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SOME GROWTH PROPERTIES OF ANALYTIC FUNCTIONS RELATING TO (α, β, γ)-NEVANLINNA ORDER AND (α, β, γ)-NEVANLINNA TYPE IN THE UNIT DISC

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Abstract. Growth analysis of analytic functions is very important part of research in the field of complex analysis and many researchers are involved in this area during past decades. Collecting ideas from Heittokangas et al. (Meromorphic functions of finite φ-order and linear q-difference equations, J. Difference Equ. Appl., **27**(9) (2021), 1280-1309) and Belaïdi et al. (Study of complex oscillation of solutions of a second order linear differential equation with entire coefficients of (α, β, γ)-order, WSEAS Trans. Math., **21**(2022), 361-370), here in this paper, we have defined the (α, β, γ)-Nevanlinna order and (α, β, γ)-Nevanlinna type of an analytic function f in the unit disc U. We have also established some growth properties of the composition of two analytic functions in the unit disc on the basis of their (α, β, γ)-Nevanlinna order, (α, β, γ)-Nevanlinna lower order, (α, β, γ)-Nevanlinna type and (α, β, γ)-Nevanlinna weak type as compared to the growth of their corresponding left and right factors, where α, β, γ are continuous non-negative functions defined on $(-\infty, +\infty)$.

1. Introduction

A function f, analytic in the unit disc $U = \{z : |z| < 1\}$ is said to have finite Nevanlinna order [8] if there exists a number μ such that the Nevanlinna characteristic function of f denoted by

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta$$

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satisfies $T_f(r) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers μ is called the Nevanlinna order of f. Thus the Nevanlinna order $\rho(f)$ of f is given by

$$\rho(f) = \limsup_{r\to 1} \frac{\log T_f(r)}{-\log(1-r)}.$$

Similarly, the Nevanlinna lower order $\lambda(f)$ of f is given by

$$\lambda(f) = \liminf_{r\to 1} \frac{\log T_f(r)}{-\log(1-r)}.$$

However during the last several years many authors have investigated about Nevanlinna theory in the field of unit disc in different directions, e.g., [2, 3, 10, 11, 12].

Now, let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b) + c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m \geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0, 1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1-t)\cdot 0) \\ &\geq t\alpha(x) + (1-t)\alpha(0) \geq t\alpha(x), \end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$,

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) + \alpha\left(\frac{b}{a+b}(a+b)\right) \\ &= \alpha(a) + \alpha(b),\ a, b \geq 0. \end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \to 1$. Throughout the present paper we take $\alpha, \alpha_1, \alpha_2, \alpha_3 \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

Heittokangas et al. [7] have introduced a new concept of φ-order of entire and meromorphic functions considering φ as subadditive function. Later on Belaïdi et al. [4] have extended the above idea and have introduced the definitions of (α, β, γ)-order and (α, β, γ)-lower order for entire and meromorphic function. Using these concepts, one may define the (α, β, γ)-Nevanlinna order and (α, β, γ)-Nevanlinna lower order of an analytic function f in the unit disc U in the following ways:

Definition 1.1. *The (α,β,γ)-Nevanlinna order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ and (α,β,γ)-Nevanlinna lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of an analytic function f in the unit disc U are defined as:*

$$\begin{aligned} \rho_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to 1}\frac{\alpha(\log T(r,f))}{\beta\left(\log(\gamma(\frac{1}{1-r}))\right)} \\ \text{and } \lambda_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to 1}\frac{\alpha(\log T(r,f))}{\beta\left(\log(\gamma(\frac{1}{1-r}))\right)}. \end{aligned}$$

Belaïdi et al. [5] have also introduced the definitions of another growth indicators, called (α,β,γ)-type and (α,β,γ)-lower type for entire and meromorphic functions. Using that concepts, one can define (α,β,γ)-Nevanlinna type and (α,β,γ)-Nevanlinna lower type of an analytic function f in the unit disc U in the following way:

Definition 1.2. [5] *The (α,β,γ)-Nevanlinna type denoted by $\sigma_{(\alpha,\beta,\gamma)}[f]$ and (α,β,γ)-Nevanlinna lower type denoted by $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$, of an analytic function f in the unit disc U having finite positive (α,β,γ)-Nevanlinna order $\left(0<\rho_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ are defined as:*

$$\begin{aligned} \sigma_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to 1}\frac{\exp(\alpha(\log T(r,f)))}{\left(\exp\left(\beta\left(\log(\gamma(\frac{1}{1-r}))\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}} \\ \text{and } \overline{\sigma}_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to 1}\frac{\exp(\alpha(\log T(r,f)))}{\left(\exp\left(\beta\left(\log(\gamma(\frac{1}{1-r}))\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}}. \end{aligned}$$

It is obvious that $0\le\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]\le\sigma_{(\alpha,\beta,\gamma)}[f]\le+\infty$.

Analogously, to determine the relative growth of two analytic functions having same non-zero finite (α,β,γ) -Nevanlinna lower type, one can introduce the definitions of (α,β,γ)-Nevanlinna weak type and (α,β,γ)-Nevanlinna upper weak type of a analytic function f in the unit disc U having finite positive (α,β,γ)-Nevanlinna lower order, which are as follows:

Definition 1.3. *The (α,β,γ)-Nevanlinna weak type denoted by $\tau_{(\alpha,\beta,\gamma)}[f]$ and (α,β,γ)-Nevanlinna upper weak type denoted by $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ of an analytic function f in the unit disc U having finite positive (α,β,γ)-Nevanlinna lower order $\left(0<\lambda_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ are defined as:*

$$\begin{aligned} \overline{\tau}_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to 1}\frac{\exp(\alpha(\log T(r,f)))}{\left(\exp\left(\beta\left(\log(\gamma(\frac{1}{1-r}))\right)\right)\right)^{\lambda_{(\alpha,\beta,\gamma)}[f]}} \\ \text{and } \tau_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to 1}\frac{\exp(\alpha(\log T(r,f)))}{\left(\exp\left(\beta\left(\log(\gamma(\frac{1}{1-r}))\right)\right)\right)^{\lambda_{(\alpha,\beta,\gamma)}[f]}}. \end{aligned}$$

It is obvious that $0\le\tau_{(\alpha,\beta,\gamma)}[f]\le\overline{\tau}_{(\alpha,\beta,\gamma)}[f]\le+\infty$.

Here, in this paper, our aim is to investigate some growth properties relating to the composition of two analytic functions in the unit disc U on the basis of (α,β,γ)-Nevanlinna order, (α,β,γ)-Nevanlinna type and (α,β,γ)-Nevanlinna weak type as compared to the growth of their corresponding left and right factors. We do not explain the standard definitions and notations in the theory of analytic functions as those are available in [1, 6, 8, 9].

2. Main results

In this section, the main results of the paper are presented.

Theorem 2.1. *Let f and g be two analytic functions in the unit disc U such that $0 < \lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] \leq \rho_{(\alpha_1,\beta,\gamma)}[f \circ g] < +\infty$ and $0 < \lambda_{(\alpha_2,\beta,\gamma)}[f] \leq \rho_{(\alpha_2,\beta,\gamma)}[f] < +\infty$. Then*

$$\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]} \leq \liminf_{r \to 1} \frac{\alpha_1(\log T(r, f \circ g))}{\alpha_2(\log T(r, f))}$$
$$\leq \min\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\rho_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}\right\}$$
$$\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\rho_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}\right\}$$
$$\leq \limsup_{r \to 1} \frac{\alpha_1(\log T(r, f \circ g))}{\alpha_2(\log T(r, f))} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}.$$

Proof. From the definitions of $\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g]$, $\rho_{(\alpha_1,\beta,\gamma)}[f \circ g]$, $\lambda_{(\alpha_2,\beta,\gamma)}[f]$ and $\rho_{(\alpha_2,\beta,\gamma)}[f]$, we have for arbitrary positive ε and for all sufficiently large values of r (< 1) such that

$$\alpha_1\left(\log T(r, f \circ g)\right) \geqslant \left(\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] - \varepsilon\right) \beta(\log(\gamma(\frac{1}{1-r}))), \tag{1}$$

$$\alpha_1\left(\log T(r, f \circ g)\right) \leq \left(\rho_{(\alpha_1,\beta,\gamma)}[f \circ g] + \varepsilon\right) \beta(\log(\gamma(\frac{1}{1-r}))), \tag{2}$$

$$\alpha_2\left(\log T(r, f)\right) \geqslant \left(\lambda_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon\right) \beta(\log(\gamma(\frac{1}{1-r}))) \tag{3}$$

$$\text{and } \alpha_2\left(\log T(r, f)\right) \leq \left(\rho_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon\right) \beta(\log(\gamma(\frac{1}{1-r}))). \tag{4}$$

Again for a sequence of values of r tending to 1,

$$\alpha_1\left(\log T(r, f \circ g)\right) \leq \left(\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] + \varepsilon\right) \beta(\log(\gamma(\frac{1}{1-r}))), \tag{5}$$

$$\alpha_1\left(\log T(r, f \circ g)\right) \geqslant \left(\rho_{(\alpha_1,\beta,\gamma)}[f \circ g] - \varepsilon\right) \beta(\log(\gamma(\frac{1}{1-r}))), \tag{6}$$

$$\alpha_2\left(\log T(r, f)\right) \leq \left(\lambda_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon\right) \beta(\log(\gamma(\frac{1}{1-r}))) \tag{7}$$

$$\text{and } \alpha_2\left(\log T(r, f)\right) \geqslant \left(\rho_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon\right) \beta(\log(\gamma(\frac{1}{1-r}))). \tag{8}$$

Now from (1) and (4) it follows for all sufficiently large values of r (< 1), that

$$\frac{\alpha_1\left(\log T(r, f \circ g)\right)}{\alpha_2\left(\log T(r, f)\right)} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] - \varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r \to 1} \frac{\alpha_1\left(\log T(r, f \circ g)\right)}{\alpha_2\left(\log T(r, f)\right)} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{9}$$

Combining (5) and (3) , we have for a sequence of values of r tending to 1 that

$$\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\leq\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to 1}\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\leq\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{10}$$

Again from (1) and (7), for a sequence of values of r tending to 1, we get

$$\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\geq\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\geq\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{11}$$

Now, it follows from (3) and (2) , for all sufficiently large values of r (<1) that

$$\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\leq\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\leq\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{12}$$

Now from (2) and (8) , it follows for a sequence of values of r tending to 1, that

$$\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\leq\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\leq\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{13}$$

So combining (4) and (6) , we get for a sequence of values of r tending to 1, that

$$\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\geqslant\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{\alpha_1\left(\log T(r,f\circ g)\right)}{\alpha_2\left(\log T(r,f)\right)}\geqslant\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{14}$$

Thus the theorem follows from (9) , (10) , (11), (12) , (13) and (14) . □

Remark 2.1. *If we take "$0<\lambda_{(\alpha_3,\beta,\gamma)}[g]\leq\rho_{(\alpha_3,\beta,\gamma)}[g]<+\infty$" instead of "$0<\lambda_{(\alpha_2,\beta,\gamma)}[f]\leq\rho_{(\alpha_2,\beta,\gamma)}[f]<+\infty$" and other conditions remain same, the conclusion of Theorem 2.1 remains true with "$\lambda_{(\alpha_3,\beta,\gamma)}[g]$", "$\rho_{(\alpha_3,\beta,\gamma)}[g]$" and "$\alpha_3\left(\log T(r,g)\right)$" in replace of "$\lambda_{(\alpha_2,\beta,\gamma)}[f]$", "$\rho_{(\alpha_2,\beta,\gamma)}[f]$" and "$\alpha_2\left(\log T(r,f)\right)$" respectively in the denominators.*

Theorem 2.2. *Let f and g be two non-constant analytic functions in the unit disc U such that $0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$ and $\lambda_{(\alpha,\beta,\gamma)}[f \circ g] = +\infty$. Then*

$$\lim_{r\to 1} \frac{\alpha(\log T(r, f \circ g))}{\alpha(\log T(r, f))} = +\infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to 1,

$$\alpha(\log T(r, f \circ g)) \leq \Delta \cdot \alpha(\log T(r, f)). \tag{15}$$

Again from the definition of $\rho_{(\alpha,\beta,\gamma)}[f]$, it follows for all sufficiently large values of r (< 1) that

$$\alpha(\log T(r, f)) \leq (\rho_{(\alpha,\beta,\gamma)}[f] + \epsilon)\beta(\log(\gamma(\frac{1}{1-r}))). \tag{16}$$

From (15) and (16), for a sequence of values of r tending to 1,we have

$$\alpha(\log(T(r, f \circ g))) \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f] + \epsilon)\beta(\log(\gamma(\frac{1}{1-r}))),$$

$$i.e., \ \frac{\alpha(\log T(r, f \circ g))}{\beta(\log(\gamma(\frac{1}{1-r})))} \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f] + \epsilon),$$

$$i.e., \ \liminf_{r\to 1} \frac{\alpha(\log T(r, f \circ g))}{\beta(\log(\gamma(\frac{1}{1-r})))} < +\infty,$$

$$i.e., \ \lambda_{(\alpha,\beta,\gamma)}[f \circ g] < +\infty.$$

This is a contradiction.

Thus the theorem follows. □

Remark 2.2. *If we take " $0 < \lambda_{(\alpha,\beta,\gamma)}[g] \leq \rho_{(\alpha,\beta,\gamma)}[g] < +\infty$" instead of " $0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$" and other conditions remain same, the conclusion of Theorem 2.2 remains true with "$\alpha(\log T(r, g))$" in replace of "$\alpha(\log T(r, f))$" in the denominators.*

Remark 2.3. *Theorem 2.2 and Remark 2.2 are also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta,\gamma)}[f \circ g] = +\infty$" is replaced by "$\rho_{(\alpha,\beta,\gamma)}[f \circ g] = +\infty$" and the other conditions remain the same.*

Theorem 2.3. *Let f and g be two analytic functions in the unit disc U such that $0 < \overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f \circ g] \leq \sigma_{(\alpha_1,\beta,\gamma)}[f \circ g] < +\infty$, $0 < \overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] \leq \sigma_{(\alpha_2,\beta,\gamma)}[f] < +\infty$ and $\rho_{(\alpha_1,\beta,\gamma)}[f \circ g] = \rho_{(\alpha_2,\beta,\gamma)}[f]$. Then*

$$\begin{aligned}
\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]} &\leq \liminf_{r\to 1} \frac{\exp(\alpha_1(\log T(r, f \circ g)))}{\exp(\alpha_2(\log T(r, f)))} \\
&\leq \min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\
&\leq \max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\
&\leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\log T(r, f \circ g)))}{\exp(\alpha_2(\log T(r, f)))} \leq \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}.
\end{aligned}$$

Proof. From the definitions of $\sigma_{(\alpha_2,\beta,\gamma)}[f], \overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f], \sigma_{(\alpha_1,\beta,\gamma)}[f\circ g]$ and $\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f\circ g]$, we have for arbitrary positive $\varepsilon(>0)$ and for all sufficiently large values of r (<1) that

$$\exp(\alpha_1(\log T(r,f\circ g))) \le \left(\sigma_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon\right)(\exp(\beta(\log(\gamma(\frac{1}{1-r})))))^{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}, \tag{17}$$

$$\exp(\alpha_1(\log T(r,f\circ g))) \ge \left(\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon\right)(\exp(\beta(\log(\gamma(\frac{1}{1-r})))))^{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}, \tag{18}$$

$$\exp(\alpha_2(\log T(r,f))) \le \left(\sigma_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon\right)(\exp(\beta(\log(\gamma(\frac{1}{1-r})))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}, \tag{19}$$

$$\exp(\alpha_2(\log T(r,f))) \ge \left(\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon\right)(\exp(\beta(\log(\gamma(\frac{1}{1-r})))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{20}$$

Again for a sequence of values of r tending to 1, we get that

$$\exp(\alpha_1(\log T(r,f\circ g))) \geqslant (\sigma_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon)(\exp(\beta(\log(\gamma(\frac{1}{1-r})))))^{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}, \tag{21}$$

$$\exp(\alpha_1(\log T(r,f\circ g))) \le \left(\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon\right)(\exp(\beta(\log(\gamma(\frac{1}{1-r})))))^{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}, \tag{22}$$

$$\exp(\alpha_2(\log T(r,f))) \le \left(\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon\right)(\exp(\beta(\log(\gamma(\frac{1}{1-r})))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}, \tag{23}$$

$$\exp(\alpha_2(\log T(r,f))) \ge \left(\sigma_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon\right)(\exp(\beta(\log(\gamma(\frac{1}{1-r})))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{24}$$

Now from (18), (19) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_2,\beta,\gamma)}[f]$, it follows for all sufficiently large values of r (<1) that

$$\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))} \geqslant \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\sigma_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to 1}\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))} \geqslant \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}. \tag{25}$$

Combining (22) and (20) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_2,\beta,\gamma)}[f]$, we get for a sequence of values of r tending to 1 that

$$\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))} \le \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to 1}\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))} \le \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}. \tag{26}$$

Now from (18), (23) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_2,\beta,\gamma)}[f]$, we obtain for a sequence of values of r tending to 1, that

$$\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))} \ge \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))} \ge \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}. \tag{27}$$

In view of the condition $\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_2,\beta,\gamma)}[f]$, it follows from (20) and (17) for all sufficiently large values of $r\ (<1)$ that

$$\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))}\le\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to1}\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))}\le\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}. \tag{28}$$

Now from (17), (24) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_2,\beta,\gamma)}[f]$, it follows for a sequence of values of r tending to 1, that

$$\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))}\le\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\sigma_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to1}\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))}\le\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}. \tag{29}$$

So combining (19) and (21) and in view of the condition $\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_2,\beta,\gamma)}[f]$, we get for a sequence of values of r tending to 1, that

$$\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))}\geqslant\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\sigma_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to1}\frac{\exp(\alpha_1(\log T(r,f\circ g)))}{\exp(\alpha_2(\log T(r,f)))}\geqslant\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}. \tag{30}$$

Thus the theorem follows from (25) , (26) , (27), (28) , (29) and (30) . □

Remark 2.4. *If we take "*$0<\overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g]\le\sigma_{(\alpha_3,\beta,\gamma)}[g]<+\infty$*" and "*$\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_3,\beta,\gamma)}[g]$*" instead of "*$0<\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]\le\sigma_{(\alpha_2,\beta,\gamma)}[f]<+\infty$*" and "*$\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_2,\beta,\gamma)}[f]$*" and other conditions remain same, the results of Theorem 2.3 remain true with "*$\sigma_{(\alpha_3,\beta,\gamma)}[g]$*", "*$\overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g]$*" and "*$\exp(\alpha_3(\log T(r,g)))$*" instead of "*$\sigma_{(\alpha_2,\beta,\gamma)}[f]$*", "*$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$*" and "*$\exp(\alpha_2(\log T(r,f)))$*" respectively in the denominators.*

Remark 2.5. *If we take "*$0<\tau_{(\alpha_2,\beta,\gamma)}[f]\le\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]<+\infty$*" and "*$\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\lambda_{(\alpha_2,\beta,\gamma)}[f]$*" instead of "*$0<\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]\le\sigma_{(\alpha_2,\beta,\gamma)}[f]<+\infty$*" and "*$\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_2,\beta,\gamma)}[f]$*" and other conditions remain same, the results of Theorem 2.3 remain true with "*$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$*" and "*$\tau_{(\alpha_2,\beta,\gamma)}[f]$*" in place of "*$\sigma_{(\alpha_2,\beta,\gamma)}[f]$*" and "*$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$*" respectively in the denominators.*

Remark 2.6. *If we take "*$0<\tau_{(\alpha_3,\beta,\gamma)}[g]\le\overline{\tau}_{(\alpha_3,\beta,\gamma)}[g]<+\infty$*" and "*$\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\lambda_{(\alpha_3,\beta,\gamma)}[g]$*" instead of "*$0<\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]\le\sigma_{(\alpha_2,\beta,\gamma)}[f]<+\infty$*" and "*$\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]=\rho_{(\alpha_2,\beta,\gamma)}[f]$*" and other conditions remain same, the results of Theorem 2.3 remain true with "*$\tau_{(\alpha_3,\beta,\gamma)}[g]$*", "*$\overline{\tau}_{(\alpha_3,\beta,\gamma)}[g]$*" and "*$\exp(\alpha_3(\log T(r,g)))$*" in place of "*$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$*", "*$\sigma_{(\alpha_2,\beta,\gamma)}[f]$*" and "*$\exp(\alpha_2(\log T(r,f)))$*" respectively in the denominators.*

Now in the line of Theorem 2.3 , one can easily prove the following theorem using the notions of (α,β,γ)-Nevanlinna weak type and (α,β,γ)-Nevanlinna upper weak type and so the proof is omitted.

Theorem 2.4. *Let f and g be two analytic functions in the unit disc U such that $0 < \tau_{(\alpha_1,\beta,\gamma)}[f \circ g] \leq \overline{\tau}_{(\alpha_1,\beta,\gamma)}[f \circ g] < +\infty$, $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$ and $\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$. Then*

$$\begin{aligned}\frac{\tau_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]} &\leq \liminf_{r\to 1} \frac{\exp(\alpha_1(\log T(r, f \circ g)))}{\exp(\alpha_2(\log T(r, f)))} \\ &\leq \min\left\{\frac{\tau_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\tau_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\overline{\tau}_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\ &\leq \max\left\{\frac{\tau_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\tau_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\overline{\tau}_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\ &\leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\log T(r, f \circ g)))}{\exp(\alpha_2(\log T(r, f)))} \leq \frac{\overline{\tau}_{(\alpha_1,\beta,\gamma)}[f \circ g]}{\tau_{(\alpha_2,\beta,\gamma)}[f]}.\end{aligned}$$

Remark 2.7. *If we take "$0 < \tau_{(\alpha_3,\beta,\gamma)}[g] \leq \overline{\tau}_{(\alpha_3,\beta,\gamma)}[g] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] = \lambda_{(\alpha_3,\beta,\gamma)}[g]$" instead of "$0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 2.4 remain true with "$\tau_{(\alpha_3,\beta,\gamma)}[g]$", "$\overline{\tau}_{(\alpha_3,\beta,\gamma)}[g]$" and "$\exp(\alpha_3(\log T(r, g)))$" in place of "$\tau_{(\alpha_2,\beta,\gamma)}[f]$", "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" and "$\exp(\alpha_2(\log T(r, f)))$" respectively in the denominators.*

Remark 2.8. *If we take "$0 < \overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] \leq \sigma_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] = \rho_{(\alpha_2,\beta,\gamma)}[f]$" instead of "$0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 2.4 remain true with "$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$" and "$\sigma_{(\alpha_2,\beta,\gamma)}[f]$" in place of "$\tau_{(\alpha_2,\beta,\gamma)}[f]$" and "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" respectively in the denominators.*

Remark 2.9. *If we take "$0 < \overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g] \leq \sigma_{(\alpha_3,\beta,\gamma)}[g] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] = \rho_{(\alpha_3,\beta,\gamma)}[g]$" instead of "$0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f \circ g] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 2.4 remain true with "$\overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g]$", "$\sigma_{(\alpha_3,\beta,\gamma)}[g]$" and "$\exp(\alpha_3(\log T(r, g)))$" in place of "$\tau_{(\alpha_2,\beta,\gamma)}[f]$", "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" and "$\exp(\alpha_2(\log T(r, f)))$" respectively in the denominators.*

3. Conclusion

Many researchers have investigated on the growth properties of composite entire functions during last several years from different angle of view using the concepts of order, generalized order, (p, q)-th order, (p, q)-φ order, $(p.q)$-L order, relative order, generalized order (α, β) and so many. On the other hand, Belaïdi et al. [4] have introduced the definitions of (α, β, γ)-order of entire and meromorphic functions which has a great contribution in the field of differential equations and extended so many important results in this field. In this paper, we have introduced (α, β, γ)-Nevanlinna order and (α, β, γ)-Nevanlinna type of an analytic function in the unit disc U and also investigated some growth properties of the composition of two analytic functions in the unit disc on the basis of their (α, β, γ)-Nevanlinna order, (α, β, γ)-Nevanlinna lower order, (α, β, γ)-Nevanlinna type and (α, β, γ)-Nevanlinna weak type as compared to the growth of their corresponding left and right factors, where α, β, γ are continuous non-negative functions defined on $(-\infty, +\infty)$.

This concept of (α, β, γ)-Nevanlinna order and (α, β, γ)-Nevanlinna type in the unit disc may help to develop the theory of growth properties of linear differential equations whose coefficients are entire or meromorphic functions. This is a vast area of active research and left to the interested researchers.

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