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CAYLEY YOSIDA INCLUSION PROBLEM INVOLVING XOR-OPERATION IN ORDERED HILBERT SPACES

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Abstract. In this paper, we consider and study a new class of variational inclusions called the Cayley Yosida inclusion problem involving XOR operations. By demonstrating the equivalence of our proposed problem to a fixed-point equation, we establish a foundational connection. Based on this fixed-point formulation, we introduce an iterative algorithm aimed at deriving existence and convergence results for the specified problem. Through systematic analysis, we substantiate the theoretical framework supporting the convergence of our proposed algorithm. To illustrate the practical applicability of our findings, we furnish a numerical example using MATLAB, shedding light on the effectiveness and feasibility of the devised approach. This research contributes to the broader understanding of variational inclusions involving XOR operations, offering a new perspective and computational methodology through the exploration of the Cayley Yosida inclusion problem. The developed algorithm not only provides theoretical insights but also demonstrates its practical utility through the presented numerical case, emphasizing the versatility and effectiveness of the proposed solution in tackling real-world problems.

1. Introduction

As a generalization of the traditional optimization problem, Hartman and Stampacchia [9] first introduced variational inequalities in the 1960s. They provide a framework for studying the existence and uniqueness of solutions to inequality constraints. Variational inclusions generalize the concept of variational inequalities. They provide a more comprehensive framework for modeling and solving a wide range of equilibrium problems, see [3]. Various solution techniques have been developed for variational inequalities and inclusions, including projection methods, penalty methods, and fixed-point algorithms. These methods provide numerical

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approaches for finding approximate solutions; see for example, [4, 7, 11, 21]. Variational inequalities and inclusions have found applications in diverse fields, such as mathematical economics, engineering, transportation, and physics. They are used to model and solve problems involving equilibrium, optimization with constraints, and game theory, among others; see [6, 8].

In 2008, Li [12] introduced and studied a new problem known as generalized nonlinear ordered variational inequalities (the ordered equations). These were extensively studied, focusing on developing an approximation algorithm and solution for a specific class of these inequalities and equations in ordered Banach spaces. Several studies have been carried out in this direction, including references to notable works such as [13, 14, 15, 16, 17, 18, 19, 20]. These research efforts have expanded our understanding of generalized nonlinear ordered variational inequalities and equations. In recent times, Ahmed et al. [1] proposed a new approach by introducing a novel mapping called H(., .)-ordered-compression mapping. They also defined a resolvent operator and explored its properties using XOR and XNOR operations. Additionally, they developed an algorithm specifically designed for solving XOR-variational inclusion problems. Furthermore, the study of ordered variational inclusions with XOR operators has gained significant attention in various research domains. Very recent examples of this direction of research can be found in [2, 10].

Motivated by the on-going research in this direction, in this paper we introduce a novel inclusion problem known as the Cayley Yosida inclusion problem involving XOR-operation. To tackle this problem, we propose an iterative algorithm based on the fixed-point formulation. Through this algorithm, we conduct a comprehensive convergence analysis for the aforementioned problem. Finally, we provide a numerical example that satisfies our main result and show the convergence by using MATLAB.

2. Preliminaries

Let Σ be a real Hilbert space with the usual norm $\|.\|$ and the inner product $\langle ., . \rangle$. The metric induced by the norm $\|.\|$ is denoted by d. Let $\aleph$ be a cone in Σ. The partial ordering denoted by "$\leq$" is induced by cone $\aleph$. When Σ is equipped with this partial ordering, it is referred to as an ordered Hilbert space. We denote $C(\Sigma)$ as the collection of all compact subsets of Σ, and 2^{Σ} as the collection of all nonempty subsets of Σ. The Hausdorff metric on $C(\Sigma)$ is represented by $D(., .)$. For any arbitrary elements ω and ϑ belonging to Σ, $glb\,\{\omega, \vartheta\}$ and $lub\,\{\omega, \vartheta\}$ represent the greatest lower bound and the least upper bound, respectively, for the set $\{\omega, \vartheta\}$ with respect to the partial ordering "$\leq$".

We define several operations:
$\wedge$ as the AND operator, $\vee$ as the OR operator, $\oplus$ as the XOR operator, and $\odot$ as the XNOR operator. These operations are defined as follows:

(i) $\omega \wedge \vartheta = glb\,\{\omega, \vartheta\}$,
(ii) $\omega \vee \vartheta = lub\,\{\omega, \vartheta\}$,
(iii) $\omega \oplus \vartheta = (\omega - \vartheta) \vee (\vartheta - \omega)$,
(iv) $\omega \odot \vartheta = (\omega - \vartheta) \wedge (\vartheta - \omega)$.

Definition 2.1. [22] *A cone is a non-empty closed convex subset $\aleph$ of Σ, satisfying*

(i) *if $\omega \in \aleph$ and $\tau > 0$, then $\tau\omega \in \aleph$,*
(ii) *if $\omega \in \aleph$ and $-\omega \in \aleph$, then $\omega = 0$.*

Definition 2.2. [5, 22] *Suppose $\aleph$ is a cone. Then*

(i) *For arbitrary elements $\omega, \vartheta \in \Sigma$, $\omega \leq \vartheta$ if and only if $\omega - \vartheta \in \Sigma$,*
(ii) *ω and ϑ are said to be comparable to each other if and only if $\omega \leq \vartheta$ or $\vartheta \leq \omega$ and we denote it by $\omega \propto \vartheta$.*

The following definitions and results can be found in [12, 13, 14, 15, 16, 17, 18, 19, 20].

Proposition 2.1. *Let Σ denote an ordered Hilbert space, and let $\leq$ be a partial ordering defined on Σ. For any elements $\omega, \vartheta, \nu, \mu \in \Sigma$, the following conditions are fulfilled:*

(i) $\omega \oplus \vartheta = \vartheta \oplus \omega$, $\omega \oplus \omega = 0$, $\omega \odot \vartheta = \vartheta \odot \omega = -(\omega \oplus \vartheta) = -(\vartheta \oplus \omega)$,
(ii) *let τ be a real number, then $(\tau\omega) \oplus (\tau\vartheta) = |\tau| (\omega \oplus \vartheta)$,*
(iii) *if $\omega \propto \vartheta$, then $-\omega \oplus 0 \leq \omega \leq \omega \oplus 0$,*
(iv) $0 \leq \omega \oplus \vartheta$, *if $\omega \propto \vartheta$,*
(v) *if $\omega \propto \vartheta$, then $\omega \oplus \vartheta = 0$ if and only if $\omega = \vartheta$,*
(vi) $(\omega + \vartheta) \oplus (\nu + \mu) \geq (\omega \oplus \nu) - (\vartheta \oplus \mu) \vee (\omega \oplus \mu) - (\vartheta \oplus \nu)$,
(vii) $\|0 \oplus 0\| = \|0\| = 0$,
(viii) $\|\omega \oplus \vartheta\| \leq \|\omega - \vartheta\|$,
(ix) *if $\omega \propto \vartheta$, then $\|\omega \oplus \vartheta\| = \|\omega - \vartheta\|$.*

Definition 2.3. *Assume that $G : \Sigma \to \Sigma$ is a mapping. Then*

(i) *G is an ξ-order non-extended mapping if there exists a constant $\xi > 0$ such that*
$$\xi(\omega \oplus \vartheta) \leq G(\omega) \oplus G(\vartheta), \text{ for all } \omega, \vartheta \in \Sigma,$$
(ii) *G is a comparison mapping if $\omega \propto \vartheta$, then $G(\omega) \propto G(\vartheta)$, $\omega \propto G(\omega)$ and $\vartheta \propto G(\vartheta)$, for all $\omega, \vartheta \in \Sigma$,*
(iii) *G is a strongly comparison mapping, if G is comparison mapping and $G(\omega) \propto G(\vartheta)$ if and only if $\omega \propto \vartheta$, for all $\omega, \vartheta \in \Sigma$.*

Definition 2.4. *Let $G : \Sigma \to \Sigma$ be a mapping and $Q : \Sigma \to 2^{\Sigma}$ be a multi-valued mapping. Then*

(i) *Q is called a weak-comparison mapping if $\mu_\omega \in Q(\omega)$, $\omega \propto \mu_\omega$, and if $\omega \propto \vartheta$, then there exists $\mu_\vartheta \in Q(\vartheta)$ such that $\mu_\omega \propto \mu_\vartheta$, for all $\omega, \vartheta \in \Sigma$,*
(ii) *Q is called a α_G-weak-non-ordinary difference mapping with respect to G if it is a weak comparison and for each $\omega, \vartheta \in \Sigma$, there exists $\alpha_G > 0$ and $\mu_\omega \in Q(G(\omega))$ and $\mu_\vartheta \in Q(G(\vartheta))$ such that*
$$(\mu_\omega \oplus \mu_\vartheta) \oplus \alpha_G(G(\omega) \oplus G(\vartheta)) = 0,$$
(iii) *Q is called a ρ-order different weak-comparison mapping with respect to G, if there exists $\rho > 0$ and for all $\omega, \vartheta \in \Sigma$, there exists $\mu_\omega \in Q(G(\omega))$, $\mu_\vartheta \in Q(G(\vartheta))$ such that*
$$\rho(\mu_\omega - \mu_\vartheta) \propto \omega - \vartheta,$$
(iv) *A weak-comparison mapping Q is called (α_G, ρ)-weak GNODD if it is an α_G-weak-non-ordinary difference mapping and ρ-order different weak-comparison mapping associated with G, and $[G + \rho Q](\Sigma) = \Sigma$.*

Definition 2.5. *Let G be a ξ-ordered non-extended mapping and Q be a α_G-non-ordinary difference mapping with respect to G. The resolvent operator $\mathcal{J}^Q_{G,\rho} : \Sigma \to \Sigma$ associated with G and Q is defined by*

$$\mathcal{J}^Q_{G,\rho}(\omega) = [G + \rho Q]^{-1}(\omega), \text{ for all } \omega \in \Sigma, \ \rho > 0.$$

Lemma 2.1. *Let $Q : \Sigma \to 2^{\Sigma}$ be an ordered (α_G, ρ)-weak GNODD mapping and $G : \Sigma \to \Sigma$ be a ξ-ordered non-extended mapping with respect to G. Then for $\alpha_G > \frac{1}{\rho}$, the following relation holds:*

$$\mathcal{J}^Q_{G,\rho}(\omega) \oplus \mathcal{J}^Q_{G,\rho}(\vartheta) \leq \frac{1}{\xi(\alpha_G\rho - 1)}(\omega \oplus \vartheta), \text{ for all } \omega, \vartheta \in \Sigma. \tag{1}$$

Definition 2.6. *The generalized Cayley operator $\mathcal{K}^Q_{G,\rho} : \Sigma \to \Sigma$ is defined as*

$$\mathcal{K}^Q_{G,\rho}(\omega) = \left[2\mathcal{J}^Q_{G,\rho} - G\right](\omega), \text{ for all } \omega \in \Sigma.$$

Definition 2.7. *The generalized Yosida approximation operator $\Upsilon^Q_{G,\rho} : \Sigma \to \Sigma$ is defined as*

$$\Upsilon^Q_{G,\rho}(\omega) = \frac{1}{\rho}\left[G - \mathcal{J}^Q_{G,\rho}\right](\omega), \text{ for all } \omega \in \Sigma.$$

Definition 2.8. *A mapping $G : \Sigma \to \Sigma$ is called Lipschitz continuous if there exists a constant $\tau_G > 0$ such that*

$$\|G(\omega) - G(\vartheta)\| \leq \tau_G \|\omega - \vartheta\|, \text{ for all } \omega, \vartheta \in \Sigma.$$

Definition 2.9. *Let $S : \Sigma \to C(\Sigma)$ be a multi-valued mapping. Then S is called D-Lipschitz continuous if there exists a constant $\tau_D > 0$ such that*

$$D(S(\omega), S(\vartheta)) \leq \tau_D \|\omega - \vartheta\|, \text{ for all } \omega, \vartheta \in \Sigma.$$

Proposition 2.2. [10] *If $\omega \propto \vartheta$, G is τ_G-Lipschitz continuous, $\mathcal{K}^Q_{G,\rho}(\omega) \propto \mathcal{K}^Q_{G,\rho}(\vartheta)$, $G(\omega) \propto G(\vartheta)$, for all $\omega, \vartheta \in \Sigma$, then $\mathcal{K}^Q_{G,\rho}$ is $\tau_{\mathcal{K}}$-Lipschitz continuous, where $\tau_{\mathcal{K}} = \frac{2+\tau_G\xi(\alpha_G\rho-1)}{\xi(\alpha_G\rho-1)}$.*

Proposition 2.3. *If $\omega \propto \vartheta$, G is τ_G-Lipschitz continuous, $\Upsilon^Q_{G,\rho}(\omega) \propto \Upsilon^Q_{G,\rho}(\vartheta)$, $G(\omega) \propto G(\vartheta)$, for all $\omega, \vartheta \in \Sigma$, then $\Upsilon^Q_{G,\rho}$ is τ_{Υ}-Lipschitz continuous, where $\tau_{\Upsilon} = \frac{1+\tau_G\xi(\alpha_G\rho-1)}{\rho\xi(\alpha_G\rho-1)}$.*

Proof. For all $\omega, \vartheta \in \Sigma$, using Lemma 2.1, we evaluate

$$\begin{aligned}
\left\|\Upsilon^Q_{G,\rho}(\omega) \oplus \Upsilon^Q_{G,\rho}(\vartheta)\right\| &= \left\|\frac{1}{\rho}\left[G(\omega) - \mathcal{J}^Q_{G,\rho}(\omega)\right] \oplus \frac{1}{\rho}\left[G(\vartheta) - \mathcal{J}^Q_{G,\rho}(\vartheta)\right]\right\| \\
&= \frac{1}{\rho}\left\|\mathcal{J}^Q_{G,\rho}(\omega) \oplus \mathcal{J}^Q_{G,\rho}(\vartheta)\right\| + \frac{1}{\rho}\|G(\omega) \oplus G(\vartheta)\| \\
&\leq \frac{1}{\rho\xi(\alpha_G\rho - 1)}\|\omega \oplus \vartheta\| + \frac{1}{\rho}\|G(\omega) \oplus G(\vartheta)\|.
\end{aligned}$$

Since $\omega \propto \vartheta$, $\Upsilon^Q_{G,\rho}(\omega) \propto \Upsilon^Q_{G,\rho}(\vartheta)$, $G(\omega) \propto G(\vartheta)$ by (ix) of Proposition 2.1 and using Lipschitz continuity of G, we obtain

$$\left\|\Upsilon^Q_{G,\rho}(\omega) - \Upsilon^Q_{G,\rho}(\vartheta)\right\| \leq \left[\frac{1}{\rho\xi(\alpha_G\rho - 1)} + \frac{\tau_G}{\rho}\right]\|\omega - \vartheta\|,$$

that is,

$$\left\|\Upsilon^Q_{G,\rho}(\omega) - \Upsilon^Q_{G,\rho}(\vartheta)\right\| \leq \tau_{\Upsilon}\|\omega - \vartheta\|,$$

where $\tau_{\Upsilon} = \frac{1+\tau_G\xi(\alpha_G\rho-1)}{\rho\xi(\alpha_G\rho-1)}$. □

3. Formulation of the problem and convergence analysis

Let Σ be an ordered real Hilbert space. Let $f : \Sigma \times \Sigma \to \Sigma$ and $G : \Sigma \to \Sigma$ be single-valued mappings. Let $Q : \Sigma \to 2^{\Sigma}$ and $A : \Sigma \to C(\Sigma)$ be multi-valued mappings. Let for $\rho > 0, \mathcal{K}^Q_{G,\rho} : \Sigma \to \Sigma$ be generalized Cayley operator and $\Upsilon^Q_{G,\rho} : \Sigma \to \Sigma$ be generalized Yosida approximation operator. We consider the following problem:

Find $\omega \in \Sigma$ and $\mu \in A(\omega)$ such that

$$0 \in f\left(\mathcal{K}^Q_{G,\rho}(\omega) \oplus \Upsilon^Q_{G,\rho}(\omega), \mu\right) + Q(\omega). \tag{2}$$

Problem (2) is called the Cayley Yosida inclusion problem involving XOR-operation.

Lemma 3.2. *For $\omega \in H$, $\mu \in A(\omega)$ is the solution of (2) if and only if it satisfies*

$$\omega = \mathcal{J}^Q_{G,\rho}\left[G(\omega) - \rho f\left(\mathcal{K}^Q_{G,\rho}(\omega) \oplus \Upsilon^Q_{G,\rho}(\omega), \mu\right)\right]. \tag{3}$$

Proof. Suppose $\omega \in H, \ \mu \in A(\omega)$ satisfy (3), then

$$\begin{aligned}
&\omega = \mathcal{J}^Q_{G,\rho}\left[G(\omega) - \rho f\left(\mathcal{K}^Q_{G,\rho}(\omega) \oplus \Upsilon^Q_{G,\rho}(\omega), \mu\right)\right]\\
&\omega = (G+\rho Q)^{-1}\left[G(\omega) - \rho f\left(\mathcal{K}^Q_{G,\rho}(\omega) \oplus \Upsilon^Q_{G,\rho}(\omega), \mu\right)\right]\\
&G(\omega) + \rho Q(\omega) = G(\omega) - \rho f\left(\mathcal{K}^Q_{G,\rho}(\omega) \oplus \Upsilon^Q_{G,\rho}(\omega), \mu\right)\\
&0 \in f\left(\mathcal{K}^Q_{G,\rho}(\omega) \oplus \Upsilon^Q_{G,\rho}(\omega), \mu\right) + Q(\omega).
\end{aligned}$$

□

Algorithm 3.4. *For given $\omega_0 \in \Sigma$, we choose $\mu_0 \in A(\omega_0)$ and compute the sequences $\{\omega_n\}$ and $\{\mu_n\}$ by the following iterative scheme:*

$$\omega_{n+1} = (1-\alpha)\omega_n + \alpha \mathcal{J}^Q_{G,\rho}\left[G(\omega_n) - \rho f\left(\mathcal{K}^Q_{G,\rho}(\omega_n) \oplus \Upsilon^Q_{G,\rho}(\omega_n), \mu_n\right)\right]. \tag{4}$$

Let $\mu_{n+1} \in A(\omega_{n+1})$ such that

$$\|\mu_n \oplus \mu_{n+1}\| = \|\mu_n - \mu_{n+1}\| \leq D\left(A(\omega_n), A(\omega_{n+1})\right), \tag{5}$$

where $\mu_n \propto \mu_{n+1}, \ 0 \leq \alpha \leq 1, \ \rho > 0$ are constants and $n = 0, 1, 2, \cdots$.

Theorem 3.5. *Let Σ denote a real ordered Hilbert space. Consider the mappings $f : \Sigma \times \Sigma \to \Sigma$, $A : \Sigma \to C(\Sigma)$, $G : \Sigma \to \Sigma$, and $Q : \Sigma \to 2^{\Sigma}$, satisfying the following properties:*

- (i) *f is Lipschitz continuous in both arguments, with Lipschitz constants τ_{f1} and τ_{f2}, respectively.*
- (ii) *A is D-Lipschitz continuous with constant τ_D.*
- (iii) *G is ξ-ordered, non-extended and Lipschitz continuous with constant τ_G.*
- (iv) *Q is (α_G, ρ)-weak GNODD mapping.*

Let $\mathcal{J}^Q_{G,\rho}$ satisfy condition (1), the generalized Cayley operator $\mathcal{K}^Q_{G,\rho}$ is $\tau_{\mathcal{K}}$-Lipschitz continuous, and the generalized Yosida approximation operator $\Upsilon^Q_{G,\rho}$ is τ_{Υ}-Lipschitz continuous. Suppose $\omega_n \propto \omega_{n+1}, \ n = 0, 1, 2, \cdots, \ \mathcal{K}^Q_{G,\rho}(\omega) \propto \mathcal{K}^Q_{G,\rho}(\vartheta), \ G(\omega) \propto G(\vartheta), \ \Upsilon^Q_{G,\rho}(\omega) \propto \Upsilon^Q_{G,\rho}(\vartheta)$, for all $\omega, \vartheta \in \Sigma$. Let the following condition hold:

$$0 < (1-\alpha) + \alpha P(\theta)\tau_G + \alpha P(\theta)\rho\left[\tau_{f2}\tau_D + \tau_{f1}(\tau_{\mathcal{K}} + \tau_{\Upsilon})\right] < 1, \tag{6}$$

where $P(\theta) = \frac{1}{\xi(\alpha_G \rho - 1)}$ *and* $\alpha_G > \frac{1}{\rho}$. *Then problem (2) has a solution* (ω, μ), *and the sequences* $\{\omega_n\}$ *and* $\{\mu_n\}$ *generated by Algorithm 3.4 converge to* ω *and* μ, *respectively.*

Proof. Using (4) and (iv) of Proposition 2.1 as $w_{n+1} \propto w_n$, we deduce

$$\begin{aligned}
0 \leq \omega_{n+1} \oplus \omega_n = & \Big[(1-\alpha)\omega_n + \alpha \mathcal{J}^Q_{G,\rho} \Big[G(\omega_n) - \rho f \Big(\mathcal{K}^Q_{G,\rho}(\omega_n) \\
& \oplus \Upsilon^Q_{G,\rho}(\omega_n), \mu_n\Big)\Big]\Big] \oplus \Big[(1-\alpha)\omega_{n-1} + \alpha \mathcal{J}^Q_{G,\rho}\left[G(\omega_{n-1})\right. \\
& \left. - \rho f \Big(\mathcal{K}^Q_{G,\rho}(\omega_{n-1}) \oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}), \mu_{n-1}\Big)\right]\Big] \\
\leq & (1-\alpha)(\omega_n \oplus \omega_{n-1}) + \alpha \Big[\mathcal{J}^Q_{G,\rho} \Big[G(\omega_n) - \rho f \Big(\mathcal{K}^Q_{G,\rho}(\omega_n) \\
& \oplus \Upsilon^Q_{G,\rho}(\omega_n), \mu_n\Big)\Big] \oplus \mathcal{J}^Q_{G,\rho} \Big[G(\omega_{n-1}) - \rho f \Big(\mathcal{K}^Q_{G,\rho}(\omega_{n-1}) \\
& \oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}), \mu_{n-1}\Big)\Big]\Big]. \qquad (7)
\end{aligned}$$

Since $\mathcal{J}^Q_{G,\rho}$ satisfy condition (1) and using the commutativity property of the $\oplus$ operation, we can transform (7) into

$$\begin{aligned}
0 \leq \omega_{n+1} \oplus \omega_n \leq & (1-\alpha)(\omega_n \oplus \omega_{n-1}) + \alpha P(\theta) \Big[\Big[G(\omega_n) - \rho f \Big(\mathcal{K}^Q_{G,\rho}(\omega_n) \\
& \oplus \Upsilon^Q_{G,\rho}(\omega_n), \mu_n\Big)\Big] \oplus \Big[G(\omega_{n-1}) - \rho f \Big(\mathcal{K}^Q_{G,\rho}(\omega_{n-1}) \\
& \oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}), \mu_{n-1}\Big)\Big]\Big] \\
\leq & (1-\alpha)(\omega_n \oplus \omega_{n-1}) + \alpha P(\theta) \left[(G(\omega_n) \oplus G(\omega_{n-1}))\right. \\
& + \rho \Big(f \Big(\mathcal{K}^Q_{G,\rho}(\omega_n) \oplus \Upsilon^Q_{G,\rho}(\omega_n), \mu_n\Big) \\
& \left. \oplus f \Big(\mathcal{K}^Q_{G,\rho}(\omega_{n-1}) \oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}), \mu_{n-1}\Big)\Big)\right]. \qquad (8)
\end{aligned}$$

Using (viii) of Proposition 2.1, from (8), we have

$$\begin{aligned}
\|\omega_{n+1} \oplus \omega_n\| \leq & (1-\alpha)\|\omega_n \oplus \omega_{n-1}\| + \alpha P(\theta) \|(G(\omega_n) \oplus G(\omega_{n-1})) \\
& + \rho \Big(f \Big(\mathcal{K}^Q_{G,\rho}(\omega_n) \oplus \Upsilon^Q_{G,\rho}(\omega_n), \mu_n\Big) \oplus f \Big(\mathcal{K}^Q_{G,\rho}(\omega_{n-1}) \\
& \oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}), \mu_{n-1}\Big)\Big)\Big\| \\
\leq & (1-\alpha)\|\omega_n \oplus \omega_{n-1}\| + \alpha P(\theta) \|G(\omega_n) \oplus G(\omega_{n-1})\| \\
& + \alpha P(\theta)\rho \Big\| f \Big(\mathcal{K}^Q_{G,\rho}(\omega_n) \oplus \Upsilon^Q_{G,\rho}(\omega_n), \mu_n\Big) \\
& \oplus f \Big(\mathcal{K}^Q_{G,\rho}(\omega_{n-1}) \oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}), \mu_{n-1}\Big)\Big\| \\
\leq & (1-\alpha)\|\omega_n - \omega_{n-1}\| + \alpha P(\theta) \|G(\omega_n) - G(\omega_{n-1})\| \\
& + \alpha P(\theta)\rho \Big\| f \Big(\mathcal{K}^Q_{G,\rho}(\omega_n) \oplus \Upsilon^Q_{G,\rho}(\omega_n), \mu_n\Big) \\
& - f \Big(\mathcal{K}^Q_{G,\rho}(\omega_{n-1}) \oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}), \mu_{n-1}\Big)\Big\|. \qquad (9)
\end{aligned}$$

As $\omega_{n+1} \propto \omega_n$ for all n, using (ix) of Proposition 2.1 and Lipschitz continuity of G from (9), we have

$$\begin{aligned}\|\omega_{n+1}-\omega_n\| \leq &(1-\alpha)\|\omega_n-\omega_{n-1}\| + \alpha P(\theta)\tau_G\|\omega_n-\omega_{n-1}\| \\ &+ \alpha P(\theta)\rho\left\| f\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n),\mu_n\right)\right. \\ &\left. -f\left(\mathcal{K}^Q_{G,\rho}(\omega_{n-1})\oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}),\mu_{n-1}\right)\right\|. \qquad (10)\end{aligned}$$

Using Lipschitz continuity of f in both the arguments and D-Lipschitz continuity of A, we have

$$\begin{aligned}&\left\| f\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n),\mu_n\right) - f\left(\mathcal{K}^Q_{G,\rho}(\omega_{n-1})\oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}),\mu_{n-1}\right)\right\| \\ &= \left\| f\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n),\mu_n\right) - f\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n),\mu_{n-1}\right)\right. \\ &\left.\ +f\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n),\mu_{n-1}\right) - f\left(\mathcal{K}^Q_{G,\rho}(\omega_{n-1})\oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}),\mu_{n-1}\right)\right\| \\ &\leq \left\| f\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n),\mu_n\right) - f\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n),\mu_{n-1}\right)\right\| \\ &\ + \left\| f\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n),\mu_{n-1}\right) - f\left(\mathcal{K}^Q_{G,\rho}(\omega_{n-1})\oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}),\mu_{n-1}\right)\right\| \\ &\leq \tau_{f2}\|\mu_n-\mu_{n-1}\| + \tau_{f1}\left\|\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n)\right) - \left(\mathcal{K}^Q_{G,\rho}(\omega_{n-1})\oplus \Upsilon^Q_{G,\rho}(\omega_{n-1})\right)\right\| \\ &\leq \tau_{f2}D\left(A(\omega_n),A(\omega_{n-1})\right) + \tau_{f1}\left\|\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n)\right)\right. \\ &\left.\ - \left(\mathcal{K}^Q_{G,\rho}(\omega_{n-1})\oplus \Upsilon^Q_{G,\rho}(\omega_{n-1})\right)\right\| \\ &\leq \tau_{f2}\tau_D\|\omega_n-\omega_{n-1}\| + \tau_{f1}\left\|\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n)\right) - \left(\mathcal{K}^Q_{G,\rho}(\omega_{n-1})\oplus \Upsilon^Q_{G,\rho}(\omega_{n-1})\right)\right\|. \qquad (11)\end{aligned}$$

Using Lipschitz continuity of $\mathcal{K}^Q_{G,\rho}$, $\Upsilon^Q_{G,\rho}$, condition (vi) of Proposition 2.1, and the fact that $\oplus$ is commutative, we have

$$\begin{aligned}&\left\|\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n)\right) - \left(\mathcal{K}^Q_{G,\rho}(\omega_{n-1})\oplus \Upsilon^Q_{G,\rho}(\omega_{n-1})\right)\right\| \\ &= \left\|\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n)\right) - \left(\Upsilon^Q_{G,\rho}(\omega_{n-1})\oplus \mathcal{K}^Q_{G,\rho}(\omega_{n-1})\right)\right\| \\ &\leq \left\|\left(\mathcal{K}^Q_{G,\rho}(\omega_n)+ \Upsilon^Q_{G,\rho}(\omega_{n-1})\right) \oplus \left(\Upsilon^Q_{G,\rho}(\omega_n)+ \mathcal{K}^Q_{G,\rho}(\omega_{n-1})\right)\right\| \\ &\leq \left\|\left(\mathcal{K}^Q_{G,\rho}(\omega_n)+ \Upsilon^Q_{G,\rho}(\omega_{n-1})\right) - \left(\Upsilon^Q_{G,\rho}(\omega_n)+ \mathcal{K}^Q_{G,\rho}(\omega_{n-1})\right)\right\| \\ &\leq \left\|\mathcal{K}^Q_{G,\rho}(\omega_n) - \mathcal{K}^Q_{G,\rho}(\omega_{n-1})\right\| + \left\|\Upsilon^Q_{G,\rho}(\omega_n) - \Upsilon^Q_{G,\rho}(\omega_{n-1})\right\| \\ &\leq \tau_{\mathcal{K}}\|\omega_n-\omega_{n-1}\| + \tau_{\Upsilon}\|\omega_n-\omega_{n-1}\|. \qquad (12)\end{aligned}$$

Using (12) in (11), we have

$$\begin{aligned}&\left\| f\left(\mathcal{K}^Q_{G,\rho}(\omega_n)\oplus \Upsilon^Q_{G,\rho}(\omega_n),\mu_n\right) - f\left(\mathcal{K}^Q_{G,\rho}(\omega_{n-1})\oplus \Upsilon^Q_{G,\rho}(\omega_{n-1}),\mu_{n-1}\right)\right\| \\ &\leq \left[\tau_{f2}\tau_D + \tau_{f1}\left(\tau_{\mathcal{K}}+\tau_{\Upsilon}\right)\right]\|\omega_n-\omega_{n-1}\|. \qquad (13)\end{aligned}$$

From (10) and (13), we have

$$\begin{aligned}\|\omega_{n+1}-\omega_n\| &\leq (1-\alpha)\|\omega_n-\omega_{n-1}\| + \alpha P(\theta)\tau_G\|\omega_n-\omega_{n-1}\| \\ &\quad + \alpha P(\theta)\rho\left[\tau_{f2}\tau_D + \tau_{f1}(\tau_{\mathcal{K}}+\tau_{\Upsilon})\right]\|\omega_n-\omega_{n-1}\| \\ &= \Big[(1-\alpha) + \alpha P(\theta)\tau_G + \alpha P(\theta)\rho\left[\tau_{f2}\tau_D + \tau_{f1}(\tau_{\mathcal{K}}+\tau_{\Upsilon})\right]\Big] \\ &\quad \times \|\omega_n-\omega_{n-1}\| \\ &= \eta(\theta)\|\omega_n-\omega_{n-1}\|, \end{aligned} \tag{14}$$

where $\eta(\theta) = \Big[(1-\alpha) + \alpha P(\theta)\tau_G + \alpha P(\theta)\rho\left[\tau_{f2}\tau_D + \tau_{f1}(\tau_{\mathcal{K}}+\tau_{\Upsilon})\right]\Big]$.

By the condition (6), $\eta(\theta) < 1$, we can deduce from (14) that the sequence $\{\omega_n\}$ is a Cauchy sequence. Consequently, there exists an element $\omega \in \Sigma$ such that $\{\omega_n\}$ converges to ω as $n \to \infty$. Utilizing (5) and the D-Lipschitz continuity of A, we can infer that $\{\mu_n\}$ is also a Cauchy sequence in Σ. Thus, there exists an element $\mu \in \Sigma$ such that $\{\mu_n\}$ converges to μ as $n \to \infty$. Employing the continuity of all the operators involved in (2) and Lemma 3.2, we conclude that

$$\omega = \mathcal{J}^Q_{G,\rho}\left[G(\omega) - \rho f\left(\mathcal{K}^Q_{G,\rho}(\omega) \oplus \Upsilon^Q_{G,\rho}(\omega), \mu\right)\right].$$

Therefore, the result follows. □

sectionNumerical example To support Theorem 3.5, we present the following numerical example:

Example 3.6. *Let $\Sigma = \mathbb{R}$ with the usual norm and inner product.*

(i) *Let $f : \Sigma \times \Sigma \to \Sigma$ be a single-valued mapping such that*

$$f(\omega, \vartheta) = \frac{2\omega}{17} + \frac{\vartheta}{13}.$$

Then for any $\omega_1, \omega_2, \vartheta \in \Sigma$, we have

$$\begin{aligned}\|f(\omega_1,\vartheta) - f(\omega_2,\vartheta)\| &= \left\|\frac{2\omega_1}{17} + \frac{\vartheta}{13} - \frac{2\omega_2}{17} - \frac{\vartheta}{13}\right\| \\ &= \frac{2}{17}\|\omega_1-\omega_2\| \\ &\leq \frac{2}{15}\|\omega_1-\omega_2\|,\end{aligned}$$

that is, f is Lipschitz continuous with respect to its first argument, with a constant $\tau_{f1} = \frac{2}{15}$. Similarly, it can be shown that f is Lipschitz continuous with respect to its second argument, with a constant $\tau_{f2} = \frac{1}{11}$.

(ii) *Let $A : \Sigma \to C(\Sigma)$ be a multi-valued mapping defined as*

$$A(\omega) = \left\{\frac{2\omega}{17}\right\}.$$

Now

$$\begin{aligned}D(A(\omega), A(\vartheta)) &\leq max\left\{\left\|\frac{2\omega}{17} - \frac{2\vartheta}{17}\right\|, \left\|\frac{2\vartheta}{17} - \frac{2\omega}{17}\right\|\right\} \\ &= \frac{2}{17}max\{\|\omega-\vartheta\|, \|\vartheta-\omega\|\} \\ &\leq \frac{2}{13}\|\omega-\vartheta\|.\end{aligned}$$

That is, A is D-Lipschitz continuous with constant $\tau_D = \frac{2}{13}$.

(iii) *Let $G : \Sigma \to \Sigma$ be a single-valued mapping defined by*

$$G(\omega) = \frac{\omega}{5}.$$

Clearly G is Lipschitz continuous with constant $\tau_G = \frac{2}{5}$ and ξ-ordered non-extended mapping with constant $\xi = \frac{1}{7}$.

(iv) *Let $Q : \Sigma \to 2^{\Sigma}$ be a multi-valued mapping defined by*

$$Q(\omega) = \{3\omega\}.$$

For $\rho = 5$, it is clear that Q is (α_G, ρ)-weak GNODD mapping with $\alpha_G = \frac{2}{5}$.

(v) *Based on the aforementioned calculations, we can derive the resolvent operator $\mathcal{J}^{Q}_{G,\rho}$ such that*

$$\mathcal{J}^{Q}_{G,\rho}(\omega) = [G + \rho Q]^{-1}(\omega) = \frac{5}{76}\omega,$$

where $\rho = 5$.
The resolvent operator $\mathcal{J}^{Q}_{G,\rho}$ satisfies the condition (1), that is,

$$\begin{aligned}\mathcal{J}^{Q}_{G,\rho}(\omega) \oplus \mathcal{J}^{Q}_{G,\rho}(\vartheta) &= \frac{5}{76}\omega \oplus \frac{5}{76}\vartheta \\ &= \frac{5}{76}(\omega \oplus \vartheta) \\ &\leq \frac{1}{2}(\omega \oplus \vartheta),\end{aligned}$$

where $\frac{1}{\xi(\alpha_G\rho - 1)} = \frac{1}{2}$.

(vi) *Using the value of $\mathcal{J}^{Q}_{G,\rho}$, we obtain the generalized Yosida approximation operator $\Upsilon^{Q}_{G,\rho}$ as*

$$\Upsilon^{Q}_{G,\rho}(\omega) = \frac{1}{\rho}\left[G(\omega) - \mathcal{J}^{Q}_{G,\rho}(\omega)\right] = \frac{51}{1900}\omega.$$

Clearly, $\Upsilon^{Q}_{G,\rho}$ is Lipschitz continuous with a Lipschitz constant

$$\tau_\Upsilon = \frac{1 + \tau_G\xi(\alpha_G\rho - 1)}{\rho\xi(\alpha_G\rho - 1)} = \frac{37}{25}.$$

(vii) *Using the value of $\mathcal{J}^{Q}_{G,\rho}$, we obtain the generalized Cayley operator $\mathcal{K}^{Q}_{G,\rho}$ as*

$$\mathcal{K}^{Q}_{G,\rho}(\omega) = \left[2\mathcal{J}^{Q}_{G,\rho}(\omega) - G(\omega)\right] = \frac{-13}{185}\omega.$$

Clearly, $\mathcal{K}^{Q}_{G,\rho}$ is Lipschitz continuous with a Lipschitz constant

$$\tau_\mathcal{K} = \frac{2 + \tau_G\xi(\alpha_G\rho - 1)}{\xi(\alpha_G\rho - 1)} = \frac{49}{35}.$$

(viii) *For $P(\theta) = \frac{1}{2}$, $\alpha = \frac{1}{2}$, $\tau_G = \frac{2}{5}$, $\rho = 5$, $\tau_{f1} = \frac{2}{15}$, $\tau_{f2} = \frac{1}{11}$, $\tau_D = \frac{2}{13}$, $\tau_\mathcal{K} = \frac{49}{35}$ and $\tau_\Upsilon = \frac{27}{250}$, the condition (6) of Theorem 3.5 is satisfied.*

Therefore, all the conditions of Theorem 3.5 are fulfilled, and as a result, problem (2) possesses a solution (ω, μ). Consequently, $\{\omega_n\}$ and $\{\mu_n\}$ converge to ω and μ, respectively.

Table 1. Numerical values of ω_n for different initial values of ω_0.

No. of iterations	$\omega_0 = 1.0$ ω_n	*No. of iterations*	$\omega_0 = 2$ ω_n	*No. of iterations*	$\omega_0 = 4$ ω_n
1	1.0000	1	2.0000	1	4.0000
2	0.5032	2	1.0064	2	2.0128
3	0.2532	3	0.5064	3	1.0129
4	0.1274	4	0.2548	4	0.5097
5	0.0641	5	0.1282	5	0.2565
6	0.0323	6	0.0645	6	0.1291
8	0.0082	8	0.0163	8	0.0327
10	0.0021	10	0.0041	10	0.0083
12	0.0005	12	0.0010	12	0.0021
14	0.0001	14	0.0003	14	0.0005
16	0.0000	16	0.0001	16	0.0001
18	0.0000	18	0.0000	18	0.0000
19	0.0000	19	0.0000	19	0.0000
20	0.0000	20	0.0000	20	0.0000

The sequence $\{\omega_n\}$ *is computed using the following iterative scheme:*

$$\begin{aligned} \omega_{n+1} &= (1-\alpha)\omega_n + \alpha\mathcal{J}^{Q}_{G,\rho}\left[G(\omega_n) - \rho f\left(\mathcal{K}^{Q}_{G,\rho}(\omega_n) \oplus \Upsilon^{Q}_{G,\rho}(\omega_n), \mu_n\right)\right] \\ &= \frac{23766851}{47230352}\omega_n, \\ \textit{and } \mu_{n+1} &= \left(\frac{2}{17}\right)\left(\frac{23766851}{47230352}\omega_n\right) \\ &= \frac{23766851}{401457992}\omega_n. \end{aligned}$$

All the code is implemented using MATLAB R2021a. Figure 4.1 (Table 1) illustrates the convergence behavior of $\{\omega_n\}$ *for initial values* $\omega_0 = 1, 2, 4$. *Additionally, Figure 4.2 (Table 2) presents the convergence of both* $\{\omega_n\}$ *and* $\{\mu_n\}$ *for initial values* $\omega_0 = 1, 4$.

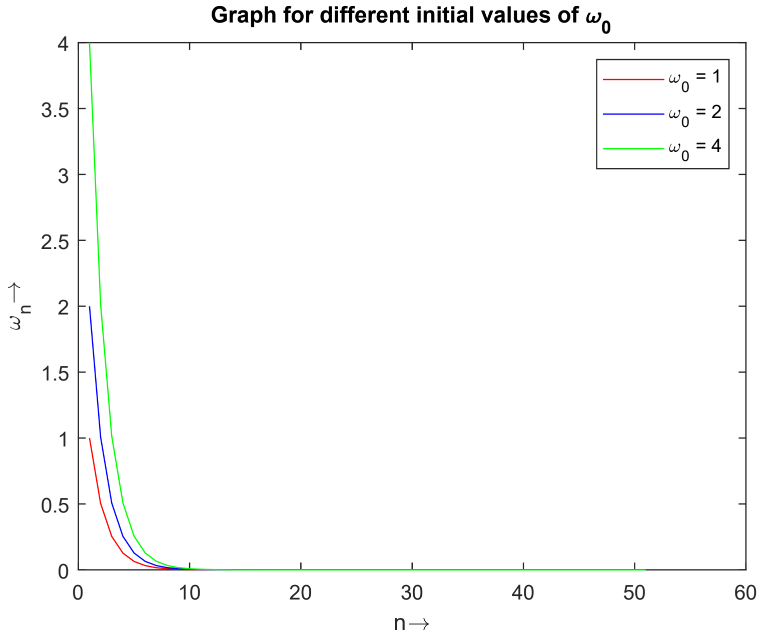

FIGURE 1. The convergence of $\{\omega_n\}$ with initial values $\omega_0 = 1,\ \omega_0 = 2$ and $\omega_0 = 4$.

TABLE 2. Numerical values of ω_n and μ_n for different initial values of ω_0.

	$\omega_0 = 1.0$			$\omega_0 = 4$	
No. of iterations	ω_n	μ_n	*No. of iterations*	ω_n	μ_n
1	*1.0000*	*0.0592*	*1*	*4.0000*	*0.2368*
2	*0.5032*	*0.0298*	*2*	*2.0128*	*0.1192*
3	*0.2532*	*0.0150*	*3*	*1.0129*	*0.0600*
4	*0.1274*	*0.0075*	*4*	*0.5097*	*0.0302*
5	*0.0641*	*0.0038*	*5*	*0.2565*	*0.0152*
6	*0.0323*	*0.0019*	*6*	*0.1291*	*0.0076*
8	*0.0082*	*0.0005*	*8*	*0.0327*	*0.0019*
10	*0.0021*	*0.0001*	*10*	*0.0083*	*0.0005*
12	*0.0005*	*0.0000*	*12*	*0.0021*	*0.0001*
14	*0.0001*	*0.0000*	*14*	*0.0005*	*0.0000*
16	*0.0000*	*0.0000*	*16*	*0.0001*	*0.0000*
18	*0.0000*	*0.0000*	*18*	*0.0000*	*0.0000*
20	*0.0000*	*0.0000*	*20*	*0.0000*	*0.0000*

4. Conclusion

The paper introduces a new class of variational inclusions called Cayley Yosida inclusion problems involving XOR operations. We present a fixed-point formulation of the problem and propose an iterative algorithm to solve it. We also conduct

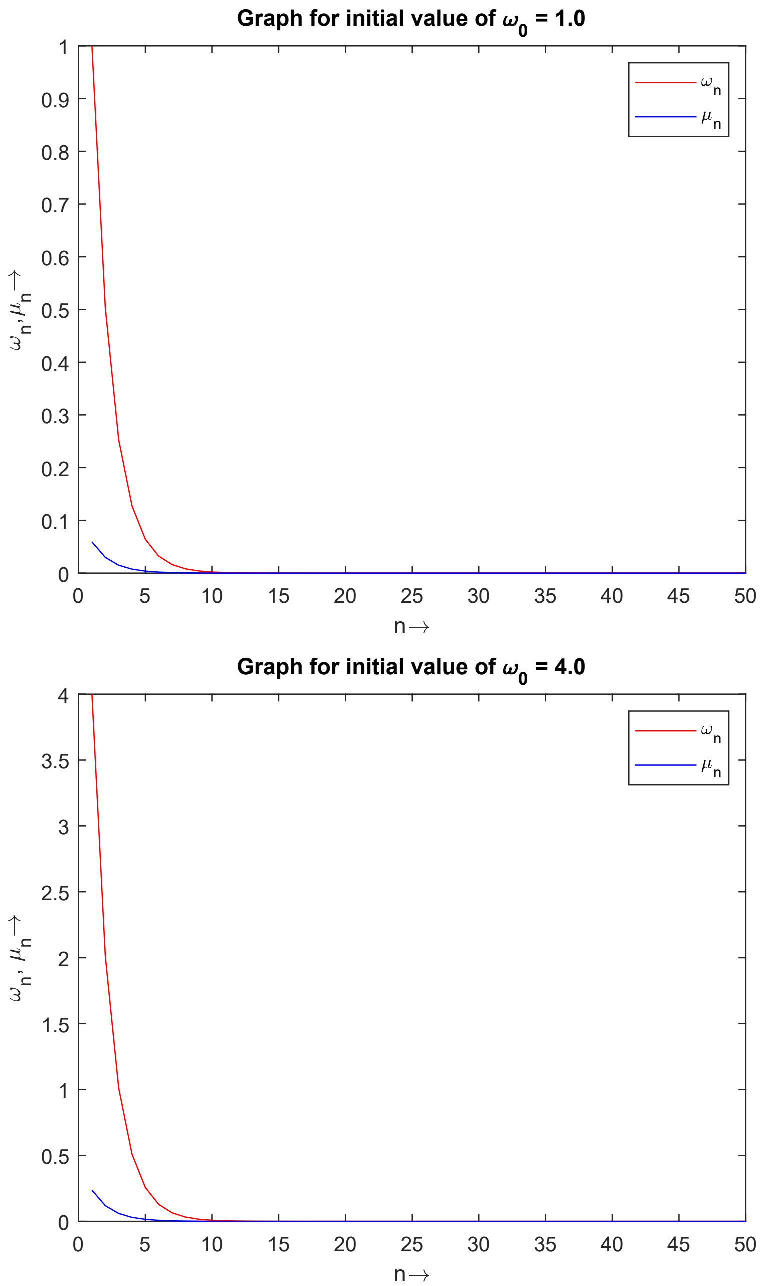

FIGURE 2. The convergence of $\{\omega_n\}$ and $\{\mu_n\}$ with initial values $\omega_0 = 1$ and $\omega_0 = 4$.

a convergence analysis of the algorithm to ensure its effectiveness. To validate our main result, we provide a numerical example that demonstrates the practical applicability of our approach. Overall, this paper contributes to the study of variational inclusions involving XOR-operation, providing a theoretical framework and a computational method for solving such problems.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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