{1}{G}). \] (47)

Here, \( N_{0}(r, \frac{1}{F}) \) represents the counting function for the zeros of \( F' \) excluding those shared with \( F \) and \( F - 1 \). Similarly, \( N_{0}(r, \frac{1}{G}) \) is defined similarly.

Applying the second fundamental theorem yields

\[ T(r, G) + T(r, F) \leq \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{F - 1}) \]

\[ + \overline{N}(r, \frac{1}{G - 1}) - N_{0}(r, \frac{1}{F}) - N_{0}(r, \frac{1}{G}) + S(r, f). \] (48)

Given that \( F \) and \( G \) share 1 IM, we deduce from (46)

\[ \overline{N}(r, \frac{1}{F - 1}) + \overline{N}(r, \frac{1}{G - 1}) = 2N_{E}^{1}(r, \frac{1}{F - 1}) + 2N_{L}(r, \frac{1}{F - 1}) \]

\[ + 2N_{E}^{2}(r, \frac{1}{F - 1}) + 2N_{E}^{2}(r, \frac{1}{F - 1}). \] (49)

From this, (16) and (47), we get

\[ \overline{N}(r, \frac{1}{F - 1}) + \overline{N}(r, \frac{1}{G - 1}) \leq N_{2}(r, \frac{1}{F}) + \overline{N}(r, G) + N_{2}(r, \frac{1}{G}) + N_{0}(r, \frac{1}{F}) \]

\[ + 3N_{L}(r, \frac{1}{F - 1}) + 3N_{L}(r, \frac{1}{G - 1}) + N_{E}^{1}(r, \frac{1}{F - 1}) \]

\[ + 2N_{E}^{2}(r, \frac{1}{F - 1}) + N_{0}(r, \frac{1}{G}) + S(r, f). \] (50)
We now note that
\[ N_L \left( r, \frac{1}{F-1} \right) + 2N_L^2 \left( r, \frac{1}{F-1} \right) + 2N_L \left( r, \frac{1}{G-1} \right) + N^{(1)}_L \left( r, \frac{1}{F-1} \right) \leq N \left( r, \frac{1}{G-1} \right) \leq T(r, G) + O(1). \]  

Combining (50) and (51) yields
\[ N \left( r, \frac{1}{G-1} \right) + N \left( r, \frac{1}{F-1} \right) \leq N(r) + N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + 2N_L \left( r, \frac{1}{G-1} \right) \]
\[ + N_L \left( r, \frac{1}{G-1} \right) + T(r, G) + N_0 \left( r, \frac{1}{F} \right) \]
\[ + N_0 \left( r, \frac{1}{G} \right) + S(r, f). \]  

Employing (52) within (48) and (42), results in
\[ T(r, F) \leq 3N(r, G) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + 2N_L \left( r, \frac{1}{G-1} \right) \]
\[ + S(r, f). \]  

Utilizing (53) and (23) yields
\[ T(r, \Psi(z, f)) \leq N \left( r, \frac{1}{\Psi(z, f)} \right) + 3N(r, G) + N \left( r, \frac{1}{\Psi} \right) + 2N_L \left( r, \frac{1}{G-1} \right) \]
\[ + S(r, f), \]
\[ T(r, f)d(\Psi) \leq N \left( r, \frac{1}{\Psi} \right) + 3N(r, f) + (d(\Psi) - m \left( r, \frac{1}{\Psi} \right) d^*(\Psi)) + N \left( r, \frac{1}{f(d(\psi))} \right) \]
\[ + 2N_L \left( r, \frac{1}{G-1} \right) \]  
\[ + S(r, f). \]  

From (15), (23), and (40), we get
\[ 2N_L \left( r, \frac{1}{F-1} \right) + N_L \left( r, \frac{1}{G-1} \right) \leq 2N \left( r, \frac{1}{F^*} \right) + N \left( r, \frac{1}{G^*} \right) \]
\[ \leq N(r, f)(2Q^* + 3) + (2d(\Psi) + n)N \left( r, \frac{1}{\Psi} \right) \]
\[ + 2m \left( r, \frac{1}{\Psi} \right) (d(\Psi) - d^*(\Psi)) + S(r, f). \]  

Again using (55) in (54), we get
\[ T(r, f)d(\Psi) \leq nN \left( r, \frac{1}{f} \right) + 3N(r, G) + (d(\Psi) - d^*(\Psi))m \left( r, \frac{1}{\Psi} \right) + N \left( r, \frac{1}{f(d(\Psi))} \right) \]
\[ + N(r, f)(2Q^* + 3) + N \left( r, \frac{1}{\Psi} \right) (2d(\Psi) + n) \]
\[ + 2(d(\Psi) - d^*(\Psi))m \left( r, \frac{1}{\Psi} \right) + S(r, f), \]
\[ T(r, f)(3d^*(\Psi) - 2d(\Psi)) \leq N(r, f)(2Q^* + 6) + N \left( r, \frac{1}{\Psi} \right) (3d^*(\Psi) + 2n) + S(r, f). \]

Therefore, we obtain
\[ \Theta(\infty, f)(2Q^* + 6) + \delta(0, f)(3d^*(\Psi) + 2n) \leq 2(Q^* + n + d(\Psi)) + 6, \]  
which contradicts (10).
Thus, \( \varphi = 0 \).

Integrating \( \varphi \) results in

\[
\frac{1}{G - 1} = \frac{A}{F - 1} + B.
\]  

(57)

Here, \( (A \neq 0) \) and \( B \) are constants. Consequently,

\[
G = \frac{(B + 1)F + (A - B - 1)}{BF + (A - B)}, \quad F = \frac{(B - A)G + (A - B - 1)}{BG - (B + 1)}.
\]  

(58)

We examine the following three cases.

**Case 1.** Suppose \( B \neq 0, -1 \). According to (58), we have

\[
N \left( r, \frac{1}{G - (\frac{B+1}{B})} \right) = N(r, F).
\]  

(59)

From this, along with the second fundamental theorem, we have

\[
T(r, G) \leq N \left( r, \frac{1}{G - (\frac{B+1}{B})} \right) + N(r, G) + N \left( r, \frac{1}{G} \right) + S(r, f),
\]

\[
nT(r, f) \leq (2Q^* + 6)N(r, f) + (3d^*(\Psi) + 2n)N \left( r, \frac{1}{f} \right) + S(r, f),
\]

Therefore, we have

\[
\Theta(\infty, f)(2Q^* + 6) + \delta(0, f)(3d^*(\Psi) + 2n) \leq 3d^*(\Psi) + 2Q^* + 7,
\]

which contradicts (10).

**Case 2.** If \( B = 0 \), then according to (58), we have

\[
G \equiv \frac{F + (A - 1)}{A}, \quad F = AG - (A - 1).
\]  

(60)

Our assertion is that \( A = 1 \). Assuming \( A \neq 1 \), then from (60), we obtain

\[
N \left( r, \frac{1}{F} \right) = N \left( r, \frac{1}{G - (\frac{A-1}{A})} \right).
\]  

(61)

With this and the Nevanlinna second fundamental theorem, we obtain

\[
T(r, G) \leq N \left( r, \frac{1}{G} \right) + N \left( r, \frac{1}{G - (\frac{A-1}{A})} \right) + N(r, G) + S(r, f),
\]

\[
\leq N(r, f) + N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{\Psi(z, f)} \right) + S(r, f),
\]

\[
[n - d(\Psi) + d^*(\Psi)]T(r, f) \leq (Q^* + 1)N(r, f) + (n + d(\Psi))N \left( r, \frac{1}{f} \right) + S(r, f).
\]

So, we have

\[
(Q^* + 1)\Theta(\infty, f) + (n + d(\Psi))\delta(0, f) \leq Q^* + 2d(\Psi) - d^*(\Psi) + 1,
\]

this contradicts (10).

Hence, \( A = 1 \). According to (60), we have \( G \equiv F \).

Thus, \( p(f(z)) \equiv \Psi(z, f) \).

**Case 3.** If \( B = -1 \), then according to (58), we have

\[
G = \frac{A}{-F + A + 1}, \quad F = \frac{(1 + A)G - A}{G}.
\]  

(62)

In case \( A \neq -1 \), we deduce from (62) that

\[
N \left( r, \frac{1}{G - (\frac{A-1}{A})} \right) = N \left( r, \frac{1}{F} \right).
\]  

(63)
Using the same reasoning as in case 2, we arrive at a contradiction.

Hence, \( A = -1 \).

From (62), we obtain

\[
GF = 1.
\]

That is,

\[
p(f) \Psi(z, f) = a^2.
\]

From (65), we have

\[
N \left( r, \frac{1}{f} \right) + N(r, f) = S(r, f).
\]

Employing (62), (65), Lemma 2.10, and the Nevanlinna first fundamental theorem, we derive

\[
T(r, f)(d(\Psi) + n) = T \left( r, \frac{1}{f(\Psi)+n} \right) = T \left( r, \frac{\Psi(z, f)}{f(\Psi)+a^2} \right) + S(r, f) \\
\leq T \left( r, \frac{1}{f} \right) (d(\Psi) - d^*(\Psi)) + S(r, f).
\]

We have,

\[
(d^*(\Psi) + n)T(r, f) \leq S(r, f),
\]

This leads to a contradiction.

Thus, the proof of Theorem 1.2 is complete.

**Proof of Theorem 1.3.** Consider the definitions of \( F \) and \( G \) as given in (40).

From the theorem’s hypothesis, it follows that \( F \) and \( G \) share 1 CM. Hence,

\[
N_L \left( r, \frac{1}{F-1} \right) = N_L \left( r, \frac{1}{G-1} \right) = 0.
\]

Continuing similarly to the Proof of Theorem 1.1, we arrive at (54), which is:

\[
T(r, f)(d(\Psi) + n) \leq nN \left( r, \frac{1}{F-1} \right) + 3N(r, G) + m \left( r, \frac{1}{F} \right) (d(\Psi) - d^*(\Psi)) + N \left( r, \frac{1}{f(\Psi)} \right) \\
+ 2NL \left( r, \frac{1}{F-1} \right) + NL \left( r, \frac{1}{G-1} \right) + S(r, f).
\]

Using (68) in (54), we get

\[
T(r, f)d(\Psi) \leq N \left( r, \frac{1}{f(\Psi)} \right) + 3N(r, G) + m \left( r, \frac{1}{F} \right) (d(\Psi) - d^*(\Psi)) + nN \left( r, \frac{1}{F} \right) \\
+ S(r, f) \\
\leq 3N(r, f) + (d(\Psi) - d^*(\Psi)) \left[ T(r, f) - N \left( r, \frac{1}{F} \right) \right] + d(\Psi)N \left( r, \frac{1}{F} \right) \\
+ nN \left( r, \frac{1}{F} \right) + S(r, f),
\]

\[
T(r, f)d^*(\Psi) \leq (d^*(\Psi) + n)N \left( r, \frac{1}{F} \right) + 3N(r, f) + S(r, f).
\]

Thus, we have

\[
3\Theta(\infty, f) + \delta(0, f)(d^*(\Psi) + n) \leq 3 + n,
\]

This contradicts (11).

Therefore, \( \varphi \equiv 0 \). Following a similar approach to the Proof of theorem 1.2, we establish Theorem 1.3.

Thus, the proof of Theorem 1.3 is concluded.
3.1. **Proof of Theorem 1.4.** Given the hypothesis that \( f(z) \) is a non-constant entire function, we can employ \( N(r, f) = S(r, f) \) in the Proof of Theorem 1.2 to derive the proof of Theorem 1.4.

3.2. **Proof of Theorem 1.5.** Given the hypothesis that \( f(z) \) is a non-constant entire function, we can utilize \( N(r, f) = S(r, f) \) in the Proof of Theorem 1.3 to derive the proof of Theorem 1.5.

**Open Question 1.1.** Considering the non-constant meromorphic function \( f^p_1 \), where \( f_1 = z - c \) for some \( c \in \mathbb{C} \), along with the differential-difference polynomial \( \Psi(z, f) \), what implications arise if they share a value \( a \) with finite weight?

**References**


**Harina P. Waghamore**

Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore, India - 560056

*Email address:* harinapw@gmail.com, harina@bub.ernet.in

**Manjunath B. E.**

Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore, India - 560056

*Email address:* manjunath.bebub@gmail.com, manjunathbe@bub.ernet.in