Electronic Journal of Mathematical Analysis and Applications
Vol. 12(2) July 2024, No.8.
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

GENERALIZED FUZZY BICOMPLEX NUMBERS AND SOME OF THEIR PROPERTIES

RITAM BISWAS, TANMAY BISWAS

Abstract. At present time, theory of complex numbers is a very renowned subject area in Mathematics as well as various other fields of science and technology. In nineteenth century, different number systems had been introduced for examining algebra regarding multiple imaginary units. The notion of bicomplex numbers, perhaps, would be the most important among those. Later on, as an extension of this number system, generalized bicomplex number had been introduced. On the other hand, the concept of fuzzy logic has been considered to be significant in Mathematics to solve problems having imprecise spectrum of data. In this paper, our intension is to fuzzify generalized bicomplex number. For this purpose, here we first define generalized fuzzy bicomplex numbers from two alternative aspects and then, based upon these we propose some basic mathematical tools such as generalized fuzzy bicomplex norm, basic arithmetic operations etc. which help to study some fundamental properties in this regard.

1. Introduction

The notion and the fundamental properties of bicomplex numbers were first introduced by Corrado Segre in the year 1892 [12]. However, from time to time, this theory has been enriched by the subsequent works of several mathematicians viz. Spampinato, Dragoni etc. [3, 13, 14]. The most inclusive study in this regard would probably be the book written by Price [9]. Afterwards, Karakus [6] introduced generalized bicomplex numbers and their elementary theory as further extension in this area. Recently, Belaïdi et al. [2] discussed some other basic properties in this regard which significantly extend some results of [7] and [11]. On the other hand, in 1988, Buckley [1] published a paper on fuzzy complex numbers where he revealed different algebraic properties of this class as a development of real fuzzy

2010 *Mathematics Subject Classification.* 08A72, 11E88, 13A18, 15A63, 15A66, 30G35.
Key words and phrases. Bicomplex number, generalized bicomplex number, generalized fuzzy bicomplex number, real fuzzy numbers.
Submitted March 12, 2024, Revised April 14, 2024.

numbers based upon the extension principle. To know more in detail about real fuzzy arithmetic, one may see [4, 5, 15]. Later on, Qiu et al. [10] came up with a modified approach to fuzzy complex analysis and discussed its consequences. The present paper is written to unveil generalized fuzzy bicomplex numbers with the objective of fuzzification of generalized bicomplex numbers and studying some of their algebraic properties which substantially reshape some works of [1] and [8].

2. Preliminaries

Karakus [6] defined a generalized bicomplex number in the following way:
A generalized bicomplex number s is expressed as

$$s = s_1 + s_2 i_1 + s_3 i_2 + s_4 j,$$

where $s_1, s_2, s_3, s_4 \in \mathbb{R}$, the set of all real numbers and i_1, i_2, j are units satisfying the following conditions:

$$\begin{aligned} i_1^2 &= -\alpha, i_2^2 = -\beta, j^2 = \alpha\beta, \\ i_2 i_1 &= i_1 i_2 = j, i_2 j = j i_2 = -\beta i_1, i_1 j = j i_1 = -\alpha i_2, \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$.

The set of all generalized bicomplex numbers are denoted by $\mathbb{T}_{\alpha\beta}$.

If $s = s_1 + s_2 i_1 + s_3 i_2 + s_4 j \in \mathbb{T}_{\alpha\beta}$ and $t = t_1 + t_2 i_1 + t_3 i_2 + t_4 j \in \mathbb{T}_{\alpha\beta}$, then the addition rule and multiplication rule are defined on $\mathbb{T}_{\alpha\beta}$ respectively as:

$$\begin{aligned} s + t &= (s_1 + t_1) + (s_2 + t_2) i_1 + (s_3 + t_3) i_2 + (s_4 + t_4) j, \\ s \cdot t &= (s_1 t_1 - \alpha s_2 t_2 - \beta s_3 t_3 + \alpha\beta s_4 t_4) + (s_1 t_2 + s_2 t_1 - \beta s_3 t_4 - \beta s_4 t_3) i_1 \\ &\quad + (s_1 t_3 - \alpha s_2 t_4 + s_3 t_1 - \alpha s_4 t_2) i_2 + (s_1 t_4 + s_2 t_3 + s_3 t_2 + s_4 t_1) j. \end{aligned}$$

Remark 1. *From the definition of generalized bicomplex numbers, the following special cases arise:*
1) If $\alpha = \beta = 1$, then $\mathbb{T}_{\alpha\beta}$ forms the algebra of bicomplex numbers.
2) If $\alpha = 1, \beta = 0$, then $\mathbb{T}_{\alpha\beta}$ is the algebra of complex numbers.
3) If $\alpha = \beta = 0$, then $\mathbb{T}_{\alpha\beta}$ is considered as the field of real numbers.

Definition 2.1. *For a generalized bicomplex number $s = s_1 + s_2 i_1 + s_3 i_2 + s_4 j$, the real number*

$$|s| = \sqrt{s_1^2 + \alpha s_2^2 + \beta s_3^2 + \alpha\beta s_4^2}$$

will be determined as the real modulus of s.

3. Definitions and Main results

A generalized fuzzy bicomplex number $\overline{s} = \overline{s}_1 + \overline{s}_2 i_1 + \overline{s}_3 i_2 + \overline{s}_4 j$ is determined by its membership function $\mu(s|\overline{s})$ which is a mapping from $\mathbb{T}_{\alpha\beta}$ into $[0, 1]$. A weak a-cut of $\overline{s}$ is defined as

$$\overline{s}(a) = \{s|\mu(s|\overline{s}) > a\},$$

where $0 < a \leq 1$. By the notation $\overline{s}(0)$ we represent the closure of the union of $\overline{s}(a)$ for $0 < a \leq 1$. A strong a-cut of $\overline{s}$ is defined as

$$\overline{s}(\overline{a}) = \{q|\mu(s|\overline{s}) > a\},$$

where $0 \leq a < 1$. We define $\overline{s}\left(\overline{1}\right) = \{s|\mu(s|\overline{s}) = 1\}$.

Now we are in the position to define generalized fuzzy bicomplex numbers along the following lines:

Definition 3.2. *$\overline{s}$ is a generalized fuzzy bicomplex number if and only if*
i) $\mu(s|\overline{s})$ is upper semi-continuous (u.s.c.)
ii) $\overline{s}(\overline{a})$ is bounded, open, compact and arcwise connected for $0 \le a \le 1$ and
iii) $\overline{s}(\overline{1})$ is non-empty.

The set of all generalized fuzzy bicomplex numbers is denoted by the symbol $\mathbb{T}^F_{\alpha\beta}$.

One may define the norm $\|\overline{s}\|$ of a generalized fuzzy bicomplex numbers $\overline{s}$ in terms of strong a-cut in the following way:

Definition 3.3. *The generalized fuzzy bicomplex norm $\|\overline{s}\|$ is defined as*

$$\mu(r|\,\|\overline{s}\|) = \sup\{\mu(s|\overline{s}) : |s| = r\} \tag{1}$$

where $\overline{s} \in \mathbb{T}^F_{\alpha\beta}$.

3.1. **The basic arithmetic operations on the set of generalized fuzzy bicomplex numbers.**

Let $f(s_1, s_2) = r$ be any mapping from $\mathbb{T}_{\alpha\beta} \times \mathbb{T}_{\alpha\beta}$ into $\mathbb{T}_{\alpha\beta}$. We can extend f to $\mathbb{T}^F_{\alpha\beta} \times \mathbb{T}^F_{\alpha\beta}$ into $\mathbb{T}^F_{\alpha\beta}$ using the extension principle. We write $f(\overline{s}_1, \overline{s}_2) = \overline{r}$, if

$$\mu(r|\overline{r}) = \sup[\min\{\mu(s_1|\overline{s}_1), \mu(s_2|\overline{s}_2)\}\,|f(s_1, s_2) = r]. \tag{2}$$

As for examples one can obtain $\overline{r} = \overline{s}_1 + \overline{s}_2$ or $\overline{r} = \overline{s}_1\overline{s}_2$, by using $f(s_1, s_2) = s_1 + s_2$ or $f(s_1, s_2) = s_1 s_2$, respectively.

For subtraction we first define $-\overline{s}$ as

$$\mu(s| - \overline{s}) = \mu(-s|\overline{s}), \tag{3}$$

and then set

$$\overline{s}_1 - \overline{s}_2 = \overline{s}_1 + (-\overline{s}_2). \tag{4}$$

In order to define division, it is essential to first define reciprocal $\overline{s}^{-1}$ of $\overline{s}$ as

$$\mu\left(s|\overline{s}^{-1}\right) = \mu\left(s^{-1}|\overline{s}\right) \tag{5}$$

in some open disk $D(0, \varepsilon)$ with center at $0\ (= 0 + 0i_1 + 0i_2 + 0j) \in \mathbb{T}_{\alpha\beta}$ such that $D(0, \varepsilon) \cap \overline{s}(\overline{0}) = \phi$, an empty set. Moreover, if $\overline{s}(\overline{0})$ is unbounded away from 0, then $\overline{s}^{-1}$ remains undefined. When zero belongs to the closure of $\overline{s}(\overline{0})$, then the closure of $\overline{s}^{-1}(0)$ will not be bounded and by the definition of generalized fuzzy bicomplex numbers, $\overline{s}^{-1} \notin \mathbb{T}^F_{\alpha\beta}$. Otherwise, the division of two generalized fuzzy bicomplex numbers can be defined as

$$\frac{\overline{s}_1}{\overline{s}_2} = \overline{s}_1\overline{s}_2^{-1}. \tag{6}$$

3.1.1. *The addition and multiplication of generalized fuzzy bicomplex numbers using strong a-cuts:*

We define two sets $S(a)$ and $P(a)$ based upon the idea of strong a-cut in the following way:

$$S(a) = \{s_1 + s_2|\,(s_1, s_2) \in \overline{s}_1(\overline{a}) \times \overline{s}_2(\overline{a})\}, \tag{7}$$

$$P(a) = \{s_1 s_2|\,(s_1, s_2) \in \overline{s}_1(\overline{a}) \times \overline{s}_2(\overline{a})\}. \tag{8}$$

Theorem 3.1. *Let $\overline{s}_1, \overline{s}_2 \in \mathbb{T}^F_{\alpha\beta}$. If $\overline{r} = \overline{s}_1 + \overline{s}_2$, then $\overline{r}(\overline{a}) = S(a)$. Also, if $\overline{r} = \overline{s}_1\overline{s}_2$, then $\overline{r}(\overline{a}) = P(a)$.*

Proof. **Case-I** First we assume that $0 \leq a < 1$.

If $r \in \overline{r}\left(\overline{a}\right)$, then by (7) we may find s_1, s_2 so that $s_1 + s_2 = r$ and $\min\left\{\mu\left(s_1|\overline{s}_1\right), \mu\left(s_2|\overline{s}_2\right)\right\} > a$, since $\mu\left(r|\overline{r}\right) > a$.

This implies that $\mu\left(s_i|\overline{s}_i\right) > a$, for $i = 1, 2$, which implies that $\left(s_1, s_2\right) \in \overline{s}_1\left(\overline{a}\right) \times \overline{s}_2\left(\overline{a}\right)$ and $r \in S\left(a\right)$.

Now if $r \in S\left(a\right)$, then by (7), $r = s_1 + s_2$ with $\mu\left(s_i|\overline{s}_i\right) > a$ for $i = 1, 2$.

Therefore $\min\left\{\mu\left(s_1|\overline{s}_1\right), \mu\left(s_2|\overline{s}_2\right)\right\} > a$ and it follows that $\mu\left(r|\overline{r}\right) > a$.

So $r \in \overline{r}\left(\overline{a}\right)$.

Hence $\overline{r}\left(\overline{a}\right) = S\left(a\right)$, for $0 \leq a < 1$.

Case-II Now let $a = 1$.

If $r \in S\left(1\right)$, then by (7), there are s_1 and s_2 so that $r = s_1 + s_2$ and $\min\left\{\mu\left(s_1|\overline{s}_1\right), \mu\left(s_2|\overline{s}_2\right)\right\} = 1$.

Hence $\mu\left(r|\overline{r}\right) = 1$ and $r \in \overline{r}\left(\overline{1}\right)$.

Now suppose that $r \in \overline{r}\left(\overline{1}\right)$.

Therefore for each $n = 2, 3, ...$ we can find a sequence s_{1n} in $\overline{s}_1\left(\overline{0}\right)$ and s_{2n} in $\overline{s}_2\left(\overline{0}\right)$ so that $s_{1n} + s_{2n} = r$ and $\min\left\{\mu\left(s_{1n}|\overline{s}_{1n}\right), \mu\left(s_{2n}|\overline{s}_{2n}\right)\right\} > 1 - \frac{1}{n}$.

Since $\overline{s}_1\left(\overline{a}\right)$ and $\overline{s}_2\left(\overline{a}\right)$ are compact for $0 \leq a \leq 1$, we may choose subsequences $\left\{s_{1n_k}\right\}$ and $\left\{s_{2n_k}\right\}$ both converging to s_1 and s_2 respectively with $s_1 + s_2 = r$ and $\min\left\{\mu\left(s_1|\overline{s}_1\right), \mu\left(s_2|\overline{s}_2\right)\right\} \geq 1$ because each $\mu\left(s_1|\overline{s}_1\right), \mu\left(s_2|\overline{s}_2\right)$ is upper semi-continuous.

This implies that $s_i \in \overline{s}_i\left(\overline{1}\right)$ for $i = 1, 2$ and thus, $r \in S\left(1\right)$ by means of (7).

Hence $\overline{r}\left(\overline{1}\right) = S\left(1\right)$.

The second part of the proposition may be derived in the same way as the first part has been proved. $\square$

Theorem 3.2. *If $\overline{s}_1$ and $\overline{s}_2$ are generalized fuzzy bicomplex numbers such that either $\overline{r} = \overline{s}_1 + \overline{s}_2$ or $\overline{r} = \overline{s}_1\overline{s}_2$, then $\overline{r}\left(\overline{a}\right)$ is open.*

Proof. First we consider $\overline{r} = \overline{s}_1 + \overline{s}_2$ and let $r \in \overline{r}\left(\overline{a}\right)$ for some $0 \leq a < 1$.

From Theorem3.1, there are $s_i \in \overline{S}_i\left(\overline{a}\right)$ for $i = 1, 2$ such that $s_1 + s_2 = r$.

Since $\overline{s}_2\left(\overline{a}\right)$ is open, an open disk $O\left(s_2, \varepsilon\right)$ at center at s_2 and radius $\varepsilon > 0$ can be chosen, which is contained in $\overline{s}_2\left(\overline{a}\right)$.

Then $s_1 + O\left(s_2, \varepsilon\right)$ is an open set, containing r, wholly inside $\overline{r}\left(\overline{a}\right)$ since $\overline{r}\left(\overline{a}\right) = S\left(\overline{a}\right)$.

Therefore $\overline{r}\left(\overline{a}\right)$ is open.

If $\overline{r} = \overline{s}_1\overline{s}_2$ then with minor modification in the argument of the first part, for example, $s_1 s_2 = r$ and $s_1.O\left(s_2, \varepsilon\right)$ replacing $s_1 + s_2 = r$, and $s_1 + O\left(s_2, \varepsilon\right)$ respectively, it can be shown that $\overline{r}\left(\overline{a}\right)$ is open. $\square$

Theorem 3.3. *Suppose that $\overline{r} = \overline{s}_1 + \overline{s}_2$ or $\overline{r} = \overline{s}_1\overline{s}_2$, where $\overline{s}_1$ and $\overline{s}_2$ are generalized fuzzy bicomplex numbers. Also suppose that for $r_n \in \overline{r}\left(\overline{0}\right)$, the sequence $\left\{r_n\right\}$ converges to r and $\mu\left(r_n|\overline{r}\right)$ converges to λ in $[0, 1]$. Then $\mu\left(r|\overline{r}\right) \geq \lambda$.*

Proof. First we suppose that $\overline{r} = \overline{s}_1 + \overline{s}_2$.

So, for every $\varepsilon > 0$ there is a sequence $\left\{s_{1n}\right\}$ in $\overline{s}_1\left(\overline{0}\right)$ and a sequence $\left\{s_{2n}\right\}$ in $\overline{s}_2\left(\overline{0}\right)$ such that $s_{1n} + s_{2n} = r_n$ and

$$\mu\left(r_n|\overline{r}\right) \geq \min\left\{\mu\left(s_{1n}|\overline{s}_1\left(\overline{0}\right)\right), \mu\left(s_{2n}|\overline{s}_2\left(\overline{0}\right)\right)\right\} > \mu\left(r_n|\overline{r}\right) - \varepsilon. \tag{9}$$

Now all the s_{1n}, all the s_{2n} and all the r_n belong to $\overline{s}_1\left(\overline{0}\right), \overline{s}_2\left(\overline{0}\right)$ and $\overline{r}\left(\overline{0}\right)$ which are compact sets.

So a subsequence may be chosen such that $s_{1n_k} \to s_1, s_{2n_k} \to s_2, r_{n_k} \to r_n$ and $s_1 + s_2 = r$ and $\lambda \geq \min\{\mu(s_1|\overline{s}_1), \mu(s_2|\overline{s}_2)\} > \lambda - \varepsilon$, because min function is continuous. As ε is arbitrarily chosen, we see that $\lambda = \min\{\mu(s_1|\overline{s}_1), \mu(s_2|\overline{s}_2)\}$ which implies $\lambda \leq \mu(r|\overline{r})$.

The next case, i.e., $\overline{r} = \overline{s}_1\overline{s}_2$ may similarly be derived in the line of the first case. □

Theorem 3.4. *If $\overline{s}_1$ and $\overline{s}_2$ are generalized fuzzy bicomplex numbers, then so are $\overline{s}_1 + \overline{s}_2, \overline{s}_1\overline{s}_2, \overline{s}_1 - \overline{s}_2$ and $\overline{s}_1/\overline{s}_2$.*

Proof. a) First we consider that $\overline{r} = \overline{s}_1 + \overline{s}_2$. We show that $\mu(r|\overline{r})$ is upper semi-continuous.

Let $\{r_n\}$ be a sequence chosen in $\overline{r}\left(\overline{0}\right)$. As $\mu(r_n|\overline{r}) \in [0,1]$ we can have a subsequence $\mu(r_{n_k}|\overline{r})$ converging to some $\lambda \in [0,1]$.

Now we know that

$$\liminf_{n\to\infty} \mu(r_n|\overline{r}) \leq \lambda \leq \limsup_{n\to\infty} \mu(r_n|\overline{r}). \tag{10}$$

So there is a subsequence $\mu\left(r_{n_j}|\overline{r}\right)$ converging to $\limsup\limits_{n\to\infty} \mu(r_n|\overline{r})$.

Theorem 3.3 implies that

$$\mu(r|\overline{r}) \geq \limsup_{n\to\infty} \mu(r_n|\overline{r}). \tag{11}$$

So from (11), we conclude that $\mu(r|\overline{r})$ is upper semi-continuous.

Next we prove that $\overline{r}(\overline{a})$ is compact and arcwise connected for $0 \leq a \leq 1$.

Since $\overline{s}_1(\overline{a})$ and $\overline{s}_2(\overline{a})$ are compact, arcwise connected, so is $\overline{s}_1(\overline{a}) \times \overline{s}_2(\overline{a})$, for $0 \leq a \leq 1$. As $S(a)$ is continuous image of $\overline{s}_1(\overline{a}) \times \overline{s}_2(\overline{a})$, it follows that $\overline{r}(\overline{a}) = S(a)$ is compact and arcwise connected, for $0 \leq a \leq 1$.

Hence $\overline{r}$ satisfies all the conditions of a generalized fuzzy bicomplex number.

b) The proof of $\overline{r} = \overline{s}_1\overline{s}_2$ is similar as that of $\overline{r} = \overline{s}_1 + \overline{s}_2$, so it is omitted.

c) If $\overline{s}$ is a generalized fuzzy bicomplex number, then so is $-\overline{s}$ and therefore $\overline{s}_1 - \overline{s}_2$ is also a generalized fuzzy bicomplex number.

d) Since the mapping $s \to s^{-1}$, $s \neq 0$, is continuous and $\overline{s}^{-1}(\overline{a}) = (\overline{s}(\overline{a}))^{-1}$ we see that $\overline{s}^{-1}$ is a generalized fuzzy bicomplex number. Therefore $\overline{s}_1/\overline{s}_2$ is a generalized fuzzy bicomplex number. □

Theorem 3.5. $\|\overline{s}\|(\overline{a}) = \|\overline{s}(\overline{a})\|$, *for $0 \leq a \leq 1$, where $\overline{s} \in \mathbb{T}^F_{\alpha\beta}$.*

Proof. Here two cases arise.

Case-1. First we assume that $0 \leq a < 1$.

If $r \in \|\overline{s}\|(\overline{a})$, then by (1), there exists a generalized bicomplex number s such that $|s| = r$ and $\mu(s|\overline{s}) > a$. Therefore $r \in \|\overline{s}(\overline{a})\|$.

Thus,

$$r \in \|\overline{s}\|(\overline{a}) \Rightarrow r \in \|\overline{s}(\overline{a})\|. \tag{12}$$

Next suppose that $r \in \|\overline{s}(\overline{a})\|$. Then using the definition of strong a-cut and (1) there exists an $s \in \mathbb{T}_{\alpha\beta}$ so that $|s| = r$ and $\mu(s|\overline{s}) > a$.

This implies that $\sup\{\mu(s|\overline{s}) \,|\, |s| = r\} > a$ and as such $r \in \|\overline{s}\|(\overline{a})$.

So,

$$r \in \|\overline{s}(\overline{a})\| \Rightarrow r \in \|\overline{s}\|(\overline{a}). \tag{13}$$

Therefore, from (12) and (13), it follows that

$$\|\overline{s}\|(\overline{a}) = \|\overline{s}(\overline{a})\|, \quad \text{for} \quad 0 \leq a < 1. \tag{14}$$

Case-2. Next suppose that $a = 1$.

If $r \in \left\|\overline{s}\left(\overline{1}\right)\right\|$ then by (1), there is an $s \in \mathbb{T}_{\alpha\beta}$ such that $r = |s|$ and $\mu(s|\overline{s}) = 1$. Then the supremum of $\mu(s|\overline{s})$ over all s such that $|s| = r$ is also one and $r \in \|\overline{s}\|\left(\overline{1}\right)$.

Now let $r \in \|\overline{s}\|\left(\overline{1}\right)$. For each $n = 2, 3, ...$ there is an s_n in $\overline{s}\left(\overline{0}\right)$ so that $|s_n| = r$ and $\mu(s_n|s) > 1 - \frac{1}{n}$. Then s_n belongs to the compact set $\overline{s}(\overline{a})$ for $0 \leq a < 1$, so there is a subsequence $s_{n_k} \to s$ with $|s| = r$ and $\mu(s|\overline{s}) \geq 1$. Therefore $r \in \left\|\overline{s}\left(\overline{1}\right)\right\|$.

Hence

$$\|\overline{s}\|\left(\overline{1}\right) = \left\|\overline{s}\left(\overline{1}\right)\right\|. \tag{15}$$

Thus the theorem follows from (14) and (15). □

Theorem 3.6. *Let $\overline{s}$ be any generalized fuzzy bicomplex number. Then*

$$\|-\overline{s}\| = \|\overline{s}\|.$$

Proof. From Theorem 3.5, it follows that

$$\|\overline{s}\|(\overline{a}) = \|\overline{s}(\overline{a})\| = \{|s| : s \in \overline{s}(\overline{a})\} \tag{16}$$

Again, in view of Theorem 3.5, we have

$$\|-\overline{s}\|(\overline{a}) = \|-\overline{s}(\overline{a})\| = \{|-s| : s \in \overline{s}(\overline{a})\} \tag{17}$$

Hence the result follows from (16) and (17) as $|-s| = |s|$. □

Theorem 3.7. *Let s_1 and s_2 be two generalized fuzzy bicomplex numbers. Then*

$$\|\overline{s}_1 + \overline{s}_2\| \leq \|\overline{s}_1\| + \|\overline{s}_2\|.$$

Proof.

$$\begin{aligned} \|\overline{s}_1 + \overline{s}_2\|(\overline{a}) &= \|(\overline{s}_1 + \overline{s}_2)(\overline{a})\| \\ &= \|\overline{s}_1(\overline{a}) + \overline{s}_2(\overline{a})\| \\ & \quad \{|s_1 + s_2| : s_i \in \overline{s}_i(\overline{a}), i = 1, 2\}. \end{aligned} \tag{18}$$

and

$$\begin{aligned} (\|\overline{s}_1\| + \|\overline{s}_2\|)(\overline{a}) &= \|\overline{s}_1\|(\overline{a}) + \|\overline{s}_2\|(\overline{a}) \\ &= \|\overline{s}_1(\overline{a})\| + \|\overline{s}_2(\overline{a})\| \\ &= \{|s_1| + |s_2| : s_i \in \overline{s}_i(\overline{a}), i = 1, 2\} \end{aligned} \tag{19}$$

Since, $|s_1 + s_2| \leq |s_1| + |s_2|$, we get the result from (18) and (19). □

Theorem 3.8. *If s_1 and s_2 be two generalized fuzzy bicomplex numbers, then*

$$\|\overline{s}_1\overline{s}_2\| \leq 2^{\frac{1}{2}}\|\overline{s}_1\|\|\overline{s}_2\|.$$

Proof. We wish to show that the set $\|\overline{s}_1\overline{s}_2\|(\overline{a})$ is less than or equal to $\left(2^{\frac{1}{2}}\|\overline{s}_1\|\|\overline{s}_2\|\right)(\overline{a})$ for $0 \leq a \leq 1$. In view of Theorem 3.5 and Theorem 3.1, we obtain that

$$\begin{aligned} \|\overline{s}_1\overline{s}_2\|(\overline{a}) &= \|(\overline{s}_1\overline{s}_2)(\overline{a})\| \\ &= \|\overline{s}_1(\overline{a})\,\overline{s}_2(\overline{a})\| \\ &= \{|s_1 s_2| : (s_1, s_2) \in \overline{s}_1(\overline{a}) \times \overline{s}_2(\overline{a})\}. \end{aligned} \tag{20}$$

Also, from Theorem 3.5 and in view Theorem 3.1, we get that

$$\begin{aligned} \left(\|\overline{s}_1\| \, \|\overline{s}_2\|\right)(\overline{a}) &= \|\overline{s}_1\|(\overline{a}) \, \|\overline{s}_2\|(\overline{a}) \\ &= \|\overline{s}_1(\overline{a})\| \, \|\overline{s}_2(\overline{a})\| \\ &= \left\{|s_1| \, |s_2| : (s_1, s_2) \in \overline{s}_1(\overline{a}) \times \overline{s}_2(\overline{a})\right\}. \end{aligned} \tag{21}$$

Hence, the result follows from (20) and (21), as $|s_1 s_2| \leq \sqrt{2} \, |s_1| \, |s_2|$. □

3.2. **Alternative definition of generalized fuzzy bicomplex numbers.**

A generalized fuzzy bicomplex number $\overline{s} = \overline{s}_1 + \overline{s}_2 i_1 + \overline{s}_3 i_2 + \overline{s}_4 j$ is defined by its membership function $\mu(s|\overline{s})$ which is a mapping from $\mathbb{T}_{\alpha\beta} \to [0, 1]$ such that

$$\mu(s|\overline{s}) = \min\left\{\mu(s_1|\overline{s}_1), \mu(s_2|\overline{s}_2), \mu(s_3|\overline{s}_3), \mu(s_4|\overline{s}_4)\right\}, \tag{22}$$

where $\mu(s_1|\overline{s}_1), \mu(s_2|\overline{s}_2), \mu(s_3|\overline{s}_3), \mu(s_4|\overline{s}_4)$ all are fuzzy real numbers usually represented by $\overline{s}_1, \overline{s}_2, \overline{s}_3$ and $\overline{s}_4$ respectively, i.e., $\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4 \in \mathbb{R}_F$.

Based upon the above definition of generalized fuzzy bicomplex numbers the following theorems can be established:

Theorem 3.9. *For every $\overline{s} \in \mathbb{T}^F_{\alpha\beta}$, there exists $s \in \mathbb{T}_{\alpha\beta}$ such that $\mu(s|\overline{s}) = 1$, i.e., every generalized fuzzy bicomplex number is normal.*

Proof. Let $\overline{s} \in \mathbb{T}^F_{\alpha\beta}$. Then $\overline{s} = (\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4)$ where $\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4 \in \mathbb{R}_F$. Since every fuzzy real number is normal, there exists $s_1, s_2, s_3, s_4 \in \mathbb{R}$ such that

$$\mu(s_1|\overline{s}_1) = \mu(s_2|\overline{s}_2) = \mu(s_3|\overline{s}_3) = \mu(s_4|\overline{s}_4) = 1.$$

Let $s = s_1 + s_2 i_1 + s_3 i_2 + s_4 j \in \mathbb{T}_{\alpha\beta}$. So we conclude that $\mu(s|\overline{s}) = 1$. □

Theorem 3.10. *For every $s = s_1 + s_2 i_1 + s_3 i_2 + s_4 j \in \mathbb{T}_{\alpha\beta}$, $\overline{s}(\overline{a}) = \overline{s}_1(\overline{a}) \times \overline{s}_2(\overline{a}) \times \overline{s}_3(\overline{a}) \times \overline{s}_4(\overline{a})$, for $0 \leq a \leq 1$.*

Proof. **Case-I.** Assume $0 \leq a < 1$.

If $s \in \overline{s}(\overline{a})$, then $\min\left\{\mu(s_1|\overline{s}_1), \mu(s_2|\overline{s}_2), \mu(s_3|\overline{s}_3), \mu(s_4|\overline{s}_4)\right\} > a$ implying that each membership function exceeds a and therefore $(s_1, s_2, s_3, s_4) \in \overline{s}_1(\overline{a}) \times \overline{s}_2(\overline{a}) \times \overline{s}_3(\overline{a}) \times \overline{s}_4(\overline{a})$, where $\overline{s} = \overline{s}_1 + \overline{s}_2 i_1 + \overline{s}_3 i_2 + \overline{s}_4 j \in \mathbb{T}^F_{\alpha\beta}$.

Now suppose that $(s_1, s_2, s_3, s_4) \in \overline{s}_1(\overline{a}) \times \overline{s}_2(\overline{a}) \times \overline{s}_3(\overline{a}) \times \overline{s}_4(\overline{a})$. Then each membership function exceeds a. Therefore the minimum of the membership functions of $\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4$ exceeds a, so that $\mu(s|\overline{s}) > a$ and hence $s \in \overline{s}(\overline{a})$, where $s = s_1 + s_2 i_1 + s_3 i_2 + s_4 j \in \mathbb{T}_{\alpha\beta}$.

Case-II. Now let $a = 1$. If $(s_1, s_2, s_3, s_4) \in \overline{s}_1(\overline{1}) \times \overline{s}_2(\overline{1}) \times \overline{s}_3(\overline{1}) \times \overline{s}_4(\overline{1})$, then we easily verify that $\mu(s|\overline{s}) = 1$ for some $s = s_1 + s_2 i_1 + s_3 i_2 + s_4 j \in \mathbb{T}_{\alpha\beta}$ and therefore $s \in \overline{s}(\overline{1})$.

Next $s \in \overline{s}(\overline{1})$.

Then there are $s_1, s_2, s_3, s_4 \in \mathbb{R}$ so that $s = s_1 + s_2 i_1 + s_3 i_2 + s_4 j$ and $\mu(s_1|\overline{s}_1) = \mu(s_2|\overline{s}_2) = \mu(s_3|\overline{s}_3) = \mu(s_4|\overline{s}_4) = 1$.

Therefore $(s_1, s_2, s_3, s_4) \in \overline{s}_1(\overline{1}) \times \overline{s}_2(\overline{1}) \times \overline{s}_3(\overline{1}) \times \overline{s}_4(\overline{1})$.

Hence the result follows from Case-I and Case-II. □

Theorem 3.11. *For every $\overline{s} \in \mathbb{T}^F_{\alpha\beta}$, $\overline{s}$ is a convex fuzzy set, i.e.,*

$$\mu(\lambda s + (1 - \lambda) t|\overline{s}) \geq \min\left\{\mu(s|\overline{s}), \mu(t|\overline{s})\right\},$$

where $s, t \in \mathbb{T}_{\alpha\beta}$ and $\lambda \in [0, 1]$.

Proof. Let $s = (s_1', s_2', s_3', s_4')$, $t = (t_1', t_2', t_3', t_4')$ and $\overline{s} = (\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4)$. We know that $\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4 \in \mathbb{R}_F$ are fuzzy real numbers. Thus

$$\begin{aligned}
\mu\left(\lambda s_1' + (1-\lambda) t_1' | \overline{s}_1\right) &\geq \min\left\{\mu\left(s_1' | \overline{s}_1\right), \mu\left(t_1' | \overline{s}_1\right)\right\}, \\
\mu\left(\lambda s_2' + (1-\lambda) t_2' | \overline{s}_2\right) &\geq \min\left\{\mu\left(s_2' | \overline{s}_2\right), \mu\left(t_2' | \overline{s}_2\right)\right\}, \\
\mu\left(\lambda s_3' + (1-\lambda) t_3' | \overline{s}_3\right) &\geq \min\left\{\mu\left(s_3' | \overline{s}_3\right), \mu\left(t_3' | \overline{s}_3\right)\right\}, \\
\mu\left(\lambda s_4' + (1-\lambda) t_4' | \overline{s}_4\right) &\geq \min\left\{\mu\left(s_4' | \overline{s}_4\right), \mu\left(t_4' | \overline{s}_4\right)\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mu\left(\lambda s + (1-\lambda) t | \overline{s}\right) &= \mu\left(\lambda\left(s_1', s_2', s_3', s_4'\right) + (1-\lambda)\left(t_1', t_2', t_3', t_4'\right) | \overline{s}\right) \\
&= \mu\left(\left(\lambda s_1' + (1-\lambda) t_1'\right), \left(\lambda s_2' + (1-\lambda) t_2'\right),\right. \\
&\qquad \left.\left(\lambda s_3' + (1-\lambda) t_3'\right), \left(\lambda s_4' + (1-\lambda) t_4'\right) | \overline{s}\right) \\
&= \min\{\mu(\lambda s_1' + (1-\lambda) t_1' | \overline{s}_1), \mu(\lambda s_2' + (1-\lambda) t_2' | \overline{s}_2), \\
&\qquad \mu\left(\lambda s_3' + (1-\lambda) t_3' | \overline{s}_3\right), \mu\left(\lambda s_4' + (1-\lambda) t_4' | \overline{s}_4\right)\} \\
&\geq \min\{\mu\left(s_1' | \overline{s}_1\right), \mu\left(t_1' | \overline{s}_1\right), \mu\left(s_2' | \overline{s}_2\right), \\
&\qquad \mu\left(t_2' | \overline{s}_2\right), ..., \mu\left(s_4' | \overline{s}_4\right), \mu\left(t_4' | \overline{s}_4\right)\} \\
&\geq \min\left\{\mu\left(s | \overline{s}\right), \mu\left(t | \overline{s}\right)\right\}.
\end{aligned}$$

□

Theorem 3.12. *If* $\overline{s} \in \mathbb{T}_{\alpha\beta}^F$ *and* $\gamma_1, \gamma_2 \in (0, 1]$ *with* $\gamma_1 \leq \gamma_2$, *then* $\overline{s}\left(\overline{\gamma}_1\right) \supseteq \overline{s}\left(\overline{\gamma}_2\right)$.

Proof. Let $s \in \overline{s}\left(\overline{\gamma}_2\right)$ where $s = s_1 + s_2 i_1 + s_3 i_2 + s_4 j \in \mathbb{T}_{\alpha\beta}$ and $\overline{s} = (\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4) \in \mathbb{T}_{\alpha\beta}^F$. So according to Theorem 3.10, $s_1 \in \overline{s}_1\left(\overline{\gamma}_2\right), s_2 \in \overline{s}_2\left(\overline{\gamma}_2\right), s_3 \in \overline{s}_3\left(\overline{\gamma}_2\right), s_4 \in \overline{s}_4\left(\overline{\gamma}_2\right)$. Since $\gamma_1 \leq \gamma_2$ then $\overline{s}_1\left(\overline{\gamma}_1\right) \supseteq \overline{s}_1\left(\overline{\gamma}_2\right)$, $\overline{s}_2\left(\overline{\gamma}_1\right) \supseteq \overline{s}_2\left(\overline{\gamma}_2\right)$, $\overline{s}_3\left(\overline{\gamma}_1\right) \supseteq \overline{s}_3\left(\overline{\gamma}_2\right)$, $\overline{s}_4\left(\overline{\gamma}_1\right) \supseteq \overline{s}_4\left(\overline{\gamma}_2\right)$. Thus $s_1 \in \overline{s}_1\left(\overline{\gamma}_1\right), s_2 \in \overline{s}_2\left(\overline{\gamma}_1\right), s_3 \in \overline{s}_3\left(\overline{\gamma}_1\right), s_4 \in \overline{s}_4\left(\overline{\gamma}_1\right)$.
So $s \in \overline{s}\left(\overline{\gamma}_1\right)$. Hence the result. □

4. Conclusion

In recent development of Mathematics, result of applying the notion of fuzzy logic as a tool has been proved to be promising. In this paper we have tried to go one step further by proposing the concept of generalized fuzzy bicomplex numbers which can be considered as the extension of fuzzy bicomplex numbers and discussed some of their properties. However, some works of [1] and [8] may still be reinvestigated in this area and left for readers as a scope of future research.

Acknowledgement. The authors are very much thankful to the reviewer for his/her valuable suggestions to bring the paper in its present form.

References

[1] J. J. Buckley, *Fuzzy complex numbers*, Fuzzy Sets Syst. **33** (1989), no. 3, 333-345.
[2] B. Belaïdi, T. Biswas and R. Biswas, *A note on generalized bicomplex numbers*, Nonlinear Stud. **28** (2021), no. 1, 127-163.
[3] G. Scorza Dragoni, *Sulle funzioni olomorfe di una variabile bicomplessa*, Reale Accad. d'Italia, Mem. Classe Sci. Nat. Fis. Mat. **5** (1934), 597–665.
[4] R. Goetschel and W. Voxman, *Topological properties of fuzzy numbers,* Fuzzy Sets Syst. **10** (1983), no. 1, 87-99.
[5] A. Kaufmann and M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Applications*, Van Nostrand Reinhold, New York, 1985.

[6] S. Ö. Karakus, F. K. Aksoyak, *Generalized bicomplex numbers and lie groups*, Adv. Appl. Clifford Algebr. **25** (2015), no. 4, 943-963.

[7] M. E. Luna-Elizarrarás, M. Shapiro, D. C. Struppa and A. Vajiac, *Bicomplex numbers and their elementary functions*, Cubo **14** (2012), no.2, 61–80.

[8] R. P. A. Moura, F. B. Bergamaschi, R. Santiago and B. Bedregal, *Fuzzy quaternion numbers*, Conference paper in IEEE International conference on Fuzzy systems (July 2013).

[9] G. B. Price, *An Introduction to Multicomplex Spaces and Functions*, Marcel Dekker, New York, 1991.

[10] D. Qiu, L. Shu and Z. Mo, *Notes on fuzzy complex analysis,* Fuzzy Sets Syst. **160** (2009), no. 11, 1578-1589.

[11] D. Rochon and M. Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers*, An. Univ. Oradea Fasc. Mat. **11** (2004), 71–110.

[12] C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici*, Math. Ann. **40** (1892), no. 3, 413–467.

[13] N. Spampinato, *Estensione nel campo bicomplesso di due teoremi, del Levi-Civita e del Severi, per le funzioni olomorfe di due variablili bicomplesse I, II*, Reale Accad. Naz. Lincei **22** (1935), 38–43, 96–102.

[14] N. Spampinato, *Sulla Rappresentazione delle funzioni di variabile bicomplessa totalmente derivabili*, Ann. Mat. Pura Appl. **14** (1935), no. 1, 305–325.

[15] L. A. Zadeh, *Fuzzy sets*, Inf. Control **8** (1965), 338-353.

Ritam Biswas
Department of Mathematics, Krishnath College
P.O.-Berhampore, Dist-Murshidabad, PIN-742101, West Bengal, India
Email address: ritamiitr@yahoo.co.in

Tanmay Biswas
Rajbari, Rabindrapally, R. N. Tagore Road
P.O.-Krishnagar, Dist-Nadia, PIN-741101, West Bengal, India
Email address: tanmaybiswas_math@rediffmail.com