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SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS ASSOCIATED WITH BOREL DISTRIBUTION SERIES

S.A.JADHAV, P.G.JADHAV, P.THIRUPATHI REDDY

Abstract. This article aims to obtain some necessary and sufficient conditions for functions, whose coefficients are probabilities of the Borel distribution series, to belong to certain subclasses of analytic and univalent functions. In this work, we shall derive a new subclass of uniformly convex functions denoted by $H_\mu(\gamma, \beta)$ in the open unit disk $D = \{z : z \in \mathbb{C}, |z| < 1\}$ by using the Borel distribution series. Also, we obtain geometric properties like coefficient inequalities, closure theorem, partial sums, extreme points, and radius of starlikeness and convexity of functions belonging to the subclass.

1. Introduction

Let A be the class of all analytic functions of the form

$$g(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1}$$

defined on the open unit disc D. Let S be the subclass of A consisting of functions that are univalent in D.
Let T be the subclass of S consisting of functions of the form

$$g(z) = z - \sum_{m=2}^{\infty} a_m z^m, (a_m \geq 0), \tag{2}$$

defined on the open unit disk D. The class T was introduced and studied by Silverman [23].

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A function $g \in A$ is called starlike of order $\gamma(0 \leq \gamma < 1)$ if and only if

$$\Re\left(\frac{zg^{'}(z)}{g(z)}\right) > \gamma, \quad (z \in D).$$

A function $g \in A$ is called convex of order $\gamma(0 \leq \gamma < 1)$ if and only if

$$\Re\left(1 + \frac{zg^{''}(z)}{g^{'}(z)}\right) > \gamma, \quad (z \in D).$$

We denote the class of all starlike functions of order γ by $S^*(\gamma)$ and the class convex functions of order γ by $K(\gamma)$.

Kanas and Wisniowska [10] introduced the class β-uniformly convex functions. For $-1 < \gamma \leq 1$ and $\beta \geq 0$.

(i) A function $g \in A$ is said to be in the class β-uniformly starlike functions of order γ if it satisfies the condition

$$\Re\left(\frac{zg^{'}(z)}{g(z)} - \gamma\right) > \beta\left|\frac{zg^{'}(z)}{g(z)} - 1\right|, \quad (z \in D).$$

This class is denoted by $S_p(\gamma, \beta)$.

(ii)A function $g \in A$ is said to be in the class β -uniformly convex functions of order γ if it satisfies the condition

$$\Re\left(1 + \frac{zg^{''}(z)}{g^{'}(z)} - \gamma\right) \geq \beta\left|\frac{zg^{''}(z)}{g^{'}(z)}\right|, \quad (z \in D).$$

This class is denoted by $UCV(\gamma, \beta)$.
Note that $UCV(\gamma, 0) = K(\gamma)$ and $S_p(\gamma, 0) = S^*(\gamma)$.
For $\beta = 1$ corresponds to the class of uniformly convex functions introduced by Goodman [8, 9] and studied by Ronning [20, 21].
The study of operators plays an important role in Geometric Function Theory (GFT). Many differential and integral operators can be written in terms of the convolution of certain analytic functions. It is observed that this formalism makes further mathematical exploration easier, and also improves the understanding of the geometric and symmetric properties of such operators. The importance of convolution in the theory of operators may easily be understood from the work in [1, 12, **?**]. Furthermore, probability is not just about flipping coins and counting cards in a disc; it is used in a wide range of real-life areas, from insurance to meteorology and politics to economics forecasting.

The Borel distribution series has applications in Geometric Function Theory(GFT), particularly in the study of univalent functions, coefficient problems, and growth and distortion theorems. The Borel distribution series can be used to analyze the behavior of these functions, particularly in determining the distribution of their coefficients. It helps in understanding the probabilistic distribution of these coefficients, providing insights into their expected magnitude and variance. The Borel distribution series is used in modeling random holomorphic functions, which are of interest in various areas of mathematical physics and probability theory. This

approach provides insights into the typical behavior of such functions, which can then be applied to specific problems in Geometric Function Theory. In some cases, the Borel distribution series is applied to study the analytic continuation of functions. This involves understanding how functions defined by a power series can be extended beyond their radius of convergence, a topic closely related to the geometric properties of these functions.
By leveraging the statistical and probabilistic insights provided by the Borel distribution series, researchers in geometric function theory can gain a deeper understanding of the properties and behaviors of holomorphic and univalent functions. This, in turn, helps in solving classical problems in the field and discovering new phenomena related to the geometric aspects of analytic functions. See [26, 5, 16, 28] for recent studies on the Borel distribution series.

In Geometric Function Theory (GFT), the elementary distributions such as the Pascal, Poisson, logarithmic, binomial and beta negative binomial, have been partially from a theoretical point of view. For detailed study, we refer the readers to (see [2, 4, 15, 18, 19]). A discrete random variable x is said to have a Borel distribution if it takes values 1, 2, 3... with the probabilities $\frac{e^{-\mu}}{1!}$, $\frac{2\mu e^{-2\mu}}{2!}$, $\frac{9\mu^2 e^{-3\mu}}{3!}$, ..., respectively, where μ is parameter.

$$P(x=r)=\frac{(\mu r)^{r-1}e^{-\mu r}}{r!}, \quad r=1,2,\ldots$$

Wanas and Khuttar[27] introduced a series whose coefficients are probabilities of the Borel distribution:

$$\mathcal{M}(\mu,z)=z+\sum_{m=2}^{\infty}\frac{[\mu(m-1)]^{m-2}e^{-\mu(m-1)}}{(m-1)!}z^m, \quad (z\in D, \quad 0<\mu\leq 1). \tag{3}$$

From a well-known ratio test, the above series is convergent with the domain of convergence of the entire complex plane.
Now we consider the linear operator $B_m(\mu)(z):A\to A$ defined by the convolution or Hadamard product

$$B_m(\mu)g(z)=\mathcal{M}(\mu,z)*f(z)=z+\sum_{m=2}^{\infty}\psi_m a_m z^m, \quad (z\in D, \quad a_m\geq 0). \tag{4}$$

$$\psi_m(\mu)=\frac{[\mu(m-1)]^{m-2}e^{-\mu(m-1)}}{(m-1)!}. \tag{5}$$

Motivated by results on connections between various subclasses of analytic univalent functions using special functions and distribution series, we obtain some necessary and sufficient conditions for the Borel distribution series to be in the classes. Inspired by Alessa et al [3], Breaz et al [6] and Murugusundaramoorthy and Magesh [13], we define new subclass $H_\mu(\gamma,\beta)$ using Borel distribution as the following:
Definition 1.1 For $-1\leq\gamma\leq 1$, $\beta\geq 0$, we let $H_\mu(\gamma,\beta)$ consists of functions $g\in A$ satisfying the condition

$$\Re\left\{\frac{z(B_m(\mu)g(z))'}{B_m(\mu)g(z)}-\gamma\right\}>\beta\left|\frac{z(B_m(\mu)g(z))'}{(B_m(\mu)g(z))}-1\right|, \tag{6}$$

where $B_m(\mu)g(z)$ is defined in (4).
Now, we define
$TH_\mu(\gamma,\beta) = H_\mu(\gamma,\beta)\bigcap T$.
The objective of this paper is to introduce coefficient inequalities, closure theorem, extreme points and radii of starlikeness, covexity, partial sums and integral operator of functions belonging to the newly described subclass.

2. Coefficient Estimates

In the section, we discuss the coefficient estimates for the function g belonging to the class $H_\mu(\gamma,\beta)$.
Theorem 2.1 For $-1 \leq \gamma \leq 1$, $\beta \geq 0$, a function g is given by (1) is in the class $H_\mu(\gamma,\beta)$ if and only if

$$\sum_{m=2}^{\infty}[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)|a_m| \leq (1-\gamma). \tag{7}$$

Where

$$\psi_m(\mu) = \frac{[\mu(m-1)]^{m-2}e^{-\mu(m-1)}}{(m-1)!}.$$

Proof. It is sufficient to show that

$$\left|\frac{z(B_m(\mu)g(z))'}{B_m(\mu)g(z)} - 1\right| - \Re\left\{\frac{z(B_m(\mu)g(z))'}{B_m(\mu)g(z)} - 1\right\} \leq 1-\gamma.$$

We have

$$\begin{aligned}\beta\left|\frac{z(B_m(\mu)g(z))'}{B_m(\mu)g(z)} - 1\right| - \Re\left\{\frac{z(B_m(\mu)g(z))'}{B_m(\mu)g(z)} - 1\right\} \\ \leq (1+\beta)\left|\frac{z(B_m(\mu))g(z))'}{B_m(\mu)g(z)}\right| \\ \leq \frac{(1+\beta)\sum_{m=2}^{\infty}(m-1)\psi_m(\mu)|a_m||z|^{m-1}}{1-\sum_{m=2}^{\infty}\psi_m(\mu)|a_m||z|^{m-1}} \\ \leq \frac{(1+\beta)\sum_{m=2}^{\infty}(m-1)\psi_m(\mu)|a_m|}{1-\sum_{m=2}^{\infty}\psi_m(\mu)|a_m|}.\end{aligned}$$

The above expression is bounded above by $(1-\gamma)$ if

$$(1+\beta)\sum_{m=2}^{\infty}(m-1)\psi_m(\mu)|a_m| \leq (1-\gamma)\left(1-\sum_{m=2}^{\infty}\psi_m(\mu)|a_m|\right).$$

Which is equivalent to

$$\sum_{m=2}^{\infty}[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)|a_m| \leq (1-\gamma).$$

This is true by hypothesis and the proof is complete.

Theorem 2.2 Let $\beta \geq 0$, $-1 \leq \gamma < 1$ then g is of the form (2) to be in the class $TH_\mu(\gamma, \beta)$ if and only if

$$\sum_{m=2}^{\infty} [m(1+\beta) - (\gamma+\beta)]\psi_m(\mu)a_m \leq (1-\gamma), \tag{8}$$

where $\psi_m(\mu)$ is given by (5).

Proof. In the view of Theorem 2.1, we have to prove only necessity
If $g \in TH_\mu(\gamma, \beta)$ and z real then

$$\Re\left\{\frac{1 - \sum\limits_{m=2}^{\infty} m\psi_m(\mu)a_m z^{m-1}}{1 - \sum\limits_{m=2}^{\infty} \psi_m(\mu)a_m z^{m-1}} - \gamma\right\} > \left|\frac{\sum\limits_{m=2}^{\infty} (m-1)\psi_m(\mu)a_m z^{m-1}}{1 - \sum\limits_{m=2}^{\infty} \psi_m(\mu)a_m z^{m-1}}\right|,$$

as $z \to 1$ along real axis, we obtain the desired inequality

$$1 - \gamma - \sum_{m=2}^{\infty} (m-\gamma)\psi_m(\mu)a_m \geq \sum_{m=2}^{\infty} (m-1)\psi_m(\mu)a_m.$$

That is

$$\sum_{m=2}^{\infty} [m(1+\beta) - (\gamma+\beta)]\psi_m(\mu)a_m \leq (1-\gamma).$$

Theorem 2.3 The class $TH_\mu(\gamma, \beta)$ is a convex set.

Proof. Let the function

$$g_j(z) = z - \sum_{m=2}^{\infty} a_{m,j} z^m, \quad (a_{m,j} \geq 0,\ j = 1, 2), \tag{9}$$

be in the class $TH_\mu(\gamma, \beta)$. We have to show that the function $k(z)$ defined by $k(z) = \eta g_1(z) + (1-\eta)g_2(z)$, $(0 \leq \eta < 1)$ is in the class $TH_\mu(\gamma, \beta)$.

$$k(z) = z - \sum_{m=2}^{\infty} [\eta a_{m,1} + (1-\eta)a_{m,2}]z^m.$$

Using Theorem 2.2, we get

$$\begin{aligned}\sum_{m=2}^{\infty} [m(1+\beta) - (\gamma+\beta)]\eta\psi_m(\mu)a_{m,1} + \sum_{m=2}^{\infty} [m(1+\beta) - (\gamma+\beta)](1-\eta)\psi_m(\mu)a_{m,2} \\ \leq \eta(1-\gamma) + (1-\eta)(1-\gamma) \\ \leq (1-\gamma),\end{aligned}$$

which implies $k(z) \in TH_\mu(\gamma, \beta)$. Hence $TH_\mu(\gamma, \beta)$ is convex.
Theorem 2.4 Let $g_1(z) = z$ and

$$g_m(z) = z - \frac{1-\gamma}{[m(1+\beta) - (\gamma+\beta)]\psi_m(\mu)} z^m, \quad (m \geq 2). \tag{10}$$

Then $g(z) \in TH_\mu(\gamma,\beta)$ if and only if it can be expressed in the form

$$g(z) = \sum_{m=1}^{\infty} \xi_m g_m(z) \quad (\xi_m \geq 0), \quad \sum_{m=1}^{\infty} \xi_m = 1. \tag{11}$$

Proof. Suppose $g(z)$ can be written as in (10), then

$$g(z) = z - \sum_{m=2}^{\infty} \xi_m \frac{1-\gamma}{[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)} z^m.$$

Now,

$$\begin{aligned}\sum_{m=2}^{\infty} \xi_m \frac{(1-\gamma)[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)}{(1-\gamma)[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)} &= \sum_{m=2}^{\infty} \xi_m \\ &= 1-\xi_1 \leq 1.\end{aligned}$$

Thus $g(z) \in TH_\mu(\gamma,\beta)$.
Conversely,
let $g(z) \in TH_\mu(\gamma,\beta)$ then by using (11), we get

$$\xi_m = \frac{[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)}{1-\gamma} a_m, \quad (m \geq 2)$$

and $\xi_1 = 1 - \sum\limits_{m=2}^{\infty} \xi_m$, then we have $g(z) = \sum\limits_{m=1}^{\infty} \xi_m g_m(z)$ and the proof is complete.

Theorem 2.5 Let the function $g_j(z), (j = 1, 2 \ldots e)$ be in the classes $TH_\mu(\gamma_j,\beta)$ then the function
$k(z) = z - \sum\limits_{m=2}^{\infty} d_m z^m$ is in the class $TH_\mu(\gamma,\beta)$, where

$$\gamma = \min_{1\leq j\leq e} \gamma_j, \quad g_j(z) = z - \sum_{m=2}^{\infty} a_{m,j} z^m, \quad a_{m,j} \geq 0, \quad (j = 1, 2, \ldots e)$$

$$d_m = \frac{1}{e} \sum_{j=1}^{e} a_{m,j}, \qquad (a_{m,j} \geq 0).$$

Proof. Since $g_j(z)$ is in the class $TH_\mu(\gamma,\beta)$ by using Theorem 2.1, we have

$$\sum_{m=2}^{\infty} [m(1+\beta)-(\gamma+\beta)]\psi_m(\mu) a_{m,j} \leq 1-\gamma.$$

Now,

$$\begin{aligned}\sum_{m=2}^{\infty}[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)d_m &= \sum_{m=2}^{\infty}[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)\left(\frac{1}{e}\sum_{j=1}^{e}a_{m,j}\right)\\ &= \frac{1}{e}\sum_{j=1}^{e}\left(\sum_{m=2}^{\infty}[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)a_{m,j}\right)\\ &= 1-\gamma.\end{aligned}$$

Observe that

$$\sum_{m=2}^{\infty}\frac{[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)\frac{1}{e}\sum\limits_{j=1}^{e}a_{m,j}}{1-\gamma}\leq 1.$$

Thus, $k(z)\in TH_\mu(\gamma,\beta)$.
The proof is completed.

3. Radius of Starlikeness and Convexity

Now, we obtain the radii of convexity and starlikeness for the class $TH_\mu(\gamma,\beta)$.
Theorem 3.1 Let the function $g(z)$ defined by (2) belong to the class $TH_\mu(\gamma,\beta)$ then $g(z)$ is starlike of order $\zeta(0\leq\zeta<1)$ in the disc $|z|<R_2$, where

$$R_2=\inf_{m\geq 2}\left[\frac{(1-\zeta)[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)}{(m-\zeta)(1-\gamma)}\right]^{\frac{1}{m-1}},\ (m\geq 2). \tag{12}$$

Proof. Given $g\in T$ and g is starlike of order ζ, we have

$$\left|\frac{zg'(z)}{g(z)}-1\right|<(1-\zeta). \tag{13}$$

For the left hand side of (13) we have,

$$\left|\frac{zg'(z)}{g(z)}-1\right|\leq\frac{\sum\limits_{m=2}^{\infty}(m-1)a_m|z|^{m-1}}{1-\sum\limits_{m=2}^{\infty}a_m|z|^{m-1}}.$$

The last expression is less than $1-\zeta$ if

$$\sum_{m=2}^{\infty}\frac{m-\zeta}{1-\zeta}a_m|z|^{m-1}<1.$$

using the fact that $g(z)\in TH_\mu(\gamma,\beta)$ if and only if

$$\sum_{m=2}^{\infty}\frac{[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)}{1-\gamma}a_m\leq 1.$$

(13) is true if

$$\frac{m-\zeta}{1-\zeta}|z|^{m-1}\leq\frac{[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)}{1-\gamma}.$$

Equivalently

$$|z|^{m-1} \leq \left\{ \frac{(1-\zeta)[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)}{(m-\zeta)(1-\gamma)} \right\}^{\frac{1}{m-1}}.$$

Which yields the starlikeness of the family.

Theorem 3.2 Let the function $g(z)$ defined by (2) belong to the class $TH_\mu(\gamma,\beta)$, then $g(z)$ is convex of order $\zeta (0 \leq \zeta < 1)$ in the disc $|z| < R_2$, where

$$R_2 = \inf_{m\geq 2} \left[\frac{(1-\zeta)[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)}{m(m-\zeta)(1-\gamma)} \right]^{\frac{1}{m-1}}, \ (m \geq 2). \tag{14}$$

Proof. Since g is convex if and only if zg' is starlike we can prove this theorem similar to Theorem 3.1.

4. Partial Sums

In this section, We derive the sharp lower bounds for quotients as follows

$$\Re\left\{\frac{g(z)}{g_k(z)}\right\}, \quad \Re\left\{\frac{g_k(z)}{g(z)}\right\}, \quad \Re\left\{\frac{g'(z)}{g'_k(z)}\right\} \quad \Re\left\{\frac{g'_k(z)}{g'(z)}\right\}.$$

Let us use earlier methods of Silverman [24] and Silvia [25] and also (see [7, 11, 14, 17]) on partial sums of analytic functions:

Theorem 4.1 Let $g(z) \in H_\mu(\gamma,\beta)$ given by (1) and let its partial sum be given by

$$g_1(z) = z, \ g_k(z) = z + \sum_{m=2}^{k} a_m z^m. \tag{15}$$

Suppose that $\sum\limits_{m=2}^{\infty} \alpha_m z^m \leq 1$ where

$$\alpha_m \geq \begin{cases} 1, & (m = 1, 2, \ldots k) \\ \alpha_k, & (m = k+1, k+2, \ldots), \end{cases}$$

and

$$\alpha_m = \frac{[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)}{1-\gamma}.$$

Then

$$\Re\left\{\frac{g(z)}{g_k(z)}\right\} > 1 - \frac{1}{\alpha_{k+1}}, \quad (z \in D) \tag{16}$$

and

$$\Re\left\{\frac{g_k(z)}{g(z)}\right\} > \frac{\alpha_{k+1}}{1+\alpha_{k+1}}, \quad (z \in D). \tag{17}$$

The result is sharp for every $n \in \mathbb{N}_0$, with the extremal function given by

$$g(z) = z + \frac{1}{\alpha_{k+1}} z^{k+1} \quad (z \in D). \tag{18}$$

Proof. We have

$$\sum_{m=2}^{k}|a_m| + \alpha_{k+1}\sum_{m=k+1}^{\infty}|a_m| \le \sum_{m=2}^{\infty}\alpha_m|a_m| \le 1. \tag{19}$$

By setting,

$$\begin{aligned} h_1(z) &= \alpha_{k+1}\left\{\frac{g(z)}{g_k(z)} - \left(1-\frac{1}{\alpha_{k+1}}\right)\right\} \\ &= \frac{1+\alpha_{k+1}\sum\limits_{m=k+1}^{\infty}a_m z^{m-1} + \sum\limits_{m=1}^{k}a_m z^{m-1}}{1+\sum\limits_{m=2}^{k}a_m z^{m-1}}. \end{aligned}$$

Now using (19) we get

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| \le \frac{\alpha_{k+1}\sum\limits_{m=k+1}^{\infty}a_m}{2-2\sum\limits_{m=2}^{k}a_m - \alpha_{k+1}\sum\limits_{m=k+1}^{\infty}a_m} \le 1,$$

which gives proof of (16) of Theorem 4.1. To see that function $g(z) = z + \frac{z^{k+1}}{\alpha_{k+1}}$ gives sharp result. For $z = re^{\frac{i\pi}{k}}$ $(0 < r < 1)$

$$\frac{g(z)}{g_k(z)} = 1 - \frac{r^{k+2}}{\alpha_{k+1}} \to 1 - \frac{1}{\alpha_{k+1}}$$

as $r \to 1^-$.
Similarly take

$$\begin{aligned} h_2(z) &= (1+\alpha_{k+1})\left\{\frac{g_k(z)}{g(z)} - \left(1-\frac{\alpha_{k+1}}{1+\alpha_{k+1}}\right)\right\} \\ &= \frac{(1-\alpha_{k+1})\sum\limits_{m=k+1}^{\infty}a_m z^{m-1} + \sum\limits_{m=1}^{k}a_m z^{m-1}}{1+\sum\limits_{m=2}^{\infty}a_m z^{m-1}}. \end{aligned}$$

Now,

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \le \frac{(1+\alpha_{k+1})\sum\limits_{m=k+1}^{\infty}a_m}{2-2\sum\limits_{m=1}^{k}a_m - (\alpha_{k+1}-1)\sum\limits_{m=k+1}^{\infty}a_m}.$$

$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \le 1$ if and only if

$$2\alpha_{k+1}\sum_{m=k+1}^{\infty}a_m \le 2-2\sum_{m=1}^{k}a_m,$$

which is equivalent to

$$\sum_{m=1}^{k} a_m + \alpha_{n+1} \sum_{m=k+1}^{\infty} a_m \leq 1.$$

Which gives assertion (17) of the Theorem 4.1.

We can similarly prove Theorem 4.2 below.

Theorem 4.2 Let $g(z) \in TH_\mu(\gamma, \beta)$ given by (1) and let its partial sum be given by (15) then

$$\Re\left\{\frac{g'(z)}{g'_k(z)}\right\} > 1 - \frac{k+1}{\alpha_{k+1}}$$

and

$$\Re\left\{\frac{g'_k(z)}{g'(z)}\right\} > \frac{\alpha_{k+1}}{k+1+\alpha_{k+1}}.$$

bounds are sharp for extremal function given by (18).

5. Hadamard Product and Integral Operator

Let the functions g(z) and h(z) be defined by $g(z) = z - \sum\limits_{m=2}^{\infty} a_m z^m$, $h(z) = z - \sum\limits_{m=2}^{\infty} b_m z^m$, $(z \in D)$. The Hadamard product of $g(z)$ and $h(z)$ be defined by

$$(g * h)(z) = z - \sum_{m=2}^{\infty} a_m b_m z^m.$$

Theorem 5.1 Let the function $g(z)$ be defined by (2) be in the class $TH_\mu(\gamma, \beta)$. Let $h(z) = z - \sum\limits_{m=2}^{\infty} b_m z^m$ for $|b_m| \leq 1$. Then $(g * h)(z) \in TH_\mu(\gamma, \beta)$.

Proof. Consider,

$$\begin{aligned}\sum_{m=2}^{\infty} [m(1+\beta) - (\gamma+\beta)]\psi_m(\mu)|a_m b_m| &\leq \sum_{m=2}^{\infty} [m(1+\beta) - (\gamma+\beta)]\psi_m(\mu)|a_m||b_m| \\ &\leq \sum_{m=2}^{\infty} [m(1+\beta) - (\gamma+\beta)]\psi_m(\mu)|a_m| \\ &\leq (1-\gamma).\end{aligned}$$

That implies $(g * h)(z) \in TH_\mu(\gamma, \beta)$.

Theorem 5.2 Let $g(z) \in TH_\mu(\gamma, \beta)$ then $F(z) \in TH_\mu(\gamma, \beta)$. Where

$$F(z) = \frac{\lambda+1}{z^\lambda} \int_0^\lambda t^{\lambda-1} g(t)dt, \quad (\lambda > -1).$$

Proof. After simplification $F(z)$ becomes

$$F(z) = z - \sum_{m=2}^{\infty} \frac{1+\lambda}{m+\lambda} a_m z^m, \quad (\lambda > -1).$$

Since $\lambda > -1$ we have $0 \leq \frac{1+\lambda}{m+\lambda}|a_m| \leq |a_m|$.

$$\sum_{m=2}^{\infty}[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)\frac{1+\lambda}{m+\lambda}|a_m| \leq \sum_{m=2}^{\infty}[m(1+\beta)-(\gamma+\beta)]\psi_m(\mu)|a_m| \leq (1-\gamma).$$

By Theorem 2.2, we have $F(z) \in TH_\mu(\gamma,\beta)$.

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S.A.Jadhav

Department of Mathematics, New Arts, Commerce and Science College, Ahmednagar-414 001, Maharashtra, India.

Email address: jadhavsonali108@gmail.com

P.G. Jadhav

Department of Mathematics, Balasaheb Jadhav Arts, Commerce and Science College, Pune-412 411, Maharashtra, India.

Email address: pgjmaths1@gmail.com

P,Thirupathi Reddy

Department of Mathematics, DRK Institute of Science and Technology, Bowrampet, Hyderabad- 500 043, Telangana, India.

Email address: reddypt2@gmail.com